Algebras over monoidal complexes

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Sources

- M. Brun, W. Bruns and T. Römer, *Cohomology of partially ordered sets and local cohomology of section rings*. Adv. Math. **208**, 210–235 (2007).
- B. Ichim and T. Römer, *On toric face rings*. J. Pure Appl. Algebra **210**, 249–266 (2007).
- W. Bruns, R. Koch and T. Römer, *Gröbner bases and Betti numbers of monoidal complexes*, Michigan Math. J., to appear
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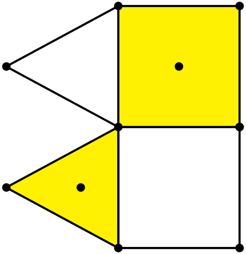
The objects

A monoidal complex $\mathscr{M}$ consists of

- a combinatorial skeleton, represented by a conical complex $\mathscr{C}$
- discrete data, namely affine monoids M_c, $c \in \mathscr{C}$ glued together along the combinatorial structure of $\mathscr{C}$.



cross-section of a fan

marked points

$\leftrightarrow$ monoid generators

Given a coefficient ring (field) $\Bbbk$, we want to define the algebra $\Bbbk[\mathscr{M}]$.

Best known example: a simplicial complex Δ defines a conical complex $\mathscr{C}(\Delta)$ (with cross-section Δ). If we take the free monoid of rank $\dim f + 1$ as M_f, $f \in \Delta$, then

$$\Bbbk[\Delta] = \Bbbk[\mathscr{C}(\Delta)]$$

is the Stanley-Reisner ring of Δ.

Formally:

Definition

A conical complex $\mathscr{C}$ is a finite collection of finitely generated cones C such that

- $C \in \mathscr{C} \Rightarrow D \in \mathscr{C}$ for each face D of $\mathscr{C}$,
- $C \cap C'$ is a face of C and of C' for all $C, C' \in \mathscr{C}$.

each C is contained in a space $\mathbb{R}^{d_C}$, but we do not require that the whole collection lives in such a space. If it does, then $\mathscr{C}$ is called a fan.

Definition

A monoidal complex $\mathscr{M}$ on $\mathscr{C}$ consists of a collection M_C, $C \in \mathscr{C}$, of affine monoids such that

- $M_C \subset C$ and $C = \mathbb{R}_+ M_C$,
- $M_D \subset M_C$ for cones $D \subset C$ in $\mathscr{C}$,
- $M_{C \cap C'} = M_C \cap M_{C'}$ for $C, C' \in \mathscr{C}$.

For each (affine) monoid $(M,+)$ we have the monoid algebra $\Bbbk[M]$ defined in the usual way: it is the free module with a "monomial basis" X^x, $x \in M$, and multiplication $X^x X^y = X^{x+y}$ (extended bilinearly).

Two ways to define $\Bbbk[\mathscr{M}]$:

Via the ring structure: For every face D of a cone C we have an epimorphism

$$\pi_{C,D} : \Bbbk[M_C] \to \Bbbk[M_D],$$

$\pi_{C,D}(X^x) = X^x$ for $x \in M_D$, $\quad \pi_{C,D}(X^x) = 0$ else.

The $\pi_{C,D}$ form an inverse system (relative to the face poset), and we set

$$\begin{aligned}\Bbbk[\mathscr{M}] &= \varprojlim K[M_C] \\ &= \Big\{(x_C) \in \prod_{C\in\mathscr{C}} \Bbbk[M_C] : \pi_{C,D}(x_C) = x_D \text{ for all } C \supset D\Big\}.\end{aligned}$$

Via the monoid structure: For every face D of a cone C we have an injection $\iota : M_D \to M_C$. They form a direct system, and we can consider the set

$$|\mathscr{M}| = \varinjlim M_C.$$

This is a partial monoid only:

$$x + y \in |\mathscr{M}| \iff \text{there exists } C \text{ with } x, y \in C.$$

This gives rise to the algebra

$$\Bbbk[|\mathscr{M}|] = \bigoplus_{x \in |\mathscr{M}|} \Bbbk X^x, \qquad X^x X^y = \begin{cases} X^{x+y} & x + y \text{ defined} \\ 0 & \text{else} \end{cases}$$

Proposition

$$\Bbbk[\mathscr{M}] = \Bbbk[|\mathscr{M}|]$$

is reduced. Its monomial prime ideals (those generated by subsets of $|\mathscr{M}|$) correspond 1:1 to the (complements of) the faces of $\mathscr{C}$. In particular, its minimal primes correspond to the maximal cones in $\mathscr{C}$.

Examples

- affine monoid algebras (trivial incidence structure),
- Stanley-Reisner rings (simplicial (conical) complex, trivial monoid structure),
- toric face rings ($\mathscr{C}$ is a fan in $\mathbb{R}^d$ and ${}_C = C \cap \mathbb{Z}^d$),
- polyhedral algebras (B. & Gubeladze, $\mathscr{C}$ defined by a complex of lattice polytopes, monoids are the corresponding polytopal monoids).

The defining ideal

$\mathscr{M}$ monoidal complex on $\mathscr{C}$. Choose $E \subset |\mathscr{M}|$ such that $E \cap M_C$ generates M_C or each $C \in \mathscr{C}$. Then $\{X^e : e \in E\}$ generates $\Bbbk[\mathscr{M}]$, and we can consider the epimorphism

$$\varphi : \Bbbk[X_e : e \in E] \to \Bbbk[\mathscr{M}]$$

Proposition

The defining ideal $I_{\mathscr{M}} = \operatorname{Ker}\varphi$ is generated by

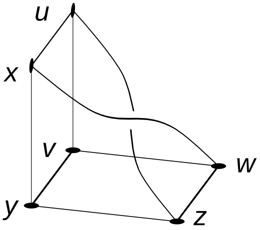
- *The squarefree monomial ideal defined by the incidence structure of $\mathscr{C}$, namely*

$$A_{\mathscr{M}} = \Big(X_{e_1}\cdots X_{e_k} : e_1 + \cdots + e_k \textit{ undefined}\Big)$$

- *and the sum $B_{\mathscr{M}}$ of the binomial ideals defining $\Bbbk[M_C]$, $C \in \mathscr{C}$,*

$$B_{M_C} = \ker\Big(\Bbbk[X_e : e \in E \cap M_C] \to \Bbbk[M_C]\Big).$$

An example: The Möbius strip



3 unit squares forming a non-embeddable polytopal complex

$(X = X_x, Y = X_y, \dots)$

With $E = \{u, v, w, x, y, z\}$ we have (modulo $I_{\mathscr{M}}$)

$$UVW = UVZ = UWY = VWX = UYZ = VXZ = WXY = XYZ = 0$$

and

$$XZ = UW, \quad YW = VZ, \quad XV = UY$$

Stability under Gröbner deformation

If we replace the defining ideal of $\Bbbk[\mathscr{M}]$ by the radical of one of its initial ideals, then the latter defines again a monoidal complex.

$\Rightarrow$ Sturmfels correspondence

Theorem

Let $(w_e)_{e\in E}$ be a weight vector. Then $\operatorname{Rad}(\operatorname{in}_w(I_{\mathscr{M}}))$ is the defining ideal of the regular subdivision of $\mathscr{M}$ defined by w.

First proved by Brun and Römer. No genericity condition is needed.

A generalization

Let

- $\mathscr{P}$ be a finite poset,
- $\mathscr{R}$ a sheaf of rings on $\mathscr{P}$,
- i. e. an inverse system ring homomorphisms $\pi_{pq} : R_p \to R_q$, $p \le q$.
- Then the section ring of $\mathscr{R}$ is

$$\Gamma(\mathscr{R}) = \varprojlim R_p.$$

With a combinatorial background discussed by Yuzvinsky. He proved a Cohen-Macaulay criterion.

Computation of local cohomology by a Hochster type formula in Brun-B-Römer.

The standard case of a section ring

- $\mathscr{P}$ is a set of ideals in a ring S
- partially ordered by $I \leq J \iff I \supset J$, $R_I = S/I$,
- $\pi_{I,J} : S/I \to S/J$ the natural epimorphism.

Then

$$\Gamma(\mathscr{R}) = S/\bigcap_{I \in \mathscr{P}} I,$$

provided sum and intersections of the ideals $I \in \mathscr{P}$ satisfy the *distributive laws*.

This is the case if S has a monomial vector space basis and every I is a monomial ideal.

Is there a minimal graded free resolution ?

A problem: for monoidal complexes in general it is a priori not clear what monoid-type object parametrizes the degrees in a graded minimal free resolution of $\Bbbk[\mathscr{M}]$ over $S = \Bbbk[X_e : e \in E]$ and whether such exists at all.

- It is not sufficient to consider $|\mathscr{M}|$ as soon as there are monomial relations.
- The monoid $\mathbb{Z}_+^E$ of monomials in $K[X_e : e \in E]$ cannot be used since some of them are identified by the binomial relations.

The way out: We let $\mathscr{H}$ be the quotient monoid of $\mathbb{Z}_+^E$ by the congruence defined by the binomial relations.

Both S and $\Bbbk[\mathscr{M}]$ are $\mathscr{H}$-graded.

Caution: in general $\mathscr{H}$ is not cancellative (or even close to cancellativity).

We speak of the elements of $\mathscr{H}$ as monomials and denote them by their (non-unique) preimages in $\mathbb{Z}_+^E$ or by monomials in the X_e.

The homomomorphism $S = K[X_e : e \in E] \to K[\mathscr{M}]$ decomposes

$$S \xrightarrow{\text{mod binomials}} \Bbbk[\mathscr{H}] \xrightarrow{\text{mod monomials}} \Bbbk[\mathscr{M}]$$

$\mathscr{H}$ is cancellative at 0 if $h + i = h$ for $h, i \in \mathscr{H} \Rightarrow i = 0$.

Proposition

Suppose $\mathscr{H}$ is cancellative at 0*. Then $K\mathscr{M}]$ has a minimal $\mathscr{H}$-graded free S-resolution.*

It is unclear whether all $\mathscr{M}$ have $\mathscr{H}$ cancellative at 0.

But this is obviously the case if the binomial equations are homogeneous in the standard sense.

Example: the Möbius strip.

Let I be the set of elements in $\mathscr{H}$ that go to 0 in $\Bbbk[\mathscr{M}]$. Let us say that $\mathscr{M}$ is I-cancellative if

$$i + g = i + h \text{ in } cH \Rightarrow g = h \text{ or } i \in I.$$

Proposition

Let $\mathscr{C}$ be a fan in $\mathbb{R}^d$ and suppose that $M_C \subset \mathbb{Z}^d$ for all C. Then $\mathscr{M}$ is I-cancellative.

In particular it is cancellative at 0.

The Möbius strip fails to be I-cancellative:

$$UVZ = UWY = XYZ = VWX, \qquad \text{but } UV \neq WX.$$

Divisor complexes

Let $h \in cH$ and define the squarefree divisor complex

$$\Delta_h = \left\{ F \subset E : X^g \prod_{f \in F} X^f = X^h \text{ for some } g \in \mathscr{H} \right\}$$

and a subcomplex

$$\Theta_h = \left\{ F \in \Delta_h : \prod_{f \in F} X^f = 0 \text{ in } \Bbbk[\mathscr{M}] \right\}$$

Such complexes have been considered by Hochster (according to Stanley), Campillo-Marijuan, Pison, Bruns-Herzog . . .

A Hochster type formula for graded Betti numbers

Theorem

Suppose that Σ is a rational pointed fan in $\mathbb{R}^n$ and that $M_c \subseteq \mathbb{Z}^n$ for $c \in \Sigma$.

1. *For $h \in |\mathscr{M}|$ we have*

$$\beta_{ih}^S(K[\mathscr{M}]) = \dim_{\Bbbk} \tilde{H}_{i-1}(\Delta_{\overline{h}}).$$

   Moreover, if $\in M_C$ for $C \in \Sigma$, then $\beta_{ih}^S(K[\mathscr{M}]) = \beta_{ih}^S(K[M_C])$.

2. *For $h \in \mathscr{H}$, $h \notin |\mathscr{M}|$, we have*

$$\beta_{ih}^S(K[\mathscr{M}]) = \dim_{\Bbbk} \tilde{H}_{i-1}(\Delta_h, \Theta_h).$$

$\tilde{H}$ is reduced (relative) simplicial homology with coefficients in $\Bbbk$.

Proof by analysis of Koszul homology. Its degree h component is the chain complex defining $\tilde{H}_{i-1}(\Delta_h, \Theta_h)$.

The formula fails for the Möbius strip in degree UVZ since the corresponding component of the Koszul complex is not the desired chain complex.