Wilf's conjecture by multiplicity

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Wilf's question

A numerical semigroup is a subset $S \subset \mathbb{N}$ such that

- $0 \in S$, $S + S \subset S$,
- there exists d such that $n \in S$ for all $n \geq d$ ($\iff$ $\gcd(S) = 1$)

S is has a unique minimal system A of generators. $e(S) = |A|$ is the embedding dimension of S. Usually S given by its generators:

$$S = \langle a_1, \dots, a_e \rangle = \left\{ b_1 a_1 + \dots + b_e a_e : b_1, \dots, b_e \in \mathbb{N} \right\}.$$

$\Gamma(S) = \mathbb{N} \setminus S$ is the set of gaps of S. $F(S) = \sup \Gamma(S)$ is the Frobenius number, $c(S) = F(S) + 1$ is the conductor, and the genus is $\gamma(S) = |\Gamma(S)|$.

Wilf's question (1978):

$$\frac{\gamma(S)}{c(S)} \leq 1 - \frac{1}{e(S)} \ ?$$

An example

$S = \langle 6, 10, 15 \rangle$, $e(S) = 3$

Gaps in red:

	0	1	2	3	4	5
$m(S) =$	6	7	8	9	10	11
	12	13	14	15	16	17
	18	19	20	21	22	23
	24	25	26	27	28	29 $= F(S)$
$c(S) =$	30	31	32	33	34	35
	...					

Wilf's inequality:

$$\frac{\gamma(S)}{c(S)} = \frac{15}{30} \leq 1 - \frac{1}{3} = 1 - \frac{1}{e(S)}$$

The blue numbers form the Apéry set to be defined later.

Wilf's question promoted

- $\Sigma(S) = \{x \in S : x < F(S)\}$ is the set of sporadic elements,
- $\sigma(S) = |\Sigma(S)|$ is their number.

Wilf's question reformulated and promoted:

Conjecture

For any numerical semigroup S one has $c(S) \leq e(S)\sigma(S)$.

Finally:

- $m(S) = \min\{x \in S : x > 0)\}$ is the multiplicity of S,

Our goal:

- Show that the conjecture can be decided efficiently for fixed m by polyhedral methods and
- describe an algorithm by which we have verified it for $m \leq 18$.

The Apéry set

View S as a "module" over the subsemigroup $\mathbb{N}m$, $m = m(S)$:

$$S = \bigcup_{i=0}^{m-1} \{x \in S : x \equiv i \mod m\} = \bigcup_{i=0}^{m-1} b_i + \mathbb{N}m.$$

with $b_i \in S$, $b_i \equiv i \mod m$.

Definition

The Apéry set of S is $\operatorname{Ap}(S) = \{b_0 = 0, \dots, b_{m-1}\}$.

$\operatorname{Ap}(S) \setminus \{0\}$ poset: $b_i \prec b_j \iff b_j - b_i \in S$ ($\iff b_j - b_i \in \operatorname{Ap}(S)$)

- $\operatorname{Min}_\prec \operatorname{Ap}(S) \cup \{m\}$ is the minimal system of generators.
- $\operatorname{Max}_\prec \operatorname{Ap}(S)$ is the socle of S. Its cardinality is the type $t(S)$.

We transfer the partial order: $\mathcal{P}(S) = \{1, \dots, m-1\}$ with $i \prec j \iff b_i \prec b_j$.

Known cases of Wilf's conjecture

There are many conditions that imply Wilf's inequality:

1. $e(S) = 2$
2. $m(S) = e(S)$ (maximal embedding dimension, Dobbs and Matthews)
3. $e(S) > t(S)$ (Fröberg, Gottlieb, and Häggkvist)
4. $2e(S) \geq m(S)$ (Sammartano) (Eliahou: $3e(S) \geq m(S)$)
5. $c(S) \leq 3m(S)$ (Eliahou, using Macaulay's theorem on Hilbert functions)
6. $\gamma(S) \leq 60$ (Fromentin and Hivert)

For $\gamma(S) \to \infty$ the probability of $c(S) \leq 3m(S)$ goes to 1 (Zhai). One can say: Wilf holds with probability 1.

In (1) and certain cases of (2) Wilf holds with $=$. It is unknown whether these are the only cases. (Checked for $m \leq 14$.)

The Kunz polyhedron

We fix $m = m(S)$. S has Kunz coordinates $(x_1, \dots, x_{m-1})$ with

$$b_i = x_i m + i, \qquad i = 1, \dots, m-1.$$

By the definition of $\mathrm{Ap}(S)$ they satisfy the inequalities

$$\begin{aligned} x_i + x_j &\geq x_{i+j} \qquad \text{for } i+j < m, \\ x_i + x_j + 1 &\geq x_{i+j} \qquad \text{for } i+j > m. \end{aligned}$$

These inequalities define the Kunz polyhedron $P_m \subset \mathbb{R}^{m-1}$. The Kunz cone C_m is defined by the associated homogeneous system.

Theorem (Kunz 1987, Rosales et al. 2002)

The semigroups of multiplicity m are in 1-1 correspondence with the integer points in P_m that have coordinates ≥ 1.

Identify S with $(x_1, \dots, x_{m-1})$. Note: $\gamma(S) = x_1 + \cdots + x_{m-1}$.

The Kunz polyhedron and the Kunz cone

We will have to look at the faces of the Kunz polyhedron P_m. Fortunately it diifers fromm the Kunz cone as little as possible:

Theorem

$P_m = (-1/m, -2/m, \dots, -(m-1)/m) + C_m$

So comouting the faces of P_m can be reduced to computing the faces of C_m

This is very helpful since the group $(\mathbb{Z}/(m))^*$ operates as a group of integral automorpisms on C_m: the indices in the inequalities defining C_m are taken modulo m, and multiplication by a unit of $\mathbb{Z}/(m)$ just permutes them!

Faces of the Kunz poyhedron

There exists a unique face $\mathrm{Face}(S)$ of P_m such that S lies in its interior $\mathrm{Face}(S)^\circ$.

Lemma

$\mathrm{Face}(S) = \mathrm{Face}(S') \iff \mathcal{P}(S) = \mathcal{P}(S')$

Among the inequalities defining P_m we pick the subset $E(S)$ that hold in S with $=$ and therefore define $\mathrm{Face}(S)$.

Let p be the number of x_i appearing on the LHS of any inequality in $E(S)$ and n their number on the RHS. Then:

$$e(S) = m(S) - n \qquad t(S) = m(S) - 1 - p.$$

So $\mathrm{Face}(S) = \mathrm{Face}(S') \implies e(S) = e(S'),\ t(S) = t(S')$.

But $\mathrm{Face}(S) = \mathrm{Face}(S') \not\Longrightarrow c(S) = c(S')$.

Wilf's conjecture for fixed m in finitely many steps

Strategy for (dis)proving Wilf's conjecture for fixed $m = m(S)$:

- Compute the face lattice of P_m (equivalently, of C_m)
- Select the "bad" faces ($\sim 0.4 - 1\%$) satisfying $e(S) \leq t(S)$ and $2e(S) < m(S)$: both necessary for a counterexample
- Subdivide each bad face into subpolyhedra Q_i such that x_i deternines $c(S)$ (system of linear inequalities for each i)
- Add $x_j \geq 1$ for all j
- Add the linear inequality saying that Wilf is violated
- Check the critical subpolyhedra for lattice points

For $m \leq 18$ no lattice point was found. Even more: the critical subpolyhedra are all empty!

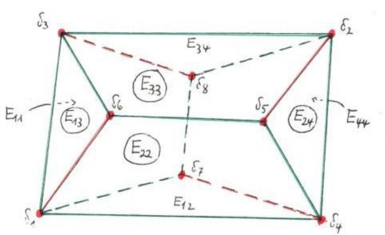
$\Longrightarrow$ Wilf's conjecture holds for $m \leq 18$

Combinatorial data of the Kunz cones – 1987

in E. Kunz, *Über die Klassifikation numerischer Halbgruppen,* Regensburger Mathematische Schriften **11**, 1987

Anhang A. Einige Daten über P_m und P_m^*.

m	3	4	5	6	7	8	9	10	11
Kantenzahl von P_m	2	4	8	11	30	47	122	225	812
Seitenzahl von P_m (ohne Spitze)	3	9	31	83	399	1347			



rot: symmetrische Halbgruppen
grün: fastvollständige Durchschnitte

Combinatorial data of the Kunz cones – 2019

$(\mathbb{Z}/(m))^*$ operates on C_m as a group opf automorphisms (but not on P_m). "Orbit" refers to this action:

m	ine	extr rays	orbits	bad orbits	faces	bad faces
7	18	30	400	0	2346	0
8	24	47	1,348	0	5,086	0
9	32	122	6,508	54	38,788	324
10	40	225	26,682	74	106,434	292
11	50	812	15,622	178	155,944	1,765
12	60	1,864	169,607	714	669,794	2,791
13	72	7,005	365,881	4,338	4,389,234	52,035
14	84	15,585	3,506,961	15,251	21,038,016	91,394
15	98	67,262	17,217,534	180,464	137,672,474	1,441,273
16	112	184,025	94,059,396	399,380	751,497,188	3,184,022
17	128	851,890	333,901,498	3,186,147	5,342,388,604	50,977,648
18	144	2,158,379	4,712,588,473	17,345,725	28,275,375,292	104,071,319
19	162	11,665,781	??	??	??	??
20	180	34,966,501	??	??	??	??
21	200	169,543,084	??	??	??	??

The Normaliz face lattice algorithm (raw version)

Every face F is the intersection of the facets $\mathbb{H}(F) = \{H \supset F\}$.
$\mathbb{E}(F)$ = extreme rays through F. C given by $\mathbb{H}(C)$.

Precomputed: $\mathbb{E}(C)$, $\mathbb{E}(H)$ for $H \in \mathbb{H}(C)$

Algorithm (simplified)

function FACELATTICE(C)
  $\mathcal{F} \leftarrow \emptyset$, $\mathcal{W} \leftarrow \{C\}$, $\mathcal{N} \leftarrow \emptyset$
  while $\mathcal{W} \neq \emptyset$ **do**
    for all $F \in \mathcal{W}$ **do** (*parallelized*)
      $\mathbb{E}(F) = \bigcap_{H \in \mathbb{H}(F)} \mathbb{E}(H)$
      for all $H \in \mathbb{H}(C)$ **do**
        compute $G = F \cap H$ *and* $\mathbb{H}(G)$, $[G \leftarrow \min \operatorname{orbit}(G)]$
        if $G \notin \mathcal{F} \cup \mathcal{W} \cup \mathcal{N}$ **then** $\mathcal{N} \leftarrow \mathcal{N} \cup \{G\}$
      end for
    end for
    $\mathcal{F} \leftarrow \mathcal{F} \cup \mathcal{W}$, $\mathcal{W} \leftarrow \mathcal{N}$, $\mathcal{N} \leftarrow \emptyset$
  end while
  return $\mathcal{F}$
end function

Some aspects of the design

A face F of C is cosimplicial if it is contained in exactly codim F facets of C. They are harmless: one ca avoid producing them more than once (at least without automorphisms) since they are obtained as an intersection of facets in a unique way.

The "difficult" faces are the degenerate, non-cosimplicial ones: even without automorphisms we don't know how to completely avoid reproducing them as an intersection of facets.

Some design details:

- Breadth first recursion: better papallelization,
- Use $\mathbb{H}(F)$ as the sugnature of F (and not $\mathbb{E}(F)$),
- avoid linear algebra as little as possible in finding the facets of a face.

Performance

m	preparation	face lattice	bad faces	total time	$\approx$ RAM
11	—	—	—	0.7 s	6 MB
12	—	—	—	2.5 s	35 MB
13	1 s	5 s	17 s	23 s	80 MB
14	3 s	37 s	39 s	1:19 m	603 MB
15	19 s	4:32 m	15 m	19:43 m	2.6 GB
16	65 s	57:43 m	37 m	1:35 h	12 GB
17	6:05 m	21:27 h	17:13 h	38:46 h	48 GB
18	19:19 m	27:13 d	1:16 d	29:05 d	720 GB

Most time consuming operations ($m = 14$):

- checking $<_{\text{lex}}$ for subsets of $\mathbb{H}(C)$ or $\mathbb{E}(C)$
- checking $\subset$

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