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Algebras Defined by Powers of Determinantal Ideals

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Let X be an $m \times n$ matrix of indeterminates over a field K, $A = K[X]$, and $I = I_t(X)$ the ideal generated by the t-minors of X. The main objects of this article are the Rees algebra $\mathscr{R} = \mathscr{R}_t(A) = \bigoplus_{i=0}^{\infty} I^i T^i \subset A[T]$, T a new indeterminate, and the associated graded ring $\mathscr{G} = \mathscr{G}_t(A) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1} = \mathscr{R}/I\mathscr{R}$. The structures of $\mathscr{R}$ and $\mathscr{G}$ are well-understood in (the simple case $t = 1$ and) for ideals of maximal minors: $t = \min(m, n)$, cf. [BV, Sect. 9] for a detailed discussion. Here we are mainly interested in the much more complicated situation $1 < t < \min(m, n)$.

The key to our results is the primary decomposition of the powers I^i determined by DeConcini, Eisenbud, and Procesi [DEP] for char $K = 0$ and extended to the situation char $K > \min(m-t, n-t, t)$ by Bruns and Vetter [BV]; these characteristics will be called *non-exceptional.* We show how this result generalizes to an arbitrary integral domain of coefficients: The intersection of primary ideals which gives I^i in non-exceptional characteristics, always is the integral closure of I^i.

It follows immediately from the primary decomposition that the powers I^i are integrally closed in non-exceptional characteristics. Therefore $\mathscr{R}$ is a normal domain, and the primary decomposition of $I\mathscr{R}$ turns out easy, giving some insight into the structure of $\mathscr{G}$. An interesting observation: The primary decomposition of the ideals I^i can be computed very quickly if one knows in advance that all these ideals are integrally closed.

The best results are obtained in characteristic 0 since one has a multiplicity free action of the linearly reductive group $GL(m, K) \times GL(n, K)$ on $K[X]$ under which I is stable, cf. [DEP] or [BV, Sect. 11]. Applying the theory of U-invariants (Kraft [Kr]) one shows that $\mathscr{R}$ has rational singularities, in particular $\mathscr{R}$ and, hence, $\mathscr{G}$ are Cohen–Macaulay rings.

We have no doubt that $\mathscr{R}$ and $\mathscr{G}$ are Cohen–Macaulay in arbitrary non-exceptional characteristic. It seems however that in exceptional characteristic they are as far as possible from this property: The case $t = 2$ indicates that one has to expect depth 0 for $\mathscr{G}$ and depth 1 for $\mathscr{R}$.

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Finally we derive results corresponding to those on $\mathscr{R}$ for the subalgebra S of $K[X]$ generated by the t-minors. This is easy since S is a retract of $\mathscr{R}$, a fact which already plays a role in the primary decomposition of $I\mathscr{R}$.

We use [BV] as a reference for the theory of determinantal ideals. Somewhat contrary to [BV] we mostly restrict ourselves to fields of coefficients; the generalization to (suitable) integral domains, where possible, is left to the reader. We indicate how to transfer the results to ideals of minors of symmetric matrices and pfaffians of alternating ones.

In the following a minor δ with row indices $a_1, ..., a_t$ and column indices $b_1, ..., b_t$ is specified by $[a_1, ..., a_t | b_1, ..., b_t]$; we call t the size of δ, $t = |\delta|$. For a graded module M over a graded K-algebra A the grade of the irrelevant maximal ideal of A with respect to M is called the *depth of M*.

1. The Integral Closures of Powers of Determinantal Ideals

The primary decomposition of the powers (more generally, products) of determinantal ideals was determined in [DEP] for characteristic 0, and generalized in [BV, (10.9), (10.13)] as follows:

(1.1) Theorem. *Let B be an integral domain, X an $m \times n$ matrix of indeterminates, $m \leqslant n$. Suppose that* $(\min(t, m-t))!$ *is invertible in B. Then*

$$I_t(X)^s = \bigcap_{j=1}^{t} I_j(X)^{((t-j+1)s)}$$

is a primary decomposition of $I_t(X)^s$ in $B[X]$. (If the intersection is only extended over the indices $j = r, ..., t$, $r = \max(1, m - s(m-t))$, the decomposition is irredundant.)

In general one only has the inclusion "$\subset$" in (1.1), and the intersection of symbolic powers is the $\mathbf{Z}$-torsion of $B[X]$ modulo $I_t(X)^s$. We say that a field K has *exceptional characteristic* (for (m, n, t)) if $0 < \operatorname{char} K \leqslant \min(t, m-t, n-t)$.

The following corollary is the key to our results on the Rees rings and associated graded rings with respect to determinantal ideals:

(1.2) Corollary. *Let K be a field of non-exceptional characteristic. Then the ideals $I_t(X)^s$ are integrally closed.*

Proof. Since intersections of integrally closed ideals are integrally closed, it is enough to show that symbolic powers of primes P in regular rings are integrally closed. Localizing one may assume that P is the maximal ideal of a regular local ring A. That the powers of such an ideal are integrally closed, is certainly a well-known fact and follows easily from the integrity of $\mathscr{G}_P(A)$ (which in fact is a polynomial ring over A/P). ▮

We will see in (2.5)(c) that one conversely derives the quantitative assertion of (1.1) from its qualitative Corollary (1.2) with almost no effort: Together with an inductive argument, the assumption that *all* the powers are integrally closed, determines their primary decomposition. In the remainder of this section we want to prove a much stronger statement.

(1.3) THEOREM. *Let B be a domain. Then*

$$\bigcap_{j=1}^{t} I_j(X)^{((t-j+1)s)}$$

is (the primary decomposition of) the integral closure of $I_t(X)^s$ in $B[X]$ (with the same remark in regard to irredundance as in (1.1)).

Proof. Let $J=\bigcap_{j=1}^{t} I_j(X)^{((t-j+1)s)}$. Since the localizations of $B[X]$ with respect to the ideals $I_j(X)$ are regular local rings (note that $B\cap I_j(X)=0$), the argument in the proof of (1.2) shows that J is integrally closed. Since $J\supset I_t(X)^s$, it remains to show that J is integral over $I_t(X)^s$. The ideal $I=I_t(X)^s$ is finitely generated. Therefore it is enough to exhibit a number e such that $J^e\subset IJ^{e-1}$. We will specify e later.

First we need a description of the symbolic powers $I_j(X)^{(k)}$ as given in [BV, (10.4)]. For a minor δ of X one puts

$$\gamma_j(\delta)=\begin{cases}0 & \text{if } |\delta|<j,\\ |\delta|-j+1 & \text{otherwise,}\end{cases}$$

and for a product $\delta_1\cdots\delta_p$ one lets $\gamma_j(\delta_1\cdots\delta_p)=\sum_{i=1}^{p}\gamma_j(\delta_i)$. Then $I_j(X)^{(k)}$ is the ideal generated by all the products $\delta_1\cdots\delta_p$ of minors such that $\gamma_j(\delta_1\cdots\delta_p)\geqslant k$.

Below we will have to transform a given product $P_1\cdots P_e$ of products P_i of minors, $P_i\in J$, in order to apply an inductive argument. The following lemma describes the necessary transformations; its part (a) is [BV, (10.11)], its part (b) is a special case of [BV, (10.10)]—and follows immediately from part (a).

(1.4) LEMMA. *Suppose that $0\leqslant u\leqslant v-2$. Then:*

(a) $\sum_{i=1}^{u+1}(-1)^i[a_1,\dots,\hat{a}_i,\dots,a_{u+1}\,|\,b_1,\dots,b_u][a_i,c_2,\dots,c_v\,|\,d_1,\dots,d_v]$ *is a **Z**-linear combination of products $\kappa\lambda$ of minors, $|\kappa|=u+1$, $|\lambda|=v-1$.*

(b) *Suppose that* $\{a_1, ..., a_u\} \subset \{c_1, ..., c_v\}$. *Then*

$$[a_1, ..., a_u | b_1, ..., b_u][c_1, ..., c_v | d_1, ..., d_v]$$

is a $\mathbf{Z}$*-linear combination of products* $\kappa\lambda$ *of minors*, $|\kappa| = u+1$, $|\lambda| = v-1$.

(1.5) LEMMA. *Let* $P = \delta_1 \cdots \delta_p$ *be a product of minors such that* $\gamma_j(P) \geqslant (t-j+1)s$ *for* $j = 1, ..., t$. *Suppose that* $|\delta_1| \leqslant \cdots \leqslant |\delta_p|$ *and* $|\delta_i| < t$ *for* $i = 1, ..., q$, $|\delta_i| \geqslant t$ *for* $i = q+1, ..., p$. *If* $|\delta_r| > t$, *then* $\gamma_j(P') \geqslant (t-j+1)s$, $j = 1, ..., t$, *for all products* $P' = \delta_1 \cdots \delta_{q-1} \kappa \delta_{q+1} \cdots \delta_{r-1} \lambda \delta_{r+1} \cdots \delta_p$ *with* $|\kappa| = |\delta_q| + 1$, $|\lambda| = |\delta_r| - 1$.

Proof of (1.5). This is of course a purely combinatorial argument on the functions γ_j. It can be extracted from the more general considerations in the proof of [BV, (10.9)]. For the convenience of the reader we give the details. One certainly has $\gamma_j(P') = \gamma_j(P)$ for $j = 1, ..., |\delta_q| + 1$ and $\gamma_j(P') = \gamma_j(P) - 1$ for $j = |\delta_q| + 2, ..., t$. Suppose that $\gamma_j(P) = (t-j+1)s$ for some j, $|\delta_q| + 2 \leqslant j \leqslant t$. Then $\gamma_i(P) = \gamma_i(\delta_{q+1} \cdots \delta_p)$ for $i = j-1$, hence $\gamma_{j-1}(P) - \gamma_j(P) = p - q$; on the other hand

$$\gamma_{j-1}(P) - \gamma_j(P) \geqslant (t-(j-1)+1)s - (t-j+1)s = s.$$

It follows immediately that $|\delta_i| = t$ for $i = q+1, ..., p$, contradicting the hypothesis $|\delta_r| > t$. ∎

Before we enter the main part of the proof of (1.3) we want to explain its essential step by means of the first nontrivial example. Let $t = 2$, $s = 2$, m, $n = 4$. Then, as is easily seen,

$$J = I_1(X)^4 \cap (I_3(X) + I_2(X)^2).$$

We claim $J^2 = IJ$. A system of generators of J is given by (the 4-minor), the products $\delta_1\delta_2$, $|\delta_i| = 2$, and the products $\delta_1\delta_2$, $|\delta_1| = 1$, $|\delta_2| = 3$. Among the generators of J^2 only the products $\delta_1\delta_2\delta'_1\delta'_2$, $|\delta_1| = |\delta'_1| = 1$, $|\delta_2| = |\delta'_2| = 3$ are critical, and after an application of (1.4)(b) (or its column version) and symmetry arguments one is left with

$$P^2, \qquad P = [1|1][2\,3\,4|2\,3\,4].$$

Put $Q = [2|1][1\,3\,4|2\,3\,4]$. By (1.4)(a)

$$P^2 \equiv PQ \qquad \text{mod } IJ$$

and by (1.4)(b)

$$PQ = ([2|1][2\,3\,4|2\,3\,4])([1|1][1\,3\,4|2\,3\,4]) \in IJ,$$

even $PQ \in I^2$.

It is quite clear that this reasoning goes through in general, and the main point is to set up a suitable formal framework for the induction. Let P be a product of minors. Then we write

$$P = FGH, \qquad \begin{aligned} F &= \delta_1 \cdots \delta_p, & |\delta_1| &\leqslant |\delta_{i+1}| < t, \\ G &= \varepsilon_1 \cdots \varepsilon_q, & |\varepsilon_i| &= t, \\ H &= \zeta_1 \cdots \zeta_r, & t &< |\zeta_i| \leqslant |\zeta_{i+1}|, \end{aligned}$$

and let

$$g(P) = \begin{cases} 0 & \text{if } H = 1, \\ |\zeta_1| & \text{if } F = 1, H \neq 1, \\ |\zeta_1| - |\delta_p| & \text{otherwise.} \end{cases}$$

For minors $\delta = [a_1, ..., a_u | b_1, ..., b_u]$, $\varepsilon = [c_1, ..., c_v | d_1, ..., d_v]$ we put

$$d(\delta, \varepsilon) = |\{a_1, ..., a_u\} \setminus \{c_1, ..., c_v\}|,$$

and, indicating by $\delta \in P$ that δ is a factor of a product P of minors, one lets

$$d(P) = \begin{cases} 0 & \text{if } G = 1 \text{ or } H = 1, \\ \min\{d(\delta, \zeta) \colon \delta, \zeta \in P, |\delta| = |\delta_p|, |\zeta| = |\zeta_1|\} & \text{otherwise.} \end{cases}$$

If $F = 1$ and $H \neq 1$, we call $(1, \zeta_1)$ an *optimal pair* of P; if $F \neq 1$ and $H \neq 1$, a pair (δ, ζ) as in the definition of $d(P)$ is called *optimal* if $d(\delta, \zeta) = d(P)$.

Let now e be a natural number greater than the cardinality of the set of pairs (δ_1, δ_2), δ_i a minor of X, and $P_1, ..., P_e$ products of minors such that $P_i \in J$.

Suppose first that $g(P_i) = 0$ for some i. Then obviously $P_i \in I$. Thus we may assume that $g(P_i) > 0$ for all i. If $d(P_i) = 0$ for some i, $d(P_1) = 0$, say, one applies (1.4)(b) to an optimal pair of P_1, obtaining an equation

$$P_1 \cdots P_e = \sum Q_j P_2 \cdots P_e,$$

where still $Q_j \in J$ by (1.5), however, $g(Q_j) < g(P_1)$. So one is left with the case in which $d(P_i) > 0$ for all i. By the definition of e, two of the optimal pairs coincide,

$$P_1 = P_1' \, \delta\zeta, \qquad P_2 = P_2' \, \delta\zeta,$$

say, $\delta = [a_1, ..., a_u | \cdots]$, $\zeta = [c_1, ..., c_v | \cdots]$. We may assume that $c_1 \notin \{a_1, ..., a_u\}$. With the choice $a_{u+1} = c_1$, (1.4)(a) yields an equation

$$\delta\zeta = \sum u_i \eta_i \vartheta_i + \sum v_j \kappa_j \lambda_j, \qquad u_i, v_j \in \mathbf{Z},$$

$$|\eta_i| = |\kappa_j| - 1 = |\delta|, \ |\vartheta_i| = |\lambda_j| + 1 = |\zeta|.$$

One substitutes this for $\delta\zeta$ in P_1 obtaining

$$P_1 \cdots P_e = \sum u_i(P'_1 \eta_i \vartheta_i)\, P_2 \cdots P_e + \sum v_j(P'_1 \kappa_j \lambda_j)\, P_2 \cdots P_e.$$

Since $g(P'_1 \kappa_j \lambda_j) < g(P_1)$, the inductive hypothesis with respect to g applies to every term in the second sum, whereas each term in the first sum can be rewritten

$$(P'_1 \eta_i \vartheta_i)\, P_2 \cdots P_e = (P'_1 \eta_i \zeta)(P'_2 \delta \vartheta_i)\, P_3 \cdots P_e,$$

and it is easy to check that $d(P'_1 \eta_i \zeta) < d(P_1)$, $d(P'_2 \delta \vartheta_i) < d(P_2)$.

(1.6) *Remark.* (a) Theorem (1.3) can be generalized to products $I_{t_1}(X) \cdots I_{t_s}(X)$; cf. [BV, (10.9)] for their primary decomposition in non-exceptional characteristics.

(b) Abeasis [Ab] and Abeasis and Del Fra [AD] established results analogous with those of [DEP] for ideals of minors of symmetric matrices and ideals of pfaffians of alternating ones. We have no doubt that their results on primary decomposition can be proved and generalized (in regard to the ring of coefficients) as (1.1) was proved in [BV] and generalized [DEP]. Furthermore one should be able to derive the companion results of (1.3).

2. Non-Exceptional Characteristics

In non-exceptional characteristic all the powers are integrally closed, cf. (1.2). Therefore one obtains immediately:

(2.1) Corollary. *Let K be a field of non-exceptional characteristic, $A = K[X]$, $I = I_t(X)$. Then the Rees algebra $\mathscr{R}_I(A)$ is normal.*

Next we want to determine the primary decomposition of $I\mathscr{R}_I(A)$, in particular the associated prime ideals of $\mathscr{G}_I(A)$. In this connection the following lemma is useful.

(2.2) Lemma. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded domain, $A_+ = \bigoplus_{i=1}^{\infty} A_i$, $f_1, \ldots, f_m$ be homogeneous elements of constant degree, and I the ideal generated by $f_1, \ldots, f_m$.*

(a) *Then $\mathscr{R} = \mathscr{R}_I(A)$ splits as an $A_0[f_1 T, \ldots, f_m T]$ module,*

$$\mathscr{R} = A_0[f_1 T, \ldots, f_m T] \oplus A_+ \mathscr{R}.$$

(b) *Furthermore the assignment* $f_i \to f_i T$ *induces an isomorphism*

$$A_0[f_1, ..., f_m] \cong A_0[f_1 T, ..., f_m T].$$

(c) *Let* $A_{\geqslant j} = \bigoplus_{i=j}^{\infty} A_i$. *Then the ideals* $A_{\geqslant j}\mathcal{R}$ *are* $A_+\mathcal{R}$*-primary for* $j \geqslant 1$.

Proof. Part (b) is obvious. Denote the A_0-submodule of A generated by the monomials of length k in $f_1, ..., f_m$ by M_k. Then one has a decomposition

$$\mathcal{R} = \bigoplus_{k=0}^{\infty} \bigoplus_{i=0}^{\infty} A_i M_k T^k.$$

Now $\bigoplus_{k=0}^{\infty} A_0 M_k T^k = A_0[f_1 T, ..., f_m T]$ and $\bigoplus_{k=0}^{\infty} \bigoplus_{i=1}^{\infty} A_i M_k T^k = A_+\mathcal{R}$. This shows (a). Since $A_+\mathcal{R} = \operatorname{Rad} A_{\geqslant j}\mathcal{R}$, part (c) is equivalent to the torsion-freeness of the associated graded ring

$$\bigoplus_{j=0}^{\infty} A_{\geqslant j}\mathcal{R}/A_{\geqslant j+1}\mathcal{R}$$

over $\mathcal{R}/A_{\geqslant j}\mathcal{R} = \mathcal{R}/A_+\mathcal{R} \cong A_0[f_1 T; ..., f_m T]$. Each of its components is isomorphic with an $A_0[f_1 T, ..., f_m T]$-submodule of $\mathcal{R}$,

$$A_{\geqslant j}\mathcal{R}/A_{\geqslant j+1}\mathcal{R} \cong \bigoplus_{k=0}^{\infty} A_j M_k T^k. \quad \blacksquare$$

(2.3) THEOREM. *Let K be a field of non-exceptional characteristic and let* $1 \leqslant t < \min(m, n)$, $A = K[X]$, $I = I_t(X)$. *Then the following hold*:

(a) $I\mathcal{R}_I(A)$ *is an unmixed, equivalently, divisorial ideal.*

(b) *It has exactly t minimal primes* $P_1, ..., P_t$, *and up to numbering,* $P_i \cap A = I_i(X)$, $i = 1, ..., t$.

(c) *The primary decomposition of* $I\mathcal{R}_I(A)$ *is*

$$I\mathcal{R}_I(A) = \bigcap_{i=1}^{t} P_i^{(t-i+1)}.$$

Proof. As above put $\mathcal{R} = \mathcal{R}_I(A)$. The ideals $I\mathcal{R}$ and $IT\mathcal{R}$ are isomorphic, and the latter is a divisorial prime ideal. This implies (a).

For (b) one invokes a standard induction argument [BV, (2.4)] after noting that (b) and (c) are trivial for $t = 1$. Let $t > 1$. Then $L = \mathcal{R}[X_{uv}^{-1}]$ is a Laurent polynomial extension of a Rees ring $\mathcal{R}_J(A')$ where $A' = K[Y]$ with an $(m-1) \times (n-1)$ matrix Y of indeterminates and $J = I_{t-1}(Y)$, $JL \cap \mathcal{R} = I\mathcal{R}$, $I_j(Y)L \cap A = I_{j+1}(X)$. By induction this shows that there is exactly one associated prime ideal P_i of $I\mathcal{R}$ such that $P_i \cap A = I_i(X)$,

$i=2, ..., t$, and if $Q \in \operatorname{Ass} \mathscr{G}_I(A)$ does not contain one of the elements X_{ij}, then $Q=P_i$ for some $i=2, ..., t$. Thus any $Q \in \operatorname{Ass} \mathscr{G}_I(A)$ different from $P_2, ..., P_t$ has to contain all the elements X_{ij}, so $Q \supset I_1(X)\mathscr{R}$. Now $I_1(X)\mathscr{R}$ is a prime ideal by (2.2), and, since $t<\min(m, n)$, even a minimal prime ideal of $I\mathscr{R}$ [BV, (10.16)]. Thus (a) finishes the proof of (b).

By the same inductive argument it is enough to determine the P_1-primary component of $I\mathscr{R}$, $P_1=I_1(X)\mathscr{R}=A_+\mathscr{R}$. Since P_1 is a divisorial prime, all the primary ideals with radical P_1 are given by the symbolic powers which by (2.2)(c) coincide with the ordinary ones. So it is enough to observe that $I\mathscr{R} \subset P_1^t$, $I\mathscr{R} \not\subset P_1^{t+1}$. ▮

The result of Simis and Trung [ST, (1.1)] allows us to describe the divisor class group of $\mathscr{R}_I(A)$:

(2.4) COROLLARY. *Let K be a field of non-exceptional characteristic, $I=I_t(X)$, $A=K[X]$, $1 \leqslant t<\min(m, n)$. Then*

$$\operatorname{Cl}(\mathscr{R}_I(A)) \cong \mathbf{Z}^t,$$

a system of generators being given by the classes of $P_1, ..., P_t$.

(2.5) *Remarks.* (a) The preceding results can be generalized to the Rees algebras with respect to ideals $I_t(X)^s$, and even to the Rees algebras of $A=K[X]/I_u(X)$ with respect to $I_t(X)^s A$, $t<u$.

(b) One would like to determine the canonical class of $\mathscr{R}_I(A)$ in the form $\operatorname{cl}(\omega)=n_1 \operatorname{cl}(P_1)+\cdots+n_t \operatorname{cl}(P_t)$. The problem of course is to find n_1, whereas induction as above easily gives $n_t=(n-t+1)(m-t+1)-2$.

(c) In proving (2.3) we only used the qualitative consequence (1.2) of (1.1), and only (1.2) is necessary to extend (2.3) to the powers $I=I_t(X)^s$. Now the primary decomposition of $I\mathscr{R}_I(A)$ yields a primary decomposition by retraction, and thus one can deduce (1.1) from (1.2): The primary components with respect to the prime ideals $I_j(X)$, $j>1$, are known by induction, and $(I_1(X)\mathscr{R})^k \cap B[X]=I_1(X)^k$.

(d) Results analogous to those of this section hold in characteristic 0 for ideals of minors of symmetric matrices and ideals of pfaffians of alternating ones as a consequence of the assertions on primary decomposition in [Ab] and [AD], cf. also (1.6)(b).

3. CHARACTERISTIC ZERO

In this section we want to apply invariant-theoretic methods in order to improve the results of Section 2 for fields of characteristic 0. Let K be an algebraically closed field of characteristic 0, and G a linearly reductive group over K. Suppose that G acts rationally on an affine domain A over K as a group of K-algebra automorphisms. A has a decomposition

$$A = \bigoplus_{\omega \in \Omega(A)} M_\omega^{e_\omega}, \qquad e_\omega \geq 1,$$

into its isotypic components, M_ω denoting an irreducible G-module of highest weight ω. We say that G *acts without multiplicities* if $e_\omega = 1$ for all $\omega \in \Omega(A)$. Let U be the unipotent radical of a maximal torus in a Borel subgroup of G. Then A shares many properties with the ring A^U of U-invariants of A (cf. [Kr] for a comprehensive treatment). If G acts without multiplicities, A^U is a semigroup ring isomorphic with $K[\Omega(A)]$. As an almost immediate consequence of results of Boutot [Bo], Brion [Br], and Hochster [Ho] one obtains the following theorem; it may have been stated elsewhere:

(3.1) THEOREM. *With the hypothesis just introduced assume that A^U is isomorphic to a semigroup ring $K[H]$, e.g., let G act without multiplicities. Then the following statements are equivalent*:

(a) *A has rational singularities.*

(b) *A is normal.*

Proof. The implication (a) $\Rightarrow$ (b) is included in the definition of rational singularities. Suppose now that A is normal. Then $A^U \cong K[H]$ is normal for elementary reasons. By [Kr, p. 190, Satz] A^U is a finitely generated K-algebra, hence H is a finitely generated semigroup. Being cancellative it can be embedded into $\mathbf{Z}^n$ for some n.

Hochster [Ho] calls a semigroup H contained in $\mathbf{N}_0^n$ ($\mathbf{N}_0$ denoting the non-negative integers) *normal* if it is finitely generated and satisfies the following condition: Let $a, b, c \in H$, $m \in \mathbf{N}$; if $ma = mb + c$, then there exists a $\tilde{c} \in H$ such that $c = m\tilde{c}$. Associating with $(a_1, ..., a_n) \in H$ the monomial $Y_1^{a_1} \cdots Y_n^{a_n}$ in the indeterminates $Y_1, ..., Y_n$ we may consider H as a semigroup of monomials in $Y_1, ..., Y_n$, and in this interpretation it is obvious that the semigroup ring $K[H]$ can only be a noetherian normal domain if H is a normal semigroup. Hochster proves the converse by exhibiting an embedding

$$S = K[H] \subsetneq K[Z_1, ..., Z_p]$$

such that there is a Reynolds operator $\rho: K[Z_1, ..., Z_p] \to S$, i.e., an S-module homomorphism ρ, $\rho \mid S = \mathrm{id}$. This is equivalent to the fact that S is a direct S-summand of $K[Z_1, ..., Z_p]$. Invoking the main result of Boutot's article [Bo] we conclude that $K[H]$ has rational singularities, in particular is a normal Cohen–Macaulay ring.

The restriction to subsemigroups $H \subset \mathbf{N}_0^n$ is not essential. (We are grateful to Hochster for pointing out this fact to us.) Let $H \subset \mathbf{Z}^n$ be a normal subsemigroup, and H_0 the maximal subgroup of H. The normality of H immediately implies that H_0 is a direct summand of $\bar{H}$, the group generated by H. Let π be the restriction to H of a projection $\bar{H} \to H_0$. Then π and the natural epimorphism $H \to H/H_0$ induce an isomorphism $H \cong H_0 \oplus H/H_0$ where $H' = H/H_0$ has no invertible element $\neq 0$, and, in addition, satisfies the conditions imposed on H. In order to show that $H' \subset \mathbf{Z}^n$ can be embedded into $\mathbf{N}_0^p$ for a suitable p one considers the cone $\mathbf{Q}_+ H'$ generated by H' in the $\mathbf{Q}$-vectorspace $V = H'\mathbf{Q}$. The polar cone

$$P = \{\varphi \in \operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q}) \colon \varphi(h) \geqslant 0 \text{ for all } h \in H'\}$$

is finitely generated by $\varphi_1, ..., \varphi_p$, say. Since no nonzero element has an inverse, $\{\varphi_1, ..., \varphi_p\}$ must contain $\dim V$ linearly independent vectors. Therefore

$$\varphi \colon H' \to \mathbf{Q}_+^p, \qquad \varphi(h) = (\varphi_1(h), ..., \varphi_p(h)),$$

is an embedding, and after multiplication with a suitable common denominator we may assume $\varphi(H') \subset \mathbf{N}_0^p$. This is exactly the map used by Hochster in order to achieve an embedding $K[H'] \subsetneq K[Z_1, ..., Z_p]$ admitting a Reynolds operator, cf. [Ho, p. 323]. (It is indeed easy to verify that $\varphi(H')$ is a full subsemigroup of $\mathbf{N}_0^p$, and this fact obviously implies the existence of a Reynolds operator.) Altogether we have an isomorphism

$$K[H] = K[H_0] \otimes K[H'],$$

and since $K[H_0]$ is regular and $K[H']$ has rational singularities, $K[H]$ has rational singularities, too. The crucial argument is now provided by the theorem of Brion [Br, p. 10]: A has rational singularities if A^U has rational singularities. ▮

The group $G = \mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$ acts without multiplicities on the K-algebra $A = K[X]$ by linear substitutions, cf. [DEP] or [BV, Sect. 11]. A determinantal ideal $I = I_t(X)$ and its powers are obviously G-stable. Therefore G acts naturally on the Rees algebra $\mathscr{R} = \mathscr{R}_I(A)$. We fix a maximal unipotent subgroup $U \subset G$. For each irreducible G-submodule M_σ of A one has $\dim_K (M_\sigma)^U = 1$, and one may choose an element $x_\sigma \in (M_\sigma)^U$ such that $H = \{x_\sigma\}_\sigma$ is a semigroup under multiplication, $A^U = K[H]$, in fact $H \cong \mathbf{N}^{\min(m,n)}$ [BV, (11.22)]. Then obviously

$$\mathscr{R}^U = K[\tilde{H}], \qquad \tilde{H} = \bigcup_{i=0}^{\infty} (H \cap I^i) T^i,$$

and one concludes from (3.1) and (2.1):

(3.2) THEOREM. *Let K be an algebraically closed field of characteristic* 0 *and* $A = K[X]$, $I = I_t(X)$. *Then* $\mathscr{R}_I(A)$ *has rational singularities.*

(3.3) COROLLARY. *Let K be a field of characteristic* 0, $A = K[X]$, $I = I_t(X)$. *Then* $\mathscr{R}_I(A)$ *and* $\mathscr{G}_I(A)$ *are Cohen–Macaulay rings.*

Proof. It is well known that $\mathscr{G}_I(A)$ is Cohen–Macaulay along with $\mathscr{R}_I(A)$ and A. Furthermore the Cohen–Macaulay property of a graded K-algebra is not affected by an extension of the field K of coefficients. Therefore one may assume that K is algebraically closed and apply (3.2). ∎

(3.4) *Remarks.* (a) We have no doubt that (3.3) holds in arbitrary non-exceptional characteristic (but it fails in exceptional characteristic, cf. (4.1) below). There seems to be no way however to go from characteristic 0 to positive characteristic. Nevertheless one should record the following fact: The Hilbert function of the graded K-algebra $\mathscr{R}_I(A)$ is the same in all non-exceptional characteristics, since each power of I has a K-basis consisting of the standard monomials it contains (since this holds for symbolic powers of the $I_j(X)$, cf. [BV, (10.4)]). In view of (1.3) the correct generalization of (3.3) to all characteristics seems to be that the integral closure of $\mathscr{R}_I(A)$ is Cohen–Macaulay.

(b) By the results of [Ab] and [AD] one immediately obtains the analogues of (3.2) for ideals of minors of symmetric $m \times m$ matrices and ideals of pfaffians of alternating ones: $\mathrm{GL}(m, K)$ acts without multiplicities in both cases and the ideals under consideration are $\mathrm{GL}(m, K)$-stable.

4. EXCEPTIONAL CHARACTERISTICS

In this section we want to show by means of an example that the results of the Sections 2 and 3 completely fail over a field of exceptional characteristic. By virtue of [BV, (10.14), (g)]

$$[1\,|\,1][2\,3\,4\,|\,2\,3\,4] \notin I_2(X)^2, \tag{$*$}$$

in $\mathbf{Z}[X]$, X of course being at least a 4×4 matrix. Since however

$$2[1\,|\,1][2\,3\,4\,|\,2\,3\,4] \in I_2(X)^2,$$

[BV, (10.10)], it follows that ($*$) also holds over a field of characteristic 2. Thus (1.3) implies that the powers of a determinantal ideal over the integers or a field of exceptional characteristic are not integrally closed in general; in particular not all of their primary components can be taken as symbolic powers.

(4.1) PROPOSITION. *Let K be a field of characteristic* 2, *X at least a* 4×4 *matrix*, $A = K[X]$, $I = I_2(X)$. *Then* depth $\mathcal{G}_I(A) = 0$ *and* depth $\mathcal{R}_I(A) = 1$.

Proof. Let $\mathcal{G} = \mathcal{G}_I(A)$. The localization argument in the proof of (2.3) which reduces the situation to an ideal of 1-minors, shows that $\mathcal{G}[x_{uv}^{-1}]$ is an integral domain for the residue class x_{uv} of any of the indeterminates. The considerations of Section 2 now immediately imply that $\mathcal{G}$ has exactly two minimal primes $P_1 = I_1(X)\mathcal{G}$ and P_2 and that every associated prime ideal different from P_2 contains all the elements x_{uv}. One has a natural homomorphism from $\mathcal{G}$ to the "associated symbolic ring" $\bigoplus_{i=0}^{\infty} I^{(i)}/I^{(i+1)}$, a domain by [BV, (10.7)]. Obviously P_2 is contained in its kernel, in particular contains none of the elements $[ij \mid uv]^*$, * denoting leading form.

The depth of a graded K-algebra is not affected by an extension of the field of coefficients. Since such an extension commutes with the construction of $\mathcal{G}$, we may now assume that K is algebraically closed. We will show below that $[ij \mid uv]^* \in I/I^2$ is a zero-divisor. As in Section 3 one lets the group $G = \mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$ act on $K[X]$ and considers the induced action on $\mathcal{G}$. The action of G permutes the finitely many associated prime ideals of $\mathcal{G}$; since G is connected, each of them is G-stable. The K-subspace V generated by the $[ij \mid uv]^*$ is G-stable, too, and as a G-submodule generated by each of the $[ij \mid uv]^*$. Thus there is an associated prime ideal P of $\mathcal{G}$ containing V. As seen above, P has to contain $I_1(X)\mathcal{G}$, too, and therefore all the generators of the irrelevant maximal ideal. This shows depth $\mathcal{G} = 0$. Then depth $\mathcal{R} = 1$, since the irrelevant maximal ideal of $\mathcal{R}$ contains the isomorphic ideals $I\mathcal{R}$ and $IT\mathcal{R}$ and depth $\mathcal{R}/IT\mathcal{R} = mn$.

It remains to prove the claim above. Let $[ij \mid uv] = [3\,4 \mid 3\,4]$, say. Expanding $[1\,3\,4 \mid 1\,3\,4]$ with respect to row 4 and applying (1.5)(b) yields $[1\,3\,4 \mid 1\,3\,4][2\,3\,4 \mid 2\,3\,4] \in I^3$. Likewise

$$[1 \mid 3][3\,4 \mid 1\,4][2\,3\,4 \mid 2\,3\,4], \qquad [1 \mid 4][3\,4 \mid 1\,3][2\,3\,4 \mid 2\,3\,4] \in I^3,$$

so

$$[3\,4 \mid 3\,4]([1 \mid 1][2\,3\,4 \mid 2\,3\,4]) \in I^3,$$

too. Since $[1 \mid 1][2\,3\,4 \mid 2\,3\,4] \notin I^2$, this shows the claim. ∎

5. ALGEBRAS GENERATED BY MINORS

Let $A = K[X]$ as usual. The subalgebra S_t generated by the t-minors of X can also be considered an algebra defined by powers of $I = I_t(X)$ since $S_t \cong \mathcal{R}_I(A)/I_1(X)\mathcal{R}_I(A)$, cf. (2.2). Choosing K-vector spaces V, W of dimensions m and n resp., one may interpret $K[X]$ as the coordinate ring of the affine space $\operatorname{Hom}_K(V, W)$, and then S_t corresponds to the coordinate ring of the Zariski closure of the image of $\operatorname{Hom}_K(V, W)$ in $\operatorname{Hom}_K(\bigwedge^t V, \bigwedge^t W)$ under the map $\varphi \to \bigwedge^t \varphi$.

The structure of S_t is very well known in the following cases:

(i) The trivial case $t = 1$, $S_t = K[X]$.

(ii) The case $t = m$ in which S_t is the homogeneous coordinate ring of the Grassmannian of m-dimensional subspaces of W.

(iii) The case $m = n = t + 1$ in which S_t is a polynomial ring over K [BV, (10.17)].

Since S_t is even a retract of $\mathscr{R}_t(A)$ we conclude from (3.2) and [Bo], elementary arguments, and (2.1):

(5.1) THEOREM. (a) *If K is algebraically closed and* $\operatorname{char} K = 0$, *$S_t$ has rational singularities.*

(b) *If* $\operatorname{char} K = 0$, *$S_t$ is Cohen–Macaulay.*

(c) *If K has non-exceptional characteristic, S_t is a normal domain.*

(5.2) *Remarks.* (a) If K has exceptional characteristic, S_t is not normal in general. As an example we again choose $\operatorname{char} K = 2$, $t = 2$, X at least a 4×4 matrix. The computations in Section 4 show that $[1 \mid 1][2\,3\,4 \mid 2\,3\,4]$ is in the field of fractions of S_2, and by (1.3) it is integral over $I_2(X)^2$. For reasons of homogeneity it must be integral over S_2.

(b) It would be desirable to find the defining relations of the algebras discussed, at least in non-exceptional characteristics. As a first step one could try to compute a representation of S_t. However, except for the cases listed above and the one to discuss now, we cannot describe a system of generators of the ideal J_t defining S_t as a residue class ring of a polynomial ring over K in a natural way. The first (and wrong) guess: J_t is generated by the relations of the t-minors obtained by equating two expansions of the $2t \times 2t$ "sub"matrices of X (with possibly multiple rows or columns). This is incorrect already for the first case not covered by the examples (i), (ii), (iii). Let $m = 3$, $n = 4$, $t = 2$, and let d_i denote the K-dimension of the degree it homogeneous component of S_t (degree measured in $K[X]$). The ideals $I_t(X)^s$ are generated by the standard monomials they contain, cf. (3.4)(a). Consequently this holds for S_t. One easily computes

$$d_1 = 18, \qquad d_2 = 165, \qquad d_3 = 1022,$$

and J_2 needs at least

$$\binom{20}{3} - 6d_1 - d_3 = 10$$

generators of degree 3 besides its 6 quadratic generators. A run of the computer program MACAULAY [BS] revealed that J_2 is indeed generated by 6 quadratic generators obtained in the way indicated above and 10 cubic generators, all of them given by 3-minors of $\bigwedge^2 X$ and $\bigwedge^2 \tilde{X}$, $\tilde{X} = XE$, $E \in \mathrm{GL}(4, K)$ an elementary transformation. They express the fact that the vectors $x_i \wedge x_j$, $x_i \wedge x_k$, $x_i \wedge x_l$ are linearly dependent if x_i, x_j, x_k, x_l are linearly dependent. It follows easily that J_2 is never generated by quadratic relations, except if $m, n \leqslant 3$.

(c) One can show that S_t is factorial only in the cases (i), (ii), (iii) listed above. In the case discussed in (b) it is not even Gorenstein. MACAULAY produced the free resolution over the coordinate ring P of $\mathrm{Hom}_K(\bigwedge^p V, \bigwedge^p W)$:

$$0 \to P(-9)^{10} \to P(-8)^{45} \to P(-7)^{66} \to P(-6)^{10} \oplus P(-5)^{66}$$
$$\to P(-4)^{60} \to P(-3)^{10} \oplus P(-2)^{6} \to P \to S_2 \to 0.$$

(d) The parts (a) and (b) of (5.1) were proved in [BV, 11.E] by a direct application of (3.1).

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