Recent Developments in Normaliz

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Abstract. The software Normaliz implements algorithms for rational cones and affine monoids. In this note we present recent developments. They include the support for (unbounded) polyhedra and semi-open cones. Furthermore, we report on improved algorithms and parallelization, which allow us to compute significantly larger examples.

Keywords: Hilbert basis, Hilbert series, rational cone, polyhedron.

1 Introduction

Normaliz [2] is a software for computations with rational cones and affine monoids. It pursues two main computational goals: finding the Hilbert basis, a minimal generating system of the monoid of lattice points of a cone; and counting elements degree-wise in a generating function, the Hilbert series. For the mathematical background we refer the reader to [1].

Normaliz (present public version 2.11) is written in C++ (using Boost and GMP/MPIR), parallelized with OpenMP, and runs under Linux, MacOs and MS Windows. It bases on its C++ library libnormaliz which offers the full functionality of Normaliz. There are file based interfaces for Singular, Macaulay 2 and Sage, and C++ level interfaces for CoCoA, polymake, Regina and GAP (in progress). There is also the GUI interface jNormaliz.

Normaliz has found applications in commutative algebra, toric geometry, combinatorics, integer programming, invariant theory, elimination theory, mathematical logic, algebraic topology and even theoretical physics.

2 Hilbert Bases and Hilbert Series

We will first describe the main functionality of Normaliz. The basic objects that constitute the input of Normaliz are a finitely generated rational cone C in $\mathbb{R}^d$ together with a sublattice L of $\mathbb{Z}^d$.

Definition 1. *A (rational)* polyhedron *P is the intersection of finitely many (rational) halfspaces. If it is bounded, then it is called a* polytope*. If all the halfspaces are linear, then P is a cone.*

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The dimension *of P is the dimension of the smallest affine subspace* aff(P) *containing P.*

An affine monoid *is a finitely generated submonoid of $\mathbb{Z}^d$ for some d.*

By the theorem of Minkowski-Weyl, $C \subset \mathbb{R}^d$ is a (rational) cone if and only if there exist finitely many (rational) vectors $x_1, \ldots, x_n$ such that

$$C = \{a_1 x_1 + \cdots + a_n x_n : a_1, \ldots, a_n \in \mathbb{R}_+\}.$$

For Normaliz, cones C and lattices L can either be specified by generators $x_1, \ldots, x_n \in \mathbb{Z}^d$ or by constraints, i.e., homogeneous systems of diophantine linear inequalities, equations and congruences. Normaliz also offers to define an affine monoid as the quotient of $\mathbb{Z}_+^n$ modulo the intersection with a sublattice of $\mathbb{Z}^n$. From version 2.11 on, Normaliz can handle rational polyhedra. This recent extension is described in Section 3.

In the following we will assume that C is pointed, i.e. $x, -x \in C \Rightarrow x = 0$. By Gordan's lemma the monoid $M = C \cap L$ is finitely generated. This affine monoid has a (unique) minimal generating system called the *Hilbert basis* Hilb(M), see Figure 1 for an example. The computation of the Hilbert basis is the first main tasks of Normaliz.

One application is the computation of the *normalization* of an affine monoid N; this explains the name Normaliz. The normalization is the intersection of the cone generated by M with the sublattice gp(M) generated by M. One calls M *normal*, if it coincides with its normalization.

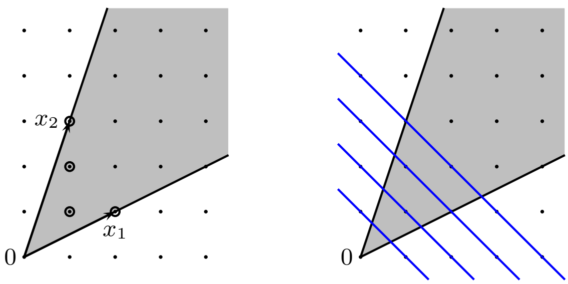

Fig. 1. A cone with the Hilbert basis (circled points) and grading

The second main task is to compute the Hilbert (or Ehrhart) series of a graded monoid. A *grading* of a monoid M is simply a homomorphism $\deg : M \to \mathbb{Z}^g$ where $\mathbb{Z}^g$ contains the degrees. The *Hilbert series* of M with respect to the grading is the formal Laurent series

$$H(t) = \sum_{u \in \mathbb{Z}^g} \#\{x \in M : \deg x = u\} t_1^{u_1} \cdots t_g^{u_g} = \sum_{x \in M} t^{\deg x},$$

provided all sets $\{x \in M : \deg x = u\}$ are finite. At the moment, Normaliz can only handle the case $g = 1$, and therefore we restrict ourselves to this case.

We assume in the following that $\deg x > 0$ for all nonzero $x \in M$ and that there exists an $x \in \operatorname{gp}(M)$ such that $\deg x = 1$. (Normaliz always rescales the grading accordingly.)

The basic fact about $H(t)$ in the $\mathbb{Z}$-graded case is that it is the Laurent expansion of a rational function at the origin:

Theorem 1 (Hilbert, Serre; Ehrhart). *Suppose that M is a normal affine monoid. Then*

$$H(t) = \frac{R(t)}{(1-t^e)^r}, \qquad R(t) \in \mathbb{Z}[t],$$

where r is the rank of M and e is the least common multiple of the degrees of the extreme integral generators of $\operatorname{cone}(M)$*. As a rational function, $H(t)$ has negative degree.*

Usually one can find denominators for $H(t)$ of much lower degree than that in the theorem, and Normaliz tries to give a more economical presentation of $H(t)$ as a quotient of two polynomials. One should note that it is not clear what the most natural presentation of $H(t)$ is in general (when $e > 1$).

A rational cone C and a grading together define the rational polytope $Q = C \cap A_1$ where $A_1 = \{x : \deg x = 1\}$. In this sense the Hilbert series is nothing but the Ehrhart series of Q.

Note that the coefficients of the Hilbert series are computed by a quasi-polynomial. Its leading coefficient is the suitably normed volume of Q.

3 Polyhedra and Inhomogeneous Systems

A main addition to the functionality of Normaliz is the direct support for (unbounded) polyhedra. For computations it is useful to homogenize coordinates by embedding $\mathbb{R}^d$ as a hyperplane in $\mathbb{R}^{d+1}$, namely via

$$\kappa : \mathbb{R}^d \to \mathbb{R}^{d+1}, \qquad \kappa(x) = (x, 1).$$

If P is a (rational) polyhedron, then the closure of the union of the rays from 0 through the points of $\kappa(P)$ is a (rational) cone $C(P)$, called the *cone over P*. The intersection $C(P) \cap (\mathbb{R}^d \times \{0\})$ can be identified with the *recession* (or tail) *cone*

$$\operatorname{rec}(P) = \{x \in \mathbb{R}^d : y + x \in P \text{ for all } y \in P\}.$$

It is the cone of unbounded directions in P. The recession cone is pointed if and only if P has a vertex. The theorem of Minkowski-Weyl can then be generalized as follows:

Theorem 2 (Motzkin). *The following are equivalent for $P \subset \mathbb{R}^d$, $P \neq \emptyset$:*

1. *P is a (rational) polyhedron;*
2. *$P = Q + C$ where Q is a (rational) polytope and C is a (rational) cone.*

If P has a vertex, then the smallest choice for Q is the convex hull of its vertices, and $C = \operatorname{rec}(P)$ is uniquely determined.

Clearly, P is a polytope if and only if $\operatorname{rec}(P) = \{0\}$. Normaliz computes the recession cone and the polytope Q if P is defined by constraints. Conversely it finds the constraints if the vertices of Q and the generators of C are specified.

Suppose that P is given by a system

$$Ax \geq b, \qquad A \in \mathbb{R}^{m\times d},\ b \in \mathbb{R}^m,$$

of linear inequalities (equations are replaced by two inequalities). Then $C(P)$ is defined by the *homogenized system*

$$Ax - x_{d+1}b \geq 0,$$

whereas the $\operatorname{rec}(P)$ is given by the *associated homogeneous system* $Ax \geq 0$. The solution set of the associated homogeneous system is always called the recession cone of the system, even if P is empty.

Via the concept of dehomogenization, Normaliz allows for a more general approach. The *dehomogenization* is a linear form δ on $\mathbb{R}^{d+1}$. For a cone $\widetilde{C}$ in $\mathbb{R}^{d+1}$ and a dehomogenization δ, Normaliz computes the polyhedron $P = \{x \in \widetilde{C} : \delta(x) = 1\}$ and the recession cone $C = \{x \in \widetilde{C} : \delta(x) = 0\}$. In particular, this allows other choices of the homogenizing coordinate.

Let $P \subset \mathbb{R}^d$ be a rational polyhedron and $L \subset \mathbb{Z}^d$ be an *affine sublattice*, i.e., a subset $w + L_0$ where $w \in \mathbb{Z}^d$ and $L_0 \subset \mathbb{Z}^d$ is a sublattice. In order to investigate (and compute) $P \cap L$ one again uses homogenization: P is extended to $C(P)$ and L is extended to $\mathcal{L} = L_0 + \mathbb{Z}(w, 1)$. Then one computes $C(P) \cap \mathcal{L}$. Via this "bridge" one obtains the following inhomogeneous version of Gordan's lemma:

Theorem 3. *Let P be a rational polyhedron with vertices and $L = w + L_0$ an affine lattice as above. Set $\operatorname{rec}_L(P) = \operatorname{rec}(P) \cap L_0$. Then there exist $x_1, \dots, x_m \in P \cap L$ such that*

$$P \cap L = \{(x_1 + \operatorname{rec}_L(P)) \cap \cdots \cap (x_m + \operatorname{rec}_L(P))\}.$$

If the union is irredundant, then $x_1, \dots, x_m$ are uniquely determined.

The Hilbert basis of $\operatorname{rec}_L(P)$ is given by $\{x : (x, 0) \in \operatorname{Hilb}(C(P) \cap \mathcal{L})\}$ and the minimal system of generators can also be read off the Hilbert basis of $C(P) \cap \mathcal{L}$: it is given by those x for which $(x, 1)$ belongs to $\operatorname{Hilb}(C(P) \cap \mathcal{L})$. Normaliz computes the Hilbert basis of $C(P) \cap L$ only at "levels" 0 and 1.

We call $M = \operatorname{rec}_L(P)$ the *recession monoid* of P with respect to L (or L_0). It is justified to say that $P \cap L$ a *module* over $\operatorname{rec}_L(P)$. In the light of the theorem, it is a finitely generated module with a unique minimal system of generators.

After the introduction of coefficients from a field K, $\operatorname{rec}_L(P)$ is turned into an affine monoid algebra, and $N = P \cap L$ into a finitely generated torsionfree module over it. As such it has a well-defined *module rank* $\operatorname{mrank}(N)$, which is computed by Normaliz via the following combinatorial description: Let $x_1, \dots, x_m$ be a system of generators of N as above; then $\operatorname{mrank}(N)$ is the cardinality of the set of residue classes of $x_1, \dots, x_m$ modulo $\operatorname{rec}_L(P)$.

Clearly, to model $P \cap L$ we need linear diophantine systems of inequalities, equations and congruences which now will be inhomogeneous in general. Conversely, the set of solutions of such a system is of type $P \cap L$.

If $\mathbb{Z}^d$ is endowed with a grading whose restriction to M satisfies our conditions, then the Hilbert series

$$H_N(t) = \sum_{x \in N} t^{\deg x}$$

is well-defined, and the qualitative statement above about rationality remains valid. However, the degree may now be ≥ 0. Again, one has an associated quasipolynomial with constant leading coefficient given by

$$q_{r-1} = \operatorname{mrank}(N) \frac{\operatorname{vol}(Q)}{(r-1)!}, \qquad Q = \operatorname{rec}(P) \cap A_1.$$

The *multiplicity* of N is $\operatorname{mrank}(N) \operatorname{vol}(Q)$.

4 Further Extensions

Normaliz now can compute the Hilbert function of a semiopen cone. Such a semiopen cone is given by $C' = C \setminus \mathcal{F}$, where C is a cone and $\mathcal{F}$ is a union of faces (not necessarily facets) of C. Typical applications come from mixed systems of homogeneous inequalities and strict inequalities. This situation could also be modeled by inhomogeneous constraints, but if only few faces are excluded it is beneficial to compute in the original cone and just exclude $\mathcal{F}$.

Additionally, we implemented two new methods of computing the lattice points of a rational polytope. One is a specialization of the so-called dual mode Hilbert basis computation to this case. The other one approximates the rational polytope by a lattice polytope.

The extension NmzIntegrate (introduced in 2.9) counts lattice points with a polynomial weight to compute the *generalized Ehrhart series*, see [4].

5 Algorithmic Improvements

Most of the algorithms in Normaliz base on a *triangulation* of the cone, i.e. a subdivision into *simplicial cones*. Simplicial cones are generated by linearly independent vectors and therefore they are much easier to handle than general cones. The improvements focus on handling large triangulations.

A triangulation is a non-disjoint decomposition of the cone, the simplicial cones intersect in lower dimensional cones. Especially for Hilbert series computations an exact (disjoint) decomposition is needed. Since version 2.7 a principle described by Köppe and Verdoolaege in [5] is used to gain it from the triangulation Γ. It allows the independent handling of the simplicial cones in Γ and thus is superior over the old method, where the simplicial cones had to be compared with each other. This exact decomposition of the cone is then used to obtain a disjoint decomposition of the monoid $M = C \cap L$ of the form

$$M = \bigcup_{\sigma \in \Gamma} \bigcup_{y} (y + M_\sigma),$$

where y runs over a special finite subset of $\sigma \cap L$ and the M_σ are free monoids. Such a disjoint union is called *Stanley decomposition*, named after R. Stanley who proved its existence in 1982.

The *pyramid decomposition* is a newly developed method to compute huge triangulations. It splits the cone in smaller pieces, the pyramids, and handles them completely independent of each other. The result is an algorithm following the "divide-and-conquerer" principle. It gives formidable improvements for larger examples, both in computation time and memory usage, and enables Normaliz to handle triangulations with more than 10^{11} simplicial cones. We refer the reader to [3] for an exact description.

For Hilbert basis computations of combinatorial examples we had introduced a *partial triangulation* in version 2.5. It has now been tuned to check the normality of even larger monoids. For example, the exact decomposition is used to avoid duplicate points in the intersections of the simplicial cones. It reduces computation time and memory requirements, together with intermediate reductions; see [7] for more details.

Together with the parallelization of the algorithms, these improvements enable us to compute significantly larger examples. One interesting class are the cut monoids of graphs for which Sturmfels and Sullivant conjecture normality if the graph is free of K_5-minors (K_5 is the complete graph on 5 vertices). With the partial triangulation implementation of Normaliz 2.5 we were able to validate the conjecture for all graphs up to 8 vertices. The recent version could verify the conjecture for all graphs up to 10 vertices (see [7]), using a result of Ohsugi [6]. The biggest of these examples produced a partial triangulation with more than $15 \cdot 10^9$ simplicial cones, almost $7 \cdot 10^8$ candidates for the Hilbert basis, and took 30 hours with 20 threads on our compute server.

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