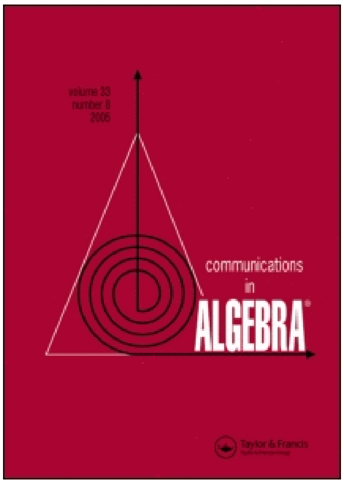

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THE BUCHSBAUM-EISENBUD STRUCTURE THEOREMS AND ALTERNATING SYZYGIES

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The purpose of this article is to embed the Buchsbaum-Eisenbud structure theorems [3] into a more general theory of "orientations" and "alternating syzygies". In the introduction the following situation will be considered: R is a commutative noetherian ring, $f: F \to G$ is a homomorphism of free R-modules, split in depth 1, $M = \operatorname{Im} f$, and $N = \operatorname{Coker} f$ is a torsionfree R-module:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[dr, "f"'] & G \arrow[r, "g"] & N \arrow[r] & 0 \\ & M \arrow[u] & & \end{tikzcd}$$

An *orientation* on an (almost arbitrary) R-module M of rank r is a linear map $\mu\colon \overset{r}{\wedge}M \to R$ such that grade $\operatorname{Im}\mu \geq 2$. (Equivalently one may require that $(\overset{r}{\wedge}M)^{**} \cong R$.) Such an orientation induces a duality

$$\mu^i\colon \overset{i}{\wedge}M \to (\overset{r-i}{\wedge}M)^*,\ \mu^i(x)(y) = \mu(x \wedge y).$$

The first structure theorem of Buchsbaum and Eisenbud [3, Theorem 3.1] may be viewed as describing explicitely how orientations are passed along finite free resolutions, or, more generally, simply from M to N in the situation above. When an orientation γ on G is given (for example by choice of a basis) then an orientation μ on M induces an orientation ν on N, and μ and ν are connected through the formula

$$\gamma(y \wedge (\overset{r}{\wedge}f(x))) = \mu(\overset{r}{\wedge}f(x))\ \nu(\overset{s}{\wedge}g(y))$$

for all $x \in \overset{r}{\wedge}F$, $y \in \overset{s}{\wedge}G$, $s = \operatorname{rank} N$. Because of this equation we prefer to call the first structure theorem the factorization theorem. The term on the left side runs through the r-minors of a matrix of f if bases of F and G are given and x and y vary over the corresponding bases of $\overset{r}{\wedge}F$ and $\overset{s}{\wedge}G$ resp. The second structure theorem [3, Theorem 6.1] which expresses the (r-1)-minors in terms of μ then is an instant consequence of the fact that $\mu^1\colon M \to (\overset{r-1}{\wedge}M)^*$ is an isomorphism, once M is a second syzygy.

Every $\alpha \in (\overset{w}{\wedge}G)^*$ acts on $\overset{w+1}{\wedge}G$ as a homomorphism to G:

$$\alpha(x_1\wedge\ldots\wedge x_{w+1}) = \sum_{i=1}^{w+1}(-1)^{w+1-i}\alpha(x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge x_{w+1})x_i.$$

Composition with g defines a linear map

$$g^w\colon (\overset{w}{\wedge}G)^* \to \mathrm{Hom}_R(\overset{w+1}{\wedge}G,N) \ , \ g^w(\alpha) := g\circ\alpha.$$

We call the elements of $\mathrm{Ker}\, g^w$ *alternating syzygies* of N of order w. It is not hard to see that $\mathrm{Ker}\, g^w$ is isomorphic to $(\overset{t-w}{\wedge}M)^{**}$, $t = \mathrm{rank}\, G$, the isomorphism depending on the choice of an orientation of G. This fact may be considered the first and generalizable part of the factorization theorem, whereas its second part is unique for $w = s := \mathrm{rank}\, N$: $\mathrm{Ker}\, g^s$ coincides with $(\overset{s}{\wedge}N)^*$, considered as a submodule of $(\overset{s}{\wedge}G)^*$ in the natural way.

For trivial reasons an alternating syzygy α induces a map $b\colon \overset{w+1}{\wedge}G \to F$ for which

$$\begin{tikzcd}
F \arrow[r, "f"] & G \arrow[r] & N \arrow[r] & 0 \\
& \overset{w+1}{\wedge}G \arrow[ul, "b"] \arrow[u, "\alpha"'] & &
\end{tikzcd}$$

is a commutative diagram. The generalization of the second structure theorem ($\alpha = \nu\circ\overset{s}{\wedge}g$) then ties together μ,α,b, a linear form $\beta \in (\overset{r-t+w}{\wedge}M)^*$ and the $(t-w-1)$-minors of f ($\beta = 1$ for $\alpha = \nu\circ\overset{s}{\wedge}g$).

We were led to the study of alternating syzygies of arbitrary order by attempts to understand more about the generic structure of (finite) free resolutions. One of the phenomena we found is a process by which an alternating syzygy α gives rise to "new" alternating syzygies of higher order. This "reflection" is constructed from a free resolution $\mathfrak{F}$ of N, a Koszul type complex associated with α, and a comparison map. The result of the reflection usually depends on these data, but modulo certain irrelevant submodules one obtains intrinsic maps relating the alternating syzygies of order w, the cohomology module $\mathrm{Ext}_R^j(N,R)$, and the alternating syzygies of order w + j.

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1. Preparations

A. *Notations and conventions*

A subset I of the integers $\mathbb{Z}$ has $|I|$ elements and represents the sequence of its elements in ascending order. The set $\{i: 1 \le i \le n\}$ is denoted by $[1,n]$. For mutually disjoint subsets $J_1,\dots,J_n$ of $\mathbb{Z}$ we denote the sign of the permutation of $J_1\cup\dots\cup J_n$, which is given by the concatenation of $J_1,\dots,J_n$ by $\sigma(J_1,\dots,J_n)$.

An r-subset of I has r elements, and the set of all r-subsets is called $S(r,I)$ and, more specifically, $S(r,n)$ if $I = [1,n]$. The complement of J in I is denoted by $I\backslash J$ or, in case $J = \{j\}$, also by $I\backslash j$.

Usually R is supposed to be a commutative and noetherian ring with an identity element. We only consider finitely generated R-modules. For an R-module M and a prime ideal P the *depth* of M_P is the length of a maximal M_P-regular sequence in R_P, whereas the *grade* of an ideal I is the length of a maximal R-regular sequence inside I. We say that a "linear" object $\mathfrak{O}$ *has property* $\mathfrak{P}$ *in depth* n if its localizations to all prime ideals P of the underlying ring such that $\operatorname{depth} R_P \leq n$, have the property $\mathfrak{P}$. The *rank* r is attributed to a module if it is free of constant rank r in depth 0. The torsion submodule of a module M is called TM. A module is an *n-th syzygy* if there is an exact sequence

$$0 \to M \to F_{n-1} \to \cdots \to F_0$$

with free modules F_i, $i = 0,\ldots,n-1$.

For a sequence $x_1,\ldots,x_n$ of elements of a module M and a subset $I \in S(r,n)$, $I = \{i_1,\ldots,i_r\}$, $i_1 \leq \ldots \leq i_r$, we denote the element $x_{i_1} \wedge \ldots \wedge x_{i_r}$ of $\overset{r}{\wedge} M$ by x_I. In the case, in which $e_1,\ldots,e_n$ is a basis of a free module F, the elements e_I^*, $I \in S(r,n)$ form the basis of $(\overset{r}{\wedge} F)^*$ which is dual to the basis e_I, $I \in S(r,n)$, of $\overset{r}{\wedge} F$.

The ideal of s-minors of a matrix x is denoted by $I_s(x)$. For a map $f\colon F \to G$ of free R-modules, represented by a matrix x, the ideal $I_s(x)$ is an invariant of f. Consequently we put $I_s(f) := I_s(x)$ and $I(f) := I_r(x)$ for $r = \operatorname{rank} \operatorname{Im} f$.

In certain cases where there is a "natural" map $f\colon M \to N$ we will simply write g for the composition of f with a map g defined on N. This is only a measure to simplify notations and will not cause any ambiguities. If, for example, α is a linear form on $\overset{r}{\wedge} N$ and $x \in \overset{r}{\wedge} M$, then $(\alpha \circ \overset{r}{\wedge} f)(x)$ may be denoted by $\alpha(x)$.

B. *An exactness criterion*

Very frequently we will use the following exactness criterion and its corollary. We record these well-known results for easier reference.

(1.1) Proposition: *Let* $0 \to U \xrightarrow{f} M \xrightarrow{g} N$ *be a sequence of* R*-modules where* U *is a second syzygy and* M *and* N *are first syzygies. If this sequence is exact in depth* 1*, then it is exact.*

Proof: The module $(\operatorname{Im} f + \operatorname{Ker} g)/\operatorname{Ker} g$ is zero since it is isomorphic to a submodule of the torsionfree module N and, on the other hand, zero in depth 0. The same argument shows that $\operatorname{Ker} f$ is zero. Finally, for every element P of Supp Ker g/Im f one would have depth $R_P \geq 2$ by hypothesis, hence depth$(\operatorname{Ker} g/\operatorname{Im} f)_P \geq 1$, which is impossible unless Supp Ker g/Im f $= \emptyset$.

(1.2) <u>Corollary:</u> *Let* $f: M \to N$ *be a homomorphism of* R-*modules which is an isomorphism in depth* 1.

If M *is a second and* N *a first syzygy, then* f *is an isomorphism.*

The map f^* *is an isomorphism.*

Proof: Apply (1.1) to $0 \to M \to N \to 0$ and $0 \to N^* \to M^* \to 0$ resp.

C. *Some multilinear algebra*

As the material in B. the results below are included for lack of a suitable reference.

Let M be an R-module. Then a linear form $\alpha \in (\overset{w}{\wedge}M)^*$ induces homomorphisms $\lambda^u(\alpha): \overset{u}{\wedge}M \to \overset{u-w}{\wedge}M$ given by

$$\lambda^u(\alpha)(x_1\wedge\ldots\wedge x_u) := \sum_{J\in S(u-w,u)} \sigma([1,u]\setminus J,J)\alpha(x_{[1,u]\setminus J})x_J$$

for $x_1,\ldots,x_u \in M$ and $u \geq w$, and $\lambda^u(\alpha) = 0$ for $u < w$. The collection of these homomorphism extends to an endomorphism

$$\lambda(\alpha): \wedge M \to \wedge M.$$

Let $\alpha_1 \in (\overset{v_1}{\wedge}M)^*$, $\alpha_2 \in (\overset{v_2}{\wedge}M)^*$. The restriction of $\lambda(\alpha_1)\circ\lambda(\alpha_2)$ to $\overset{v_1+v_2}{\wedge}M$ is a linear form on $\overset{v_1+v_2}{\wedge}M$, which we denote by α now.

(1.3) <u>Proposition:</u> *With the notations just introduced,* $\lambda(\alpha) = \lambda(\alpha_1)\circ\lambda(\alpha_2)$.

Proof: By direct calculation one obtains for $x = x_1\wedge\ldots\wedge x_p \in \overset{p}{\wedge}M, I = [1,p]$:

$$\lambda(\alpha)(x) =$$
$$= \sum_{\substack{V_2\in S(v_2,I)\\ V_1\in S(v_1,I\setminus V_2)}} \sigma(V_1\cup V_2, I\setminus(V_1\cup V_2))\sigma(V_2,V_1)\alpha_1(x_{V_1})\alpha_2(x_{V_2})x_{I\setminus(V_1\cup V_2)}$$

and

$$\lambda(\alpha_1)\circ\lambda(\alpha_2)(x) =$$
$$= \sum_{\substack{V_2\in S(v_2,I)\\ V_1\in S(v_1,I\setminus V_2)}} \sigma(V_2,I\setminus V_2)\sigma(V_1,I\setminus(V_1\cup V_2))\alpha_1(x_{V_1})\alpha_2(x_{V_2})x_{I\setminus(V_1\cup V_2)}.$$

Equality follows from

$$\sigma(V_1\cup V_2, I\setminus(V_1\cup V_2))\sigma(V_2,V_1) = \sigma(V_2,V_1,I\setminus(V_1\cup V_2))$$
$$= \sigma(V_2,I\setminus V_2)\sigma(V_1,I\setminus(V_1\cup V_2)). -$$

We now define a multiplication $\wedge$ on the graded module

$$\wedge^*(M) := \bigoplus_{i\geq 0} (\overset{i}{\wedge}M)^*$$

by extending bilinearly the operation which associates with two homogeneous elements $\alpha_1 \in (\overset{v_1}{\wedge}M)^*$, $\alpha_2 \in (\overset{v_2}{\wedge}M)^*$ the "product" $\alpha = \alpha_1\wedge\alpha_2 \in (\overset{v_1+v_2}{\wedge}M)^*$ as above.

(1.4) Theorem: *With this multiplication* $\Lambda^*(M)$ *is an exterior algebra.*

Proof: Extending the map λ in a natural way, we obtain an R-linear injective map $\lambda: \Lambda^*(M) \to \mathrm{End}(\Lambda M)$ such that $\lambda(\alpha_1 \wedge \alpha_2) = \lambda(\alpha_1)\circ\lambda(\alpha_2)$. Therefore one may verify the defining conditions on the image $\lambda(\Lambda^*(M))$. Now bilinearity and associativity of the product are trivial, whereas the formulas in the proof of (1.3) imply that the product is anticommutative and alternating: $\lambda(\alpha_1)\circ\lambda(\alpha_2)$ and $\lambda(\alpha_2)\circ\lambda(\alpha_1)$ differ termwise by

$$\sigma(V_2,V_1,V_3)\sigma(V_1,V_2,V_3) = (-1)^{|V_1||V_2|} = (-1)^{v_1v_2}$$

for $\alpha_1 \in (\overset{v_1}{\Lambda}M)^*$, $\alpha_2 \in (\overset{v_2}{\Lambda}M)^*$. Similarly it follows that $\lambda(\alpha)\circ\lambda(\alpha) = 0$ for $\alpha \in (\overset{v}{\Lambda}M)^*$, v odd: The terms

$$\sigma(V_1,V_2,I\setminus(V_1\cup V_2))\alpha(x_{V_1})\alpha(x_{V_2})x_{I\setminus(V_1\cup V_2)} \quad \text{and}$$

$$\sigma(V_2,V_1,I\setminus(V_1\cup V_2))\alpha(x_{V_1})\alpha(x_{V_2})x_{I\setminus(V_1\cup V_2)}$$

cancel each other. -

For later application we note an almost obvious result:

(1.5) Proposition: *Suppose that* M *has rank* r, *and let* $\alpha \in (\overset{w}{\Lambda}M)^*$. *If* $\lambda^u(\alpha) = 0$ *for any* $u, w \le u \le r$, *then* $\alpha = 0$.

Proof: The equation $\alpha = 0$ can be verified in depth 0, whence we may assume that M is free, and for free modules the proposition is really trivial. -

In order to simplify notation, we shall henceforth denote the maps $\lambda^u(\alpha)$ and $\lambda(\alpha)$ simply by α.

2. Orientable modules

In this section we introduce the class of orientable modules and prove a fundamental duality for their exterior powers.

Definition: Let M be an R-module of rank r. M is called *orientable* if it is free in depth 1 and if there is a homomorphism $\mu: \overset{r}{\Lambda}M \to R$ such that grade $\operatorname{Im}\mu \geq 2$. Such a map is called an *orientation* on M. An *oriented module* is a module together with an orientation.

Examples of orientable modules:

(a) Projective modules, provided $\overset{r}{\Lambda}M \cong R$.

(b) Every module over a factorial ring, which is free in depth 1.

(c) Every module M such that grade $M \geq 2$.

(d) Finally, and most important: If M is free in depth 1 and has a finite free resolution, then M is orientable. This will be a consequence of the Buchsbaum-Eisenbud factorization theorem (cf. section 3).

(2.1) *Remark:* Bourbaki has introduced the notion of a divisor class attached to a module M over a normal domain. If M is free in depth 1, this is simply the isomorphism class of the ideal $(\overset{r}{\wedge}M)^{**}$, $r = \operatorname{rank} M$ [1, pp. 107, 108]. Therefore such a module over a normal domain is orientable if and only if its divisor class is zero:

(2.2) <u>Proposition:</u> *Let* M *be free in depth* 1. *Then* M *is orientable if and only if* $(\overset{r}{\wedge}M)^*$ $((\overset{r}{\wedge}M)^{**})$ *is a free module of rank* 1. *A linear form* $\mu \in (\overset{r}{\wedge}M)^*$ *is an orientation if and only if it is a basis of* $(\overset{r}{\wedge}M)^*$. *Orientations differ only by a unit factor.*

Proof: Let M be orientable. The orientation $\mu\colon \overset{r}{\wedge}M \to R$ satisfies the hypothesis of (1.2). Therefore μ^* and μ^{**} are isomorphisms. Clearly μ^* maps 1 to μ. Conversely, suppose that $(\overset{r}{\wedge}M)^{**}$ is free. The natural homomorphism $(\overset{r}{\wedge}M)^* \to (\overset{r}{\wedge}M)^{***}$ is an isomorphism, again because of (1.2). So we may assume that $(\overset{r}{\wedge}M)^*$ is free. Let μ be a basis of $(\overset{r}{\wedge}M)^*$. Then μ^* is an isomorphism, its kernel $(R/\operatorname{Im}\mu)^*$ as well as its cokernel $\operatorname{Ext}^1_R(R/\operatorname{Im}\mu, R)$ are zero, whence grade $\operatorname{Im}\mu \geq 2$. -

Let μ be an orientation on M. Then the linear map $\mu^i\colon \overset{i}{\wedge}M \to (\overset{r-i}{\wedge}M)^*$ is given by

$$(\mu^i(x))(y) := \mu(x \wedge y) \quad , \quad x \in \overset{i}{\wedge}M,\ y \in \overset{r-i}{\wedge}M.$$

Of course, we cannot expect μ^i to be an isomorphism in general: $(\overset{r-i}{\wedge} M)^*$ is always a second syzygy, $\overset{i}{\wedge} M$ in general is not. This however is the only obstruction, cf. (2.4).

(2.3) *Remark:* It is sometimes convenient to use the following modification of μ_i:

$$\tilde{\mu}_i : \overset{i}{\wedge} M \to (\overset{r-i}{\wedge} M)^* \ , \ \tilde{\mu}^i(x)(y) := \mu(y \wedge x).$$

$\tilde{\mu}_i$ and μ_i differ by sign only, and $\tilde{\mu}_i$ coincides with the composition

$$\overset{i}{\wedge} M \longrightarrow (\overset{i}{\wedge} M)^{**} \xrightarrow{(\mu^{r-i})^*} (\overset{r-i}{\wedge} M)^* \ .$$

Moreover the notation μ^i will also be used in case μ is just a linear form on $\overset{r}{\wedge} M$ without any further hypotheses on M or μ.

(2.4) Theorem: *Let* M *be an oriented* R-*module of rank* r. *Then:*

(a) $(\mu^i)^*$ *and* $(\mu^i)^{**}$ *are isomorphisms,* $i = 0,\dots,r$.

(b) *If* $\overset{i}{\wedge} M$ *is a second syzygy, then* μ^i *itself is an isomorphism.*

Proof: Since the definition of μ^i and dualization commute with localization, we may assume that R is local. If $\operatorname{depth} R \le 1$, then M is free, and μ^i obviously is an isomorphism. Now all the claims follow from (1.2). -

We note two important specializations:

(2.5) Corollary: *Let* M *be an oriented second syzygy of rank* r. *Then* $\mu^1: M \to (\overset{r-1}{\wedge} M)^*$ *is an isomorphism.*

(2.6) Corollary: *Let* M *be an oriented second syzygy of rank* 2. *Then* $\mu^1: M \to M^*$ *is an isomorphism. The bilinear form induced by* μ^1 *is alternating.*

The last corollary was proved by Miller [10] for modules over regular local rings by a completely different argument.

Using the dualities given by an orientation μ on an orientable module of rank r one defines bilinear maps

$$\beta_1, \beta_2, \tilde{\beta}_2 : \overset{m}{\wedge} M \times \overset{n}{\wedge} M \to \overset{m+n-r}{\wedge} M$$

by

$$\beta_1(x,y) := \mu^m(x)(y), \beta_2(x,y) := \mu^n(y)(x),$$
$$\tilde{\beta}_2(x,y) := \tilde{\mu}^n(y)(x).$$

Obviously, β_1 and $\beta_2(\tilde{\beta}_2)$ differ only by sign if m+n = r. It is extremely important to note that this remains essentially true for all m and n:

(2.7) Lemma: *For all* $x \in \overset{m}{\wedge} M$, $y \in \overset{n}{\wedge} M$ *one has*

$$\beta_1(x,y) \equiv \varepsilon_1 \beta_2(x,y) \text{ and } \beta_1(x,y) \equiv \varepsilon_2 \tilde{\beta}_2(x,y) \bmod T(\overset{m+n-r}{\wedge} M)$$

where $\varepsilon_2 := (-1)^{(m+n-r)(r-m)}$ *and* $\varepsilon_1 := \varepsilon_2 (-1)^{n(r-m)}$.

Proof: It suffices to verify the congruences in depth 0, whence one may assume that M is free and R is local. Then M has a basis $x_1,\dots,x_r$ such that $\mu(x_1\wedge\dots\wedge x_r) = 1$. It is enough to consider $x = x_I$, $I \in S(m,r)$, $y = x_J$, $J \in S(n,r)$. In case $I \cup J \neq [1,r]$ both sides of the first congruence are zero, otherwise

$$\beta_1(x,y) = \sigma(\bar{I}, I\cap J)\sigma(I,\bar{I})x_{I\cap J}, \text{ and}$$

$$\beta_2(x,y) = \sigma(\bar{J}, I\cap J)\sigma(J,\bar{J})x_{I\cap J},$$

$\bar{\ }$ denoting the complement with respect to $[1,r]$. Now $\sigma(I,\bar{I})\sigma(\bar{J},I\cap J) = \sigma(\bar{J},I\cap J,\bar{I})$ and $\sigma(J,\bar{J})\sigma(\bar{I},I\cap J) = \sigma(\bar{I},I\cap J,\bar{J}) = \varepsilon_1\sigma(\bar{J},I\cap J,\bar{I})$. The second congruence follows since $\tilde{\mu}^n = (-1)^{n(r-n)}\mu^n$. -

Though we don't itend to give a systematic treatment of orientable modules, it may be useful to list some of their properties.

(2.8) <u>Proposition:</u> *Let the modules* M, N, U *be free in depth* 1, $r := \operatorname{rank} M$.

(a) *Let* $f\colon M \to N$ *be an isomorphism in depth* 1. *Then* N *is orientable if and only if* M *is orientable.*

(b) *The following are equivalent:* (i) M *is orientable.* (ii) M/TM *is orientable.* (iii) M* *is orientable.*

(c) *If two of the modules in an exact sequence* $0 \to U \to M \to N \to 0$ *are orientable, then the third one is orientable too.*

(d) *The following are equivalent:* (i) M *is orientable.* (ii) *There is an exact sequence* $0 \to M^{**} \to R^{r+1} \to I \to 0$ *with* I *an Ideal of grade* ≥ 2. (iii) *There is an exact sequence* $0 \to R^{r-1} \to M/TM \to J \to 0$ *with* J *an ideal of grade* ≥ 2.

(e) *If* M *and* N *are orientable, then* $M \oplus N$, $M \otimes N$, $\mathrm{Hom}(M,N)$, $S^n M$, $\overset{n}{\wedge} M$ *are orientable too.*

(f) *Let* μ *be an orientation on* M. *Then the ideal* $\mathrm{Im}\,\mu$ *annihilates* $(\overset{r-i}{\wedge} M)^*$ *modulo* $\mathrm{Im}\,\mu^i$, $i = 0,\ldots,r$.

The proof of (2.7) is not difficult. Since we shall not use (2.7) in the sequel, we content ourselves with some hints: (a) follows from (1.2) applied to $\overset{r}{\wedge} f$. For (b) one uses (a) and the natural relationships between $\overset{r}{\wedge} M$, $\overset{r}{\wedge}(M/TM)$, and $\overset{r}{\wedge} M^*$. Part (c) is a generalization of (3.1) below. The ingredients of (d) are the well-known facts about the realization of first and second syzygies as extensions and first syzygies resp. of ideals [2]. The proof of (e) can be based on complexes related to the exact sequences in (d), and (f) finally is obtained as a consequence of (5.3).

3. The Buchsbaum-Eisenbud factorization theorem

In this section we discuss the propagation of orientations along complexes of free modules. The following proposition indicates how orientations are passed from left to right:

(3.1) <u>Proposition:</u> *Let* F *and* G *be free* R-*modules, and* $f\colon F \to G$ *a linear map which factors through an orientable* R-*module* M:

$$\begin{tikzcd} F \arrow[rr, "f"] \arrow[dr, "f'"'] & & G \\ & M \arrow[ur, "f''"'] & \end{tikzcd}$$

Suppose that f' *is surjective and that* $\operatorname{rank} M = \operatorname{rank} f =: r$. *Let* μ *and* γ *be orientations on* M *and* G *respectively.*

(a) *Then there exists a uniquely determined linear form* $\hat{\mu} \in (\overset{r}{\wedge}G)^{**}$ *such that the diagram*

$$\begin{tikzcd} \overset{r}{\wedge}F \arrow[rr, "\overset{r}{\wedge}f"] \arrow[dr, "\mu"'] & & \overset{r}{\wedge}G \\ & R \arrow[ur, "\hat{\mu}^*"'] & \end{tikzcd}$$

is commutative. (Here we identify $(\overset{r}{\wedge}G)^{**}$ *and* $\overset{r}{\wedge}G$*).*

(b) *Let* $N := \operatorname{Coker} f$, $s := \operatorname{rank} N$, *and* $\nu := (\gamma^s)^*(\hat{\mu}) \in (\overset{s}{\wedge}G)^*$. *Then for all* $x \in \overset{r}{\wedge}F$, $y \in \overset{s}{\wedge}G$,

$$\gamma(y \wedge \overset{r}{\wedge}f(x)) = \mu(x)\nu(y) \quad .$$

(c) *Therefore* ν *induces a linear form on* $\overset{s}{\wedge}N$: $\nu \in (\overset{s}{\wedge}N)^* \subset (\overset{s}{\wedge}G)^*$ *(in a natural way).*

(d) *If* f *splits in depth* 1, *then* ν *is an orientation on* N.

Proof: We start with the trivial factorization $\overset{r}{\wedge} f = \overset{r}{\wedge} f'' \circ \overset{r}{\wedge} f'$ and the diagram

$$\begin{tikzcd}
\overset{r}{\wedge} F \arrow[rr, "\overset{r}{\wedge} f"] \arrow[dr, "\overset{r}{\wedge} f'"] \arrow[ddr, "\mu"'] & & \overset{r}{\wedge} G \\
& \overset{r}{\wedge} M \arrow[ur, "\overset{r}{\wedge} f''"'] \arrow[d, "\mu"] & \\
& R &
\end{tikzcd}$$

Dualizing the diagram we obtain

$$\begin{tikzcd}
(\overset{r}{\wedge} G)^* \arrow[rr, "(\overset{r}{\wedge} f)^*"] \arrow[dr, "(\overset{r}{\wedge} f'')^*"] \arrow[ddr, "\hat{\mu}"'] & & (\overset{r}{\wedge} F)^* \\
& (\overset{r}{\wedge} M) \arrow[ur, "(\overset{r}{\wedge} f')^*"'] & \\
& R \arrow[u, "\mu^*"] \arrow[uur, "\mu^*"'] &
\end{tikzcd}$$

and there is exactly one choice for $\hat{\mu}$: $\hat{\mu} := (\mu^*)^{-1} \circ (\overset{r}{\wedge} f'')^*$. Let $x \in \overset{r}{\wedge} F$, $y \in \overset{s}{\wedge} G$. Then

$$\begin{aligned}
\mu(x)\nu(y) &= \mu(x)[(\gamma^s)^*(\hat{\mu})](y) = \mu(x)\hat{\mu}(\gamma^s(y)) \\
&= [\hat{\mu}^*(\mu(x))](\gamma^s(y)) = (\overset{r}{\wedge} f(x))(\gamma^s(y)) \\
&= \gamma^s(y)(\overset{r}{\wedge} f(x)) = \gamma(y \wedge \overset{r}{\wedge} f(x)).
\end{aligned}$$

For (c) we have to show that ν vanishes on the kernel of the natural epimorphism $\overset{s}{\wedge}G \to \overset{s}{\wedge}N$, which is generated by the elements $y \wedge f(x_o)$, $x_o \in F$, $y \in \overset{s-1}{\wedge} G$. Since grade $\operatorname{Im}\mu \geq 2$, it is enough to have $\mu(x)\nu(y \wedge f(x_o)) = 0$ for all $x \in \overset{r}{\wedge}F$. By the preceding equation

$$\mu(x)\nu(y \wedge f(x_o)) = \gamma(y \wedge \overset{r+1}{\wedge} f(x_o \wedge x)) = 0$$

because of $\overset{r+1}{\wedge} f = 0$.

By part (a), we always have

$$I(f) = I_1(\overset{r}{\wedge}f) = I(\mu)I(\hat{\mu}^*) = I(\mu)I(\hat{\mu}) = (\operatorname{Im}\mu)(\operatorname{Im}\hat{\mu})$$

If f splits in depth 1, then grade $I(f) \geq 2$ and grade $\operatorname{Im}\hat{\mu} \geq 2$. Since furthermore $\operatorname{Im}\nu = \operatorname{Im}\hat{\mu}$, we conclude grade $\operatorname{Im}\nu \geq 2$, as required for (d). -

Proposition (3.1) is the induction step in the proof of the Buchsbaum-Eisenbud factorization theorem [3, Theorem 3.1]. We give a slight generalization observed by Eagon and Northcott [5]. (In the original version the acyclicity of the complex $\mathcal{C}$ is required.)

(3.2) <u>Theorem:</u> *Let*

$$\mathcal{C} : 0 \to F_n \xrightarrow{f_n} F_{n-1} \to \dots \to F_k \xrightarrow{f_k} F_{k-1} \to \dots$$

be a finite or infinite complex of oriented free modules which is acyclic in depth 1. *Let* $r_k := \operatorname{rank} f_k$, *and let*

φ_k *denote the given orientation on* F_k. *Then, for every* k *there exist uniquely determined linear forms* $\mu_k \in (\overset{r_k}{\wedge} F_k)^*$ *and* $\hat{\mu}_k \in (\overset{r_k}{\wedge} F_{k-1})^{**}$ *such that*

(i) $\mu_n = \varphi_n$,

(ii) *for every* k *the diagram*

$$\begin{tikzcd} \overset{r_k}{\wedge} F_k \arrow[rr, "\overset{r_k}{\wedge} f_k"] \arrow[dr, "\mu_k"'] & & \overset{r_k}{\wedge} F_{k-1} \\ & R \arrow[ur, "\hat{\mu}_k^*"'] & \end{tikzcd}$$

is commutative, and

(iii) $\mu_{k-1} = (\varphi_{k-1}^{r_{k-1}})^*(\hat{\mu}_k)$

Furthermore, for every k *one has* $I(f_k) = I(\mu_k)I(\mu_{k-1})$.

Proof: After chosing $\mu_n := \varphi_n$ one constructs the linear forms μ_k and $\hat{\mu}_k$ by descending induction on k. Successively μ_k is uniquely determined by $\hat{\mu}_{k+1}$ because of (iii), and then $\hat{\mu}_k$ is given by (3.1). As far as there is a map f_{k-1} occuring in $\mathcal{C}$, we have grade $I(f_k) \geq 2$. Hence part (c) of (3.1) then guarantees that parts (a) and (b) of (3.1) can be applied to f_{k-1}.

The equation $I(f_k) = I(\mu_k)I(\mu_{k-1})$ was observed in (3.1) already. -

If the complex $\mathcal{C}$ is exact at F_{k-1}, there is a stronger relation between $I(f_k)$ and $I(\mu_{k-1})$:

(3.3) <u>Corollay:</u> *If the sequence* $F_k \xrightarrow{f_k} F_{k-1} \xrightarrow{f_{k-1}} F_{k-2}$ *is exact (at* F_{k-1}*) then* $\operatorname{rad} I(f_k) = \operatorname{rad} I(\mu_{k-1})$.

Proof: The inclusion $\operatorname{rad} I(f_k) \subset \operatorname{rad} I(\mu_{k-1})$ follows directly from (3.2). Consider a prime ideal P of R. After replacing R by R_P it remains to show that $I(f_k) = R$, whenever $I(\mu_{k-1}) = R$. By virtue of the construction of μ_{k-1} it may be considered as an orientation on $M := \operatorname{Coker} f_k$. Since R is local and $I(\mu_{k-1}) = R$, there are elements $x_1,\dots,x_r \in M$, $r = r_{k-1} = \operatorname{rank} M$, such that

$$\mu_k(x_1 \wedge \dots \wedge x_r) = 1.$$

By elementary linear algebra, these elements generate a free direct summand of M: $M \cong TM \oplus R^r$. The hypothesis of (5.3) implies that M is torsionfree, consequently $M \cong R^r$ and $I(f_k) = R$. -

We include a version of the Buchsbaum-Eisenbud factorization theorem for "matrices and determinants".

(3.4) <u>Corollary:</u> *Let*

$$\mathfrak{L}: 0 \to R^{b_n} \xrightarrow{a_n} R^{b_{n-1}} \to \dots \to R^{b_k} \xrightarrow{a_k} R^{b_{k-1}} \to \dots$$

be a complex of free R*-modules with given bases, the maps being represented by matrices* a_k. *Suppose that is acyclic in depth* 1, *and let* $r_k := \operatorname{rank} a_k$. *Then, for every* k, *every* r_k*-subset* V *of* $[1,b_k]$ *and every* r_k*-subset* W *of* $[1,b_{k-1}]$ *there exist uniqueley determined elements* $\mu_k(V), \hat{\mu}_k(W) \in R$ *such that*

(i) $\mu_n([1,b_n]) = 1$,

(ii) $a_k(V,W) = \mu_k(V)\hat{\mu}_k(W)$,

(iii) $\mu_{k-1}([1,b_{k-1}]\backslash W) = \sigma([1,b_{k-1}]\backslash W,W)\hat{\mu}_k(W)$.

Here $a_k(V,W)$ *denotes the minor of* a_k *corresponding to rows* V *and columns* W.

Proof: Let e and d denote bases of R^{b_k} and $R^{b_{k-1}}$ resp. Then we define $\mu_k(V) := \mu_k(e_V)$, $\hat{\mu}_k(W) := \hat{\mu}_k(d_W^*)$ where $\mu_k \in (\stackrel{r_k}{\wedge} R^{b_k})^*$, $\hat{\mu}_k \in (\stackrel{r_k}{\wedge} R^{b_{k-1}})^{**}$ are given by (3.2) relative to the orientations of the free modules in $\mathcal{C}$ determined by the given bases. Observing that $\stackrel{r_k}{\wedge} f_k(e_V) = \sum_W a_k(V,W)d_W$ and $\varphi_k^{r_{k-1}}(d_{\bar{W}}) = \sigma(\bar{W},W)d_W^*$, $\bar{W} = [1,b_{k-1}]\backslash W$, one immediately translates properties (i), (ii), and (iii) of (3.2) into the corresponding equations of (3.4).

Conversely, $\hat{\mu}_k(W)$ is uniquely determined by the equations (ii): If a,b $\in$ R satisfy

$$a_k(V,W) = \mu_k(V)a = \mu_k(V)b \quad \text{for all } V,$$

then a = b, since grade $\Sigma\, R\mu_k(V) \geq$ grade $I(a_{k+1}) \geq 2$. -

(3.5) *Remark:* It is quite obvious that the dual version of (3.2) holds as well. If $\mathcal{C}$ is a complex of free R-modules such that $\mathcal{C}^*$ satisfies the hypotheses of (3.2), then the factorizations for $\mathcal{C}^*$ given by (3.2) can be dualized into factorizations for the original complex

$\mathcal{C}$. A typical example of such a complex $\mathcal{C}$ is a free resolution

$$\ldots \to F_n \xrightarrow{f_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{f_1} F_o$$

such that grade Coker $f_o \geq 2$.

Furthermore the induction process in the proof of (3.2) can be applied to any complex

$$\mathcal{D}: F_n \xrightarrow{f_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{f_1} F_o$$

if $\mathcal{D}$ is acyclic in depth 1 and Ker f_n is orientable. In connection with the preceding remark we see that orientations and factorizations as given in (3.2) propagate along free resolutions in both directions. In particular (3.2) is essentially valid for every free resolution over a factorial domain.

4. The higher structure theorems of Buchsbaum and Eisenbud

As before let $f: F \to G$ be a linear map between free R-modules factoring through an orientable module M with orientation μ:

$$\begin{tikzcd} F \arrow[rr, "f"] \arrow[dr] & & G \\ & M \arrow[ur] & \end{tikzcd}$$

For simplicity we assume that this is just the natural factorization through $M = \operatorname{Im} f$. Let $r := \operatorname{rank} M$. Proposition (3.1) showed how to factor $\overset{r}{\wedge} f$ through $\mu\colon \overset{r}{\wedge} F \to R$. If it were possible to factor $\overset{r-i}{\wedge} f$ for $i > 0$ in a similar fashion, then these factorizations would give us strong "structure theorems" for f, in particular, we could express the lower order minors of f in terms of the ideal $I(\mu)$.

In order to find necessary and sufficient conditions for the existence of a factorization

$$\begin{tikzcd}
\overset{r-i}{\wedge} F \arrow[rr, "\overset{r-i}{\wedge} f"] \arrow[dr, "\mu^{r-i}"'] & & \overset{r-i}{\wedge} G \\
& (\overset{i}{\wedge} F)^* \arrow[ur] &
\end{tikzcd}$$

we dualize this diagram:

$$\begin{tikzcd}
(\overset{r-i}{\wedge} G)^* \arrow[rr, "(\overset{r-i}{\wedge} f)^*"] \arrow[dr] \arrow[dddr, "b^i"'] & & (\overset{r-i}{\wedge} F)^* \\
& (\overset{r-i}{\wedge} M)^* \arrow[ur] & \\
& \overset{i}{\wedge} M \arrow[u, "\tilde{\mu}^i"'] & \\
& \overset{i}{\wedge} F \arrow[u, "\overset{i}{\wedge} f"'] \arrow[uuur, "(\mu^{r-i})^*"'] &
\end{tikzcd}$$

Clearly the desired map b^i exists, if $\tilde{\mu}^i$ is an isomorphism, equivalently, if $\bigwedge^i M$ is a second syzygy. (Cf. (2.3) for the definition of $\tilde{\mu}^i$ and its relation to $(\mu^{r-i})^*$.) For i = 1 we therefore obtain:

(4.1) <u>Theorem:</u> *Let* $f\colon F \to G$ *be a map between free R-modules such that* $M := \operatorname{Im} f$ *is an orientable second syzygy of rank* r. *Then, with the notations introduced above, there is a linear map* $b\colon (\bigwedge^{r-1} G)^* \to F$ *such that the diagram*

$$\begin{tikzcd} \bigwedge^{r-1} F \arrow[rr, "\bigwedge^{r-1} f"] \arrow[dr, "\mu^{r-1}"'] & & \bigwedge^{r-1} G \\ & F^* \arrow[ur, "b^*"'] & \end{tikzcd}$$

is commutative. In particular one has $I_{r-1}(f) \subset I(\mu)$.

By the Buchsbaum-Eisenbud factorization theorem (3.2) the hypotheses of (4.1) are satisfied for every second or higher syzygy in a finite free resolution, and (4.1) then specializes to the second structure theorem of Buchsbaum and Eisenbud:

(4.2) <u>Theorem:</u> *Let*

$$\mathfrak{F}\colon 0 \to F_n \xrightarrow{f_n} F_{n-1} \to \dots \to F_1 \xrightarrow{f_1} F_0$$

be a finite free resolution of oriented free modules. Then, with the notations of (3.2), *for every* $k \geq 2$ *there exists a linear map* $b_k\colon (\Lambda^{r_k-1} F_{k-1})^* \to F_k$ *such that the diagram*

$$\begin{tikzcd}
\Lambda^{r_k-1} F_k \arrow[rr, "\Lambda^{r_k-1} f_k"] \arrow[dr, "\mu_k^{r_k-1}"'] & & \Lambda^{r_k-1} F_{k-1} \\
& F_k^* \arrow[ur, "(b_k)^*"'] &
\end{tikzcd}$$

is commutative. In particular $I_{r_k-1}(f_k) \subset I(\mu_k)$.

(4.3) *Remark:* The finiteness of the resolution in (4.2) is only used to ensure the orientability of $\operatorname{Im} f_k$ (for $k \geq 2$). Therefore (4.2) holds for every free resolution, for which $\operatorname{Im} f_k$ is orientable for $k \geq 2$. Especially, it is valid for every free resolution over a factorial ring.

In contrast to section 3 one can not relax the condition on the acyclicity of $\mathfrak{F}$: If $f_k\colon F_k \to F_{k-1}$ factors through an orientable second syzygy M such that $\operatorname{rank} M = \operatorname{rank} f_k$ and $\operatorname{grade} I(f_k) \geq 2$, then necessarily $M \cong \operatorname{Im} f_k = \operatorname{Ker} f_{k-1}$.

Theorems (3.2) and (4.2) give factorizations of $\Lambda^{r_k} f_k$ for $k \geq 1$ and of $\Lambda^{r_k-1} f_k$ for $k \geq 2$. It is quite obvious from the discussion at the beginning of this section that we can not expect factorizations of $\bigwedge^{r_k-2} f_k$ for $k \geq 3$ etc. in general. In [3, Conjecture 10.1] Buchsbaum and Eisenbud specified conditions for the existence of such factorizations, and their conjecture was proved by Lebelt [8, 9] in characterictic 0 and by Weyman in general [12]:

(4.4) <u>Theorem</u>: *Let*

$$\mathfrak{F}: 0 \to F_n \xrightarrow{f_n} F_{n-1} \to \dots \to F_1 \xrightarrow{f_1} F_0$$

be a finite free resolution of oriented free modules. Then a linear map $b_k^{r_k-j}$ *for which the diagram*

$$\begin{tikzcd}
\bigwedge^{r_k-j} F_k \arrow[rr, "\bigwedge^{r_k-j} f_k"] \arrow[dr, "\mu_k^{r_k-j}"'] & & \bigwedge^{r_k-j} F_{k-1} \\
& (\bigwedge^{j} F_k)^* \arrow[ur, "(b_k^{r_k-j})^*"'] &
\end{tikzcd}$$

is commutative, exists if

(i) $k = n-1$, $j = 0,\dots,n-2$, *or*

(ii) $k = n-2$, $r_n=1$, $j = 0,\dots,n-r$ *or*

(iii) $j \leq \frac{n-2}{n-k}$.

Proof: In each of the cases (i), (ii), (iii) the module $\bigwedge^{i}\mathrm{Im}\, f_k$ is a second syzygy; cf. [8, 9, 12]. -

Lebelt's and Weyman's results also enable us to show that a factorization $\wedge^{r_k-2} f_k$ does not exist in general for $k \geq 3$. Let K be a field, and $R = K[X_1,\ldots,X_5]$. Suppose that $f_5\colon R^2 \to R^6$ is given by the matrix

$$\begin{bmatrix} X_1 & \ldots & X_5 & 0 \\ 0 & X_1 & \ldots & X_5 \end{bmatrix}$$

Then $\operatorname{Coker} f_5$ is a 4-th syzygy. By the methods of [2] one constructs a finite free resolution

$$0 \to R^2 \xrightarrow{f_5} R^6 \xrightarrow{f_4} R^7 \xrightarrow{f_3} R^{b_2} \xrightarrow{f_2} R^{b_1} \xrightarrow{f_1} R^{b_0}$$

in which f_3 is chosen such that the dual of the embedding of $\operatorname{Coker} f_4$ into R^{b_2} is surjective. Let $N := \operatorname{Coker} f_4 \cong \operatorname{Im} f_3$. Then $\overset{2}{\wedge} N$ is a first, but not a second syzygy [9, 12], and the map ν^2 can not be surjective.

If the conclusion of (4.4) would hold for every k and every j such that $j \leq k-1$, then it would supply us with a proof of the syzygy theorem of Evans and Griffith [6, 7]: If R contains a field, then $\operatorname{Im} f_k$ is projective, whenever $r_k < k$. From this invalid generalization of (4.4) one simply would obtain

$$I(\mu_k) \supset I_{r_k - r_k}(f_k) = I_0(f_k) = R,$$

and $I(\mu_k) = R$ is equivalent to $I(f_{k+1}) = R$ (cf. (3.3)), hence to $\operatorname{Im} f_k$ being projective (cf. [3, p. 131, Remark 2]). Conversely, the theorem of Evans and Griffith shows that there is a relation between $I(\mu_k)$ and $I_{r_k-j}(f_k)$ for $j < k$ (to which it is equivalent):

(4.5) Proposition: *Let* R *contain a field, and let* $\mathfrak{F}$ *be a finite free resolution as in* (4.4). *Then, for all* $k = 1,\ldots,n$ *and* $j < k$,

$$\operatorname{rad} I_{r_k-j}(f_k) \subset \operatorname{rad} I(\mu_k).$$

Proof: Let P be a prime ideal such that $P \not\supset I_{r_k-j}(f_k)$. We replace R by R_P, and may suppose that $I_{r_k-j}(f_k) = R$. Since R is local, $\operatorname{Im} f_k$ then contains a direct summand N' of F_{k-1}, $\operatorname{rank} N' = r_k-j$. Splitting off this direct summand from $\operatorname{Im} f_k$ and F_{k-1} we obtain an exact sequence

$$0 \to N'' \to F'_{k-1} \to F_{k-2} \to \cdots \to F_1 \to F_0,$$

in which N'' still has a finite free resolution. Since $\operatorname{rank} N'' = j < k$, N'' is free by the theorem of Evans and Griffith. Thus $\operatorname{Im} f_k = N' \oplus N''$ is free, and $I(\mu_k) = R$, as required. -

5. Alternating syzygies

As in the preceding section we consider a map $f: F \to G$ of free R-modules which factors through $M = \operatorname{Im} f$, and has cokernel N:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[dr, "f"'] & G \arrow[r, "g"] & N \arrow[r] & 0 \\ & M \arrow[u] & & \end{tikzcd}$$

Let f be split in depth 1, $r := \operatorname{rank} M$, $s := \operatorname{rank} N$, and $t := \operatorname{rank} G$.

Suppose momentarily that N is torsionfree. With every $\alpha \in (\overset{s}{\wedge} G)^*$ one associates the action of α on $\overset{s+1}{\wedge} G$, giving a linear map $\alpha\colon \overset{s+1}{\wedge} G \to G$ (cf. section 1.C). The composition with g induces a homomorphism $g^s\colon (\overset{s}{\wedge} G)^* \to \operatorname{Hom}_R(\overset{s+1}{\wedge} G, N)$. If $\alpha \in (\overset{s}{\wedge} N)^* \subset (\overset{s}{\wedge} G)^*$, then $g^s(\alpha) = 0$:

$$g^s(\alpha)(x) = \alpha(\overset{s+1}{\wedge} g(x)) = 0 \quad \text{for all} \quad x \in \overset{s+1}{\wedge} G,$$

since $\operatorname{Hom}_R(\overset{s+1}{\wedge} N, N) = 0$ for N torsionfree. Let N be oriented by ν. Now $(\overset{s}{\wedge} N)^*$ is free with basis ν, and every such α is a multiple of ν. After having fixed a basis $e_1, \ldots, e_t$ we call the elements

$$\nu(e_J) = \sum_{j \in J} \sigma(J\backslash j, j)\, \nu(e_{J\backslash j})\, e_j \in \operatorname{Ker} g$$

the *divided determinantal relations of* N, for reasons apparent from the interpretation of the Buchsbaum-Eisenbud factorization theorem in terms of matrices and determinants.

Conversely, every $\alpha \in (\overset{s}{\wedge} G)^*$ such that $g^{w}(\alpha) = 0$ is in fact an element of $(\overset{s}{\wedge} N)^*$: One has to show that α

vanishes on $M \wedge \overset{s-1}{\wedge} G \subset \overset{s}{\wedge} G$, and this may be tested in depth 0, whence one may assume that R is local and G has a basis according to the splitting $G = M \oplus N$.

Now we free M from the hypothesis of being torsion-free and take a more general point of view.

<u>Definition:</u> A linear form $\alpha \in (\overset{w}{\wedge} G)^*$ is called an *alternating syzygy of* g (or N) *of order* w if $g^w(\alpha) = 0$. The module of alternating syzygies of g of order w is denoted by $A(w,g)$.

As the following proposition shows, the coincidence of the alternating syzygies of w and $(\overset{w}{\wedge} N)^*$ (for torsion-free N) is restricted to the case $w = s$:

(5.1) <u>Proposition:</u> *If* α *is an alternating syzygy of order* $w < s$, *then* $\alpha = 0$.

It suffices to prove $\alpha = 0$ in depth 0, and this is an immediate consequence of (5.2) below.

With every alternating syzygy α of order w there is of course associated a linear map b making the diagram

$$\begin{tikzcd} F \arrow[r, "f"] & G \\ & \overset{w+1}{\wedge} G \arrow[ul, "b"] \arrow[u, "\alpha"'] \end{tikzcd}$$

commutative. In general, b is not uniquely determined.

The exterior powers of f provide us with "trivial" alternating syzygies. For $x \in \overset{u}{\wedge} F$ consider the linear form

$$\alpha := \gamma^{u}(\overset{u}{\wedge} f(x)) \in (\overset{w}{\wedge} G)^{*},$$

γ being an orientation and $w := t-u$. In order to prove that α is an alternating syzygy, we show directly that $\alpha\colon \overset{w+1}{\wedge} G \to G$ factors through F. We may assume $x = x_1 \wedge \ldots \wedge x_u$, $x_i \in F$. Then, by an application of Lemma (2.7) to the orientation γ on G,

$$\alpha(y) = \gamma^{u}(\overset{u}{\wedge} f(x))(y) = (-1)^{w}\tilde{\gamma}^{w+1}(y)(\overset{u}{\wedge} f(x))$$

$$= \sum_{i=1}^{u} (-1)^{w+u-i}[\tilde{\gamma}^{w+1}(y)(\overset{u-1}{\wedge} f(x_{[1,u]\setminus i}))](x_i).$$

If $d_1,\ldots,d_n$ is a basis of F and $x = d_I$, $I \in S(u,n)$, $y = e_J$, $I \in S(w+1,n)$, then $\tilde{\gamma}^{w+1}(e_J)(\overset{u-1}{\wedge} f(d_{I\setminus i}))$ is, after multiplication by $\sigma([1,t]\setminus J,J)$, the $(u-1)$-minor of the matrix of f which corresponds to the rows $I\setminus i$ and the columns $[1,t]\setminus J$. The computation above shows that a suitable map $b\colon \overset{w+1}{\wedge} G \to F$ is given by

$$b(y) = (-1)^{w}[(\overset{u-1}{\wedge} f)^{*}(\tilde{\gamma}^{w+1}(y))](x).$$

In the case in which M = F all the alternating syzygies of N arise in this way:

(5.2) Proposition: *With the hypotheses introduced so far, let the sequence*

$$0 \to F \xrightarrow{f} G \xrightarrow{g} N \to 0$$

be exact. Then, for every w, $0 \le w \le t$, $u := t - w$, *the sequence*

$$0 \longrightarrow \overset{u}{\wedge} F \xrightarrow{\gamma^u \circ \overset{u}{\wedge} f} (\overset{w}{\wedge} G)^* \xrightarrow{g^w} \mathrm{Hom}(\overset{w+1}{\wedge} G, N)$$

is exact.

Proof: As seen above, the sequence under consideration is a zero-sequence. Since N and, consequently, $\mathrm{Hom}(\overset{w+1}{\wedge} G, N)$ are torsionfree under the hypotheses of (5.2), it suffices to prove its exactness in depth 1, where the sequence representing N is split-exact by hypothesis. Replacing R by a localization R_P, $\operatorname{depth} R_P \le 1$, we may assume that G has a basis $e_1, \dots, e_t$ such that $e_1, \dots, e_r$, $r = \operatorname{rank} F$, form a basis of F, $g(e_{r+1}), \dots, g(e_t)$ form a basis of N, and $\gamma(e_1 \wedge \dots \wedge e_t) = 1$. Then it is easy to check that a linear form

$$\sum_{W \in S(w,t)} a_W e_W^*, \quad a_W \in R,$$

is in the kernel of g^w if and only if $a_W = 0$ for all $W \not\supset \{r+1, \dots, t\}$, and that the elements e_W^*, $W \supset \{r+1, \dots, t\}$ generate $\operatorname{Im} \gamma^u \circ \overset{u}{\wedge} f$. -

In general we have to content ourselves with a statement weaker than (5.2). Let M be oriented by μ now, and $r := \operatorname{rank} M$. Then $I(\mu)\operatorname{Ker} g^w \subset \operatorname{Im}(\gamma^u \circ \overset{u}{\wedge} f)$, and part (b) of (5.3) tells us how to represent the elements of $I(\mu)\operatorname{Ker} g^w$ inside $\operatorname{Im}(\gamma^u \circ \overset{u}{\wedge} f)$.

(5.3) <u>Theorem:</u> *With the hypotheses introduced so far, the following holds:*

(a) *Let* $\tilde{g}: G \to N/TN$ *be the epimorphism induced by* $g: G \to N$. *Then the sequence*

$$0 \longrightarrow (\overset{u}{\wedge} M)^{**} \xrightarrow{\gamma^u} (\overset{w}{\wedge} G)^{*} \xrightarrow{\tilde{g}^w} \operatorname{Hom}(\overset{w+1}{\wedge} G, N/TN)$$

is exact for $0 \le w \le t$, $u := t-w$.

(b) *For each alternating syzygy* α *of* N *of order* w *there is a unique linear form* $\beta \in (\overset{r-u}{\wedge} M)^* \subset (\overset{r-u}{\wedge} F)^*$ *such that the diagram*

$$\begin{tikzcd}
\overset{r}{\wedge} F \arrow[r, "\beta"] \arrow[dr, "\mu"'] & \overset{u}{\wedge} F \arrow[r, "\overset{u}{\wedge} f"] & \overset{u}{\wedge} G \\
& R \arrow[ur, "(\gamma^u)^{-1} \circ \alpha^*"'] &
\end{tikzcd}$$

is commutative.

(c) *If* N *is torsionfree, then for every* $\beta \in (\overset{r-u}{\wedge} M)^*$ *there is a unique alternating syzygy* $\alpha \in (\overset{w}{\wedge} G)^*$, $w = t-u$ *such that the diagram in* (b) *is commutative.*

Proof: (a) Using the natural identification $(\overset{u}{\wedge} M^{**})^{**} = (\overset{u}{\wedge} M)^{**}$ and $\operatorname{Ker} \tilde{g} = M^{**}$ we may assume that N is actually torsionfree. By (5.2), the sequence is exact in depth 1, and, hence, everywhere.

For part (b) we choose β such that the composition $\gamma^u \circ (\mu^{r-u})^{*-1}$ maps β to α. This is possible since $\operatorname{Ker} g^w \subset \operatorname{Ker} \tilde{g}^w$. Since $\mu \in (\overset{r}{\wedge} M)^*$ and $\beta \in (\overset{r-u}{\wedge} M)^*$ we may first replace $\overset{r}{\wedge} F$ and $\overset{u}{\wedge} F$ by $\overset{r}{\wedge} M$ and $\overset{u}{\wedge} M$ in verifying the commutativity of the diagram. Next one may assume that R is local, and M is free with a basis $x_1, \ldots, x_r$ such that $\mu(x_1 \wedge \ldots \wedge x_r) = 1$, since it is enough to test the commutativity in depth 0. Then for $J \in S(r-u,r)$ and $\bar{J} := [1,r] \setminus J$:

$$\begin{aligned}
&(\mu^{r-u})^*(\beta(x_1 \wedge \ldots \wedge x_r))(x_J) \\
&\quad = \mu^{r-u}(x_J)(x_1 \wedge \ldots \wedge x_r) \\
&\quad = \mu(x_J \wedge \sigma(J,\bar{J})\beta(x_J)x_{\bar{J}}) \\
&\quad = \beta(x_J).
\end{aligned}$$

(Here we have identified $\overset{u}{\wedge} M$ and $(\overset{u}{\wedge} M)^{**}$.) Consequently

$$\beta(x_1 \wedge \ldots \wedge x_r) = (\mu^{r-u})^{*-1}(\beta),$$

and, by the choice of β, this equality makes the diagram in (b) commutative. On the other hand this is certainly the only possible choice of β (in depth 0, and thus everywhere).

(c) If N is torsionfree, then we can clearly reverse the construction in (b). -

Theorem (5.3) may be considered to be an extension of Proposition (3.1). The linear form ν constructed in (3.1) is just the linear form α constructed for $\beta = 1 \in (\overset{0}{\wedge}M)^* = R$.

Theorem (5.3) and the existence of a map $b\colon \overset{w+1}{\wedge} G \to F$ factoring $\alpha\colon \overset{w+1}{\wedge} G \to G$ through $f\colon F \to G$ enable us to express $\mu(x)\alpha(y)$, $x \in \overset{r}{\wedge}M$, $y \in \overset{w+1}{\wedge} G$, in two different ways as an element of $M \subset G$. By virtue of (5.3) and an application of Lemma (2.7) on G and γ we have

$$\mu(x)\alpha(y) = \alpha^*(\mu(x))(y) = \gamma^u(\beta(x))(y)$$

$$= (-1)^w\tilde{\gamma}^{w+1}(y)(\beta(x)) = (-1)^w(\tilde{\gamma}^{w+1}(y)\circ\beta)(x)$$

On the other hand, since $\mu = \mu^r$,

$$\mu(x)\alpha(y) = \mu(x)(f\circ b)(y) = [(\tilde{\mu}^1\circ f\circ b)(y)](x)$$

again by an application of (2.7), on M and μ now. These two representations are associated to the commutativity of a diagram:

(5.4) <u>Theorem:</u> *With the notations introduced so far, the diagram*

$$\begin{tikzcd}
\overset{r-1}{\wedge} F \arrow[r, "\beta"] \arrow[dr, "\mu^{r-1}"'] & \overset{u-1}{\wedge} F \arrow[r, "\overset{u-1}{\wedge} f"] & \overset{u-1}{\wedge} G \\
& F^* \arrow[ur, "\varepsilon(\gamma^{u-1})^{-1} \circ b^*"'] &
\end{tikzcd}$$

is commutative, $\varepsilon := (-1)^w$.

Proof: Let us look at the following diagram which is essentially a detailed version of the dual diagram:

$$\begin{tikzcd}
\overset{w+1}{\wedge} G \arrow[r, "\varepsilon(\gamma^{u-1})^*"] \arrow[ddrr, "b"'] & (\overset{u-1}{\wedge} G)^* \arrow[rr, "(\overset{u-1}{\wedge} f)^*"] \arrow[dr] & & (\overset{u-1}{\wedge} F)^* \arrow[r, "\beta^*"] & (\overset{r-1}{\wedge} F)^* \\
& & (\overset{u-1}{\wedge} M)^* \arrow[ur] \arrow[rr, "\beta^*"] & & (\overset{r-1}{\wedge} M)^* \arrow[u] \\
& & F \arrow[r, "f"] & M \arrow[ur, "\tilde{\mu}^1"'] &
\end{tikzcd}$$

We have to show that the images $\varkappa_1$ and $\varkappa_2$ of an element $y \in \overset{w+1}{\wedge} G$ in $(\overset{r-1}{\wedge} M)^*$ coincide:

$$\varkappa_1 = \varepsilon\beta^*(\gamma^{u-1})^*(y) = (-1)^w \tilde{\gamma}^{w+1}(y) \circ \beta \quad \text{and}$$

$$\varkappa_2 = (\tilde{\mu}^1 \circ f \circ b)(y).$$

As we have seen above, $\varkappa_1$ and $\varkappa_2$ coincide on $\overset{r}{\wedge} M$, hence $\varkappa_1 = \varkappa_2$ by (1.5). -

Suppose now that M is a second syzygy. Then N is torsionfree. So we may choose β in (5.3),(c) to be identification $\overset{0}{\wedge}M = R$, and (5.4) specializes to (4.1), giving however a much more satisfying explanation for the existence of the map b. Let us discuss the proof of (5.4) for this case in less abstract terms. Choose bases $d_1,\dots,d_n$ of F and $e_1,\dots,e_t$ of G. Then the map b is the map which assigns preimages in F to the divided determinantal relations $\nu(e_J)$, $J \in S(s+1,[1,t])$, and the commutativity of the diagram in (4.1) is simply derived from equating the coefficients in the two "natural" representations of $\mu(\overset{r}{\wedge}f(d_I))\nu(e_J)$ as linear combinations of the $f(d_i), i \in I$:

$$\mu(\overset{r}{\wedge}f(d_I))\nu(e_J) = (-1)^w \sum_{i\in I} \sigma(I\backslash i,i)\gamma(\overset{r-1}{\wedge} f(d_{I\backslash i}) \wedge e_J)f(d_i)$$

and

$$\mu(\overset{r}{\wedge}f(d_I))\nu(e_J) = \sum_{i\in I} \sigma(I\backslash i,i)\mu(\overset{r-1}{\wedge} f(d_{I\backslash i}) \wedge f(b(e_J)))f(d_i)$$

For the sake of generalization below we used (2.7) in the derivation of the second representation. It follows simply from the fact that μ is an alternating syzygy of order r on M. It should be noted that this second way of verifying (4.1) was indicated by Buchsbaum and Eisenbud in [3, Proposition 6.3] without proof.

The existence of a map b, which makes the diagram in (5.4) commutative, can of course be proved directly by extending the method of proof used for (4.1). This indicates that there are "higher" versions of (5.4).

Theorem (5.4) can be considered to be obtained by working with the alternating syzygies of order t-1 induced by α. In order to obtain "higher" versions of (5.4) we look at the alternating syzygies of order $< t-1$ induced by α:

$$\begin{tikzcd}
\overset{i}{\wedge} F \arrow[r, "\overset{i}{\wedge} f"] & \overset{i}{\wedge} M \arrow[r] & \overset{i}{\wedge} G \arrow[r, "\gamma^i"] & (\overset{t-i}{\wedge} G)^* \arrow[r] & \mathrm{Hom}(\overset{t-i+1}{\wedge} G, N) \\
 & & \overset{w+i}{\wedge} G \arrow[ull, "b^i"] \arrow[u, "\alpha"'] & &
\end{tikzcd}$$

If $\overset{i}{\wedge} M$ is a second syzygy, then, by (5.3), α may be factored through $\overset{i}{\wedge} F$ in the way indicated, and as soon as the map b^i exists, a "higher" version of (5.4) holds:

(5.5) <u>Theorem:</u> *With the notations introduced so far, suppose that there exists a map* b^i *which makes the preceding diagram commutative. Then, with*

$$\varepsilon := (-1)^{iw}$$

the following diagram is commutative:

$$\begin{tikzcd}
\overset{r-i}{\wedge} F \arrow[r, "\beta"] \arrow[dr, "\mu^{r-i}"'] & \overset{u-i}{\wedge} F \arrow[r, "\overset{u-i}{\wedge} f"] & \overset{u-i}{\wedge} G \\
 & (\overset{i}{\wedge} F)^* \arrow[ur, "\varepsilon(\gamma^{u-i})^{-1} \circ (b^i)^*"'] &
\end{tikzcd}$$

The proof of (5.5) is a direct generalization of the proof of (5.4). In order to avoid any complications regarding the torsion of $\overset{i}{\wedge}M$, one should first pass to depth 0. This is harmless because one only wants to show the equality of elements of $(\overset{r-i}{\wedge} M)^*$.

6. The reflection of alternating syzygies

As we have seen in section 5, with each map f occuring in a free resolution

$$\mathfrak{F} : \ldots \to F_n \xrightarrow{f_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{f=f_1} F_o$$

there is associated an alternating syzygy of (the lowest possible) order rank f given by an orientation on $\operatorname{Im} f$, provided such an orientation exists. In this section we want to show that every alternating syzygy gives rise to alternating syzygies of higher order. To explain the basic idea we first consider the special case in which $F_o = R$. Then we have a comparison map $\varkappa$ from the ordinary Koszul complex for f_1 to $\mathfrak{F}$:

$$\begin{tikzcd}
\ldots \arrow[r] & F_3 \arrow[r, "f_3"] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] \arrow[d, equal] & F_o = R \arrow[d, equal] \\
\ldots \arrow[r] & \overset{3}{\wedge} F_1 \arrow[r, "f_1"] \arrow[u, "\varkappa_3"'] & \overset{2}{\wedge} F_1 \arrow[r, "\alpha"] \arrow[u, "\varkappa_2"'] & F_1 \arrow[r, "f_1"] & F_o
\end{tikzcd}$$

The existence of the Koszul complex simply relies on the fact that $\alpha := f_1$ is an alternating syzygy of it-self, for $f_1^2 = 0$ in Λ^*F_1.

Let $M_i := \operatorname{Im} f_i$. Since $\varkappa$ is a comparison map one has

$$f_k \circ \varkappa_k \circ f_1 = 0,$$

in particular $(f_k \circ \varkappa_k)^*(\vartheta) \circ f_1 = 0$ for every $\vartheta \in M_k^*$. In the structure of Λ^*F_1 as an exterior algebra this means

$$\begin{aligned} 0 &= (f_k \circ \varkappa_k)^*(\vartheta) \wedge f_1 = \pm\, f_1 \wedge (f_k \circ \varkappa_k)^*(\vartheta) \\ &= \pm\, f_1 \circ (f_k \circ \varkappa_k)^*(\vartheta). \end{aligned}$$

A slight generalization of the Koszul complex leads to the same result for an arbitrary alternating syzygy of order w of f_1. The lower row of the following diagram clearly is a complex, and again one has a comparison map:

$$\begin{tikzcd} \cdots \arrow[r] & F_4 \arrow[r, "f_4"] & F_3 \arrow[r, "f_3"] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 = R \\ \cdots \arrow[r] & \overset{w+3}{\Lambda} F_1 \arrow[r, "f_1"] \arrow[u, "\varkappa_4"] & \overset{w+2}{\Lambda} F_1 \arrow[r, "f_1"] \arrow[u, "\varkappa_3"] & \overset{w+1}{\Lambda} F_1 \arrow[r, "\alpha"] \arrow[u, "\varkappa_2"] & F_1 \arrow[r, "f_1"] \arrow[u, equal] & F_0 \arrow[u, equal] \end{tikzcd}$$

Thus α and the comparison map $\varkappa$ induce a sequence of maps

$$\rho_k\colon M_k^* \to A(w+k-1,f_1), \quad k \geq 1,$$

$$\rho_k(\vartheta) := (f_k \circ \varkappa_k)^*(\vartheta).$$

We call ρ_k the k-*th reflection* of α with respect to $\varkappa$. Obviously ρ_k does not depend directly on $\varkappa_k$, but it may vary with $\varkappa_2,\ldots,\varkappa_{k-1}$. It is however unique in a weaker sence (cf. Theorem (6.5)).

Thus we see that an alternating syzygy is the root of a tree of alternating syzygies: its reflection, the reflections of its reflections etc. All the branches of this tree have finite length bounded by rank F_1 - rank f_1 - 1, since each proper reflection increases the order by at least 1.

For a special case in which it was found independently and in a completely different way by Pragacz and Weyman [11], we state part of this result formally:

(6.1) Proposition: *Let* $\mathfrak{F}: 0 \to R \to R^n \to R^n \xrightarrow{f_1} R$ *be a free resolution and* m *the greatest integer* $\leq n/2$. *Then there is a sequence* $\alpha_1,\ldots,\alpha_m$ *of alternating syzygies of orders* 1,3,...,2m-1 *given by* $\alpha_1 = f_1$ *and* $\alpha_j = \rho_3(\mathrm{id})$, *the reflection taken with respect to* α_{j-1}, j = 2,...,m.

A particularly interesting case of (6.1) is the one in which $\mathfrak{F}$ is the resolution of a Gorenstein factor ring of R. By the structure theorem of Buchsbaum-

Eisenbud for such resolutions [4, Theorem 2.1], the number n is odd and, with respect to a suitable system of generators of Ker f_1, the matrix of $R^n \to R^n$ is strictly skew-symmetric. Examples indicate that such a system of generators is always supplied by the map $\alpha_m\colon \overset{n-1}{\wedge} R^n \to R^n$, α_m given by (6.1). In fact, the sequence

$$0 \to \overset{n}{\wedge} R^n \xrightarrow{f_1} \overset{n-1}{\wedge} R^n \xrightarrow{\alpha_m} R^n \xrightarrow{f_1} R$$

is a complex, and we believe that it is a resolution in general. After identifying $\overset{n}{\wedge} R^n$ with $(\overset{0}{\wedge} R^n)^* = R$ and $\overset{r-1}{\wedge} R^n$ with $(R^n)^*$ via the orientation given by the basis chosen in R^n, it certainly has the alternating structure described by the Buchsbaum-Eisenbud theorem.

In order to define the reflection of alternating syzygies for arbitrary resolutions we have to find the right generalization of the Koszul complex:

(6.2) Theorem: *Let* N *be an orientable* R*-module of rank* s*,* ν *an orientation on* N*, and* $g\colon G \to N$ *an epimorphism from a free* R*-module* G*. Furthermore let* $\alpha \in (\overset{w}{\wedge} G)^*$ *be an alternating syzygy of order* w *of* N*. Then the following holds:*

(a) *The sequence*

$$\cdots \to \overset{w+3}{\wedge} G \otimes (\overset{2}{\otimes} \overset{s-1}{\wedge} G) \xrightarrow{\eta} \overset{w+2}{\wedge} G \otimes \overset{s-1}{\wedge} G \xrightarrow{\eta} \overset{w+1}{\wedge} G \xrightarrow{\alpha} G \xrightarrow{g} N \to 0$$

$$\eta(x \otimes (y_1 \otimes \cdots \otimes y_m)) := (\nu(x) \wedge y_1) \otimes y_2 \otimes \cdots \otimes y_m,$$

is a complex. (Here ν *is considered as an element of* $(\overset{s}{\wedge}N)^* \subset (\overset{s}{\wedge}G)^*$.)

(b) *The sequences*

$$\overset{w+k+1}{\wedge} G \otimes N^* \otimes (\overset{k-1}{\otimes} \overset{s-1}{\wedge} G) \xrightarrow{\zeta} \overset{w+k}{\wedge} G \otimes (\overset{k-1}{\otimes} \overset{s-1}{\wedge} G)$$

$$\xrightarrow{\eta} \overset{w+k-1}{\wedge} G \otimes (\overset{k-2}{\otimes} \overset{s-1}{\wedge} G),$$

$$\zeta(x \otimes \xi \otimes z) := g^*(\xi)(x) \otimes z,$$

and

$$\overset{w+2}{\wedge} G \otimes N^* \overset{\zeta}{\to} \overset{w+1}{\wedge} G \overset{\alpha}{\to} G$$

are complexes.

Proof: We shall see that ζ and η are the extreme members of a family of maps, ζ corresponding to N^* and η corresponding to $(\overset{s}{\wedge}N)^*$. The inductive proof of (a) starts at (b):

In proving the first part of (b), it certainly suffices to show that the sequence

$$\overset{w+k+1}{\wedge} G \otimes N^* \otimes \overset{s-1}{\wedge} G \longrightarrow \overset{w+k}{\wedge} G \otimes \overset{s-1}{\wedge} G \longrightarrow \overset{w+k-1}{\wedge} G$$

$$x \otimes \xi \otimes y \longrightarrow g^*(\xi)(x) \otimes y \longrightarrow \nu(g^*(\xi)(x)) \wedge y$$

is a complex. This is trivially true, since $\nu \circ g^*(\xi)$ is, via the natural inclusion $(\overset{s+1}{\wedge} N)^* \subset (\overset{s+1}{\wedge} G)^*$, an element of $(\overset{s+1}{\wedge} N)^* = 0$.

The second part of (b): $\alpha(\zeta(x \otimes \xi)) = (\alpha \circ g^*(\xi))(x)$, and, as an element of $(\overset{w+1}{\wedge} G)^*$,

$$\alpha \circ g^*(\xi) = \alpha \wedge g^*(\xi) = \pm\, g^*(\xi) \circ \alpha = \pm\, \xi \circ g \circ \alpha = 0$$

since α is an alternating syzygy of N (via g).

Generalizing (a) and (b) we now consider the compositions (i)

$$[\overset{u}{\wedge} G \otimes \overset{v}{\wedge} G \otimes (\overset{t}{\wedge} N)^*] \otimes (\overset{k-1}{\otimes} \overset{s-1}{\wedge} G) \xrightarrow{\zeta_{uv}} \overset{w+k}{\wedge} G \otimes (\overset{k-1}{\otimes} \overset{s-1}{\wedge} G) \overset{\eta}{\to} \ldots,$$

$$[x \otimes y \otimes \xi] \otimes z \longrightarrow [\xi(x) \wedge y - (-1)^{t(u-t)} x \wedge \xi(y)] \otimes z,$$

$u + v - t = w + k$, $t \geq 1$, and, for $k = 1$,

$$\text{(ii)}\quad [\overset{u}{\wedge} G \otimes \overset{v}{\wedge} G \otimes (\overset{t}{\wedge} N)^*] \xrightarrow{\zeta_{uv}} \overset{w+1}{\wedge} G \to G.$$

In fact, for $v = 0$, $t = 1$ the sequences (i) and (ii) specialize to the sequences in (b), and for $t = s$, $v = s-1$, $\zeta = \nu$ the sequence (i) specializes to the one in (a). In order to see that (i) and (ii) are complexes, it suffices to show this in depth 0, since the composition of the maps ends in a free module. Thus we may assume that N is free. Furthermore, ζ_{uv} preserves the last component of the tensor product. Therefore, and by virtue of (b), it finally suffices to show that the image of

$$\overset{u}{\wedge} G \otimes \overset{v}{\wedge} G \otimes (\overset{t}{\wedge} N)^* \xrightarrow{\zeta_{uv}} \overset{w+k}{\wedge} G$$

is contained in $\operatorname{Im}\zeta_{w+k+1,0}$ for all u,v, provided N free.

This inclusion is proved by induction on t. In case t = 1 we have

$$\begin{aligned}\zeta_{uv}(x \otimes y \otimes \xi) &= \xi(x) \wedge y - (-1)^{u-1} x \wedge \xi(y)\\ &= \xi(x \wedge y) = \zeta_{w+k+1,0}((x \wedge y) \otimes \xi).\end{aligned}$$

Let now t > 1. Since N is free, every element in $(\overset{t}{\wedge}N)^*$ is a sum of products of elements of N*. Therefore we may assume $\xi = \xi_1 \wedge \xi_2$, $\xi_1 \in (\overset{t-1}{\wedge}N)$, $\xi_2 \in N^*$. Then

$$\begin{aligned}&\zeta_{u-t+1,v}(\xi_1(x) \otimes y \otimes \xi_2) + (-1)^{u-t}\,\zeta_{u,v-1}(x \otimes \xi_2(y) \otimes \xi_1)\\ &= (\xi_2 \wedge \xi_1)(x) \wedge y - (-1)^{u-t}\xi_1(x) \wedge \xi_2(y)\\ &+ (-1)^{u-t}[(\xi_1(x) \wedge \xi_2(y)) - (-1)^{(t-1)(u-t+1)} x \wedge (\xi_1 \wedge \xi_2)(y)]\\ &= (-1)^{t-1}\zeta_{uv}(x \otimes y \otimes \xi).\end{aligned}$$

By induction hypothesis the left-hand part of this equation is in $\operatorname{Im}\zeta_{w+k+1,0}$. This completes the proof for (i), and (ii) is proved in the same vein, just specializing k = 1. -

The reflection of alternating syzygies in the general case can now be defined as follows. Let

$$\mathfrak{F}: \ldots \to F_n \xrightarrow{f_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{f_1} F_0$$

a free resolution such that $N = \operatorname{Im} f_1$. Then, with $\tilde{s} := s-1$, we have a comparison map $\varkappa$ from the complex in (6.2), (a):

$$\begin{tikzcd}
\cdots \arrow[r] & F_k \arrow[r, "f_k"] & F_{k-1} \arrow[r] & \cdots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \\
\cdots \arrow[r] & \bigwedge^{w+k-1} F_1 \otimes (\bigotimes^{k-2} \bigwedge^{\tilde{s}} F_1) \arrow[r, "\eta"] \arrow[u, "\varkappa_k"] & \bigwedge^{w+k-2} F_1 \otimes (\bigotimes^{k-3} \bigwedge^{\tilde{s}} F_1) \arrow[r] \arrow[u, "\varkappa_{k-1}"] & \cdots \arrow[r] & \bigwedge^{w+1} F_1 \arrow[r, "\alpha"] \arrow[u, "\varkappa_2"] & F_1 \arrow[u, equal]
\end{tikzcd}$$

It follows from part (b) of (6.2) that for each $k \geq 2$, $\vartheta \in M_k^*$, and $z \in \bigotimes^{k-2} \bigwedge^{s-1} F_1$ the linear form on $\bigwedge^{w+k-1} F_1$ given by

$$\rho_k(\vartheta \otimes z)(x) := (f_k \circ \varkappa_k)^*(\vartheta)(x \otimes z)$$

is an alternating syzygy of order $w+k-1$. In fact, for every linear form $\xi \in N^*$ and every $y \in \bigwedge^{w+k-1} F_1$ one has

$$\begin{aligned}
\xi(f_1(\rho_k(\vartheta \otimes z)(y))) &= f_1^*(\xi)(\rho_k(\vartheta \otimes z)(y)) \\
&= \pm\ \rho_k(\vartheta \otimes z)(f_1^*(\xi)(y)) \\
&= \pm\ \vartheta(f_k \circ \varkappa_k \circ \zeta(y \otimes \xi \otimes z)) \\
&= 0.
\end{aligned}$$

Since N is torsionfree this suffices to show $f_1(\rho_k(\vartheta \otimes z)(y)) = 0$. As in the special case $F_0 = R$ we call

$$\rho_k : M_k^* \otimes (\bigotimes^{k-2} \bigwedge^{s-1} F_1) \to A(w+k-1, f_1)$$

the k-*th reflection* of α with respect to $\varkappa$.

We want to discuss to what extent the reflections may carry relevant information. In order to avoid repetitions we fix the meaning to the symbols $N, f_1, s, \nu, \alpha, \mathfrak{F}, \varkappa$ and keep the hypotheses under which the reflection was defined. We furthermore denote an orientation of F_1 by φ, the kernel of f_1 by M, the rank of M by r, and an orientation on M by μ.

First let us state that the "trivial" alternating syzygies do not contribute anything interesting. For $0 \le i \le r$ let

$$B(\operatorname{rank} F_1 - u,\ f_1) := \varphi^u(\overset{u}{\wedge} M).$$

(More precisely φ^u is to be applied to the image of the map $\overset{u}{\wedge} M \to \overset{u}{\wedge} F_1$ induced by the embedding $M \to F_1$.)

(6.3) <u>Proposition:</u> *Let* $\alpha \in B(w, f_1)$. *Then the comparison map* $\varkappa$ *can be chosen such that* $\varkappa_k = 0$ *for* $k \ge 3$.

Proof: We choose $\varkappa_2 = b$ as computed above (5.2):

$$b(y) = (-1)^w [(\overset{u-1}{\wedge} f_2)^*(\tilde{\varphi}^{w+1}(y))](x) \quad \text{for} \quad y \in \overset{w+1}{\wedge} F_1,$$

where $x \in \overset{u}{\wedge} F_2$, $u = \operatorname{rank} F_1 - w$, such that $\gamma^u(\overset{u}{\wedge} f_2(x)) = \alpha$. It suffices to prove that $b \circ \eta = 0$, and this may be tested in depth 0. As in the proof of (6.2) it is enough to show $b \circ \zeta = 0$. Let $z \in \overset{w+2}{\wedge} F_2$, $\xi \in N^*$. Then

$$(b \circ \zeta)(z \otimes \xi) = (-1)^w [(\overset{u-1}{\wedge} f_2)^*(\tilde{\varphi}^{w+1}(f_1^*(\xi)(z)))](x) = 0$$

since $(\overset{u-1}{\wedge} f_2)^*(\tilde{\varphi}^{w+1}(f_1^*(\xi)(\underset{\sim}{z}))) \in (\overset{u-1}{\wedge} F_1)^*$ is zero. In fact, for $\tilde{u}:= u-1$ and $q \in \overset{\tilde{u}}{\wedge} F_1$ one has

$$[(\overset{\tilde{u}}{\wedge} f_2)^*(\varphi^{w+1}(f_1^*(\xi)(z)))](q) = \tilde{\varphi}^{w+1}(f_1^*(\xi)(z))(\overset{\tilde{u}}{\wedge} f_2(q))$$

$$= \pm\, \varphi^{\tilde{u}}(\overset{\tilde{u}}{\wedge} f_2(q))(f_1^*(\xi)(z)) = \pm\, \xi(f_1(\varphi^{\tilde{u}}(\overset{\tilde{u}}{\wedge} f_2(q))(z)))$$

$$= 0$$

because $\varphi^{\tilde{u}}(\overset{\tilde{u}}{\wedge} f_2(q))$ is an alternating syzygy of f_1. -

As a next step we want to investigate for which $\vartheta \in M_k^*$ the result $\rho_k(\vartheta \otimes \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1)$ of the reflection may be relevant. Let $E_k \subset M_k^*$ be the submodule of linear forms extendable to F_{k-1} and

$$C(s+i, f_1) := \nu \wedge (\overset{i}{\wedge} F_1)^*, \quad 0 \le i \le r.$$

$C(s+i, f_1) \subset (\overset{s+i}{\wedge} F_1)^*$ is certainly contained in $A(s+i, f_1)$ and consists of the alternating syzygies which are "irrelevant modulo ν". We shall see shortly that $\rho_k(E_k \otimes \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1) \subset C(w+k-1, f_1)$ for $k \ge 3$ as a consequence of a more general result. Consider any linear form τ on $\overset{w+k-2}{\wedge} F_1 \otimes \overset{k-3}{\otimes} \overset{s-1}{\wedge} F_1$. Then $\tau \circ \eta \circ \zeta = 0$ and the computation (for $(f_k \circ \varkappa_k)^*(\vartheta) \circ \zeta$) above shows that $\tau \circ \eta$ induces a map

$$\sigma\colon \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1 \longrightarrow A(w+k-1, f_1).$$

(6.4) <u>Proposition:</u> *Let* $k \ge 3$. *Then, for every* $\tau \in (\overset{w+k-2}{\wedge} F_1 \otimes \overset{k-3}{\otimes} \overset{s-1}{\wedge} F_1)^*$ *one has* $\operatorname{Im} \sigma \subset C(w+k-1, f_1)$.

In particular, $\rho_k(E_k \otimes \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1) \subset C(w+k-1, f_1)$.

Proof: Let $\vartheta \in E_k$. Then there exists an $\omega \in (F_{k-1})^*$ such that $\vartheta = \omega | M_k$, and for every $y_1, \ldots, y_{k-2} \in \overset{s-1}{\wedge} F_1$ one has

$$\rho_k(\vartheta \otimes y_1 \otimes \cdots \otimes y_{k-2})(x) = (f_k \circ \varkappa_k)^*(\vartheta)(x \otimes y_1 \otimes \cdots \otimes y_{k-2})$$

$$[(\omega \circ \varkappa_{k-1}) \circ \eta](x \otimes y_1 \otimes \cdots \otimes y_{k-2})$$

Taking $\tau = \omega \circ \varkappa_{k-1}$, we see that it is enough to prove the first statement. For all $y_2, \ldots, y_{k-2} \in \overset{s-1}{\wedge} F_1$ one finds a $\psi \in (\overset{w+k-2}{\wedge} F_1)^*$ such that

$$(\tau \circ \eta)(x \otimes y_1 \otimes \cdots \otimes y_{k-2}) = \psi(\nu(x) \wedge y_1)$$

for every $y_1 \in \overset{s-1}{\wedge} F_1$ and $x \in \overset{w+k-1}{\wedge} F_1$.

Let $v = \operatorname{rank} F_1 - (w+k-2)$. There is a $p \in \overset{v}{\wedge} F_1$ such that $\psi = \varphi^v(p)$. One easily checks that for the proper choice of $\varepsilon = \pm 1$ (independent of x)

$$\psi(\nu(x) \wedge y_1) = \varepsilon(\nu \circ \varphi^{v+s-1}(p \wedge y_1))(x). -$$

The preceding propositions suggest to consider the reductions

$$\tilde{A}(w, f_1) := A(w, f_1)/B(w, f_1) \quad \text{and}$$

$$\hat{A}(w, f_1) := A(w, f_1)/C(w, f_1).$$

Using these we can show that the reflection of alternating syzygies induces homomorphisms which do no longer depend on irrelevant choices:

(6.5) Theorem: *The reflection of alternating syzygies induces homomorphisms*

$$A(w,f_1) \otimes \mathrm{Hom}_R(M,R) \to A(w+1,f_1)$$

and, for every $k \geq 3$,

$$\tilde{A}(w,f_1) \otimes \mathrm{Ext}_R^{k-1}(N,R) \otimes \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1 \to \hat{A}(w+k-1,f_1)$$

which are independent of the choice of the free resolution and the comparison map $\varkappa$ *associated to* $\alpha \in A(w,f_1)$.

Proof: The first claim follows immediately from the construction. In proving the second claim we first fix $\alpha \in A(w,f_1)$, a free resolution $\mathfrak{F}$ and a comparison map $\varkappa$. Since

$$\mathrm{Ext}_R^{k-1}(N,R) = M_k^*/E_k$$

it follows from (6.4) that

$$\rho_k\colon M_k^* \otimes \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1 \to A(w+k-1,f_1)$$

factors through $\mathrm{Ext}_R^{k-1}(N,R) \otimes \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1$. Reducing by $C(w+k-1,f_1)$ on the right hand side we make the reflection independent of $\varkappa$. Two comparison maps $\varkappa$ and $\varkappa'$ differ by a homotopy χ:

$$\varkappa_k' = \varkappa_k + (\chi_{k-1}\circ\eta + f_{k+1} \circ \chi_k).$$

For every $\vartheta \in M_k^*$ and $z \in \overset{k-2}{\otimes} \overset{s-1}{\wedge} F_1$ this implies

$$(f_k \circ \varkappa_k')^*(\vartheta)(x \otimes z) = (f_k \circ \varkappa_k)^*(\vartheta)(x \otimes z) + (f_k \circ \chi_{k-1} \circ \eta)^*(\vartheta)(x \otimes z).$$

Again by virtue of (6.4) the second summand leads to an alternating syzygy in $C(w+k-1,f_1)$. A similar argument shows the independence of the resolution. The homomorphism

$$\operatorname{Ext}_R^{k-1}(N,R) \otimes \overset{k-2}{\otimes} \overset{s-1}{\bar{\wedge}} F_1 \rightarrow \hat{A}(w+k-1,f_1)$$

obtained so far, depends linearly on $x \in A(w,f_1)$, since for $\alpha,\alpha' \in A(w,f_1)$, $a,a' \in R$ and comparison maps $\varkappa,\varkappa'$ for α,α' resp. we may take $a\varkappa + a'\varkappa'$ as a comparison map for $a\alpha + a'\alpha'$. Because of (6.3) one may eventually reduce by $B(w,f_1)$ on the left side. -

(6.6) *Remark:* In case $w = s$ the first map in (6.5) induces a homomorphism

$$A(s,f_1) \otimes \operatorname{Ext}_R^1(N,R) \rightarrow \hat{A}(s+1,f_1)$$

since α is a multiple of ν then, and the argument for $k \geq 3$ applies to the case $k = 2$, too.

One should note that the second (besides E_k) "generic" submodule of M_k^*, that is $\mu_k^{r_k-1}(\wedge^{r_k-1} M_k)$ for $r_k = \operatorname{rank} M_k$ and an orientation μ_k on M_k, in general gives a non-zero contribution to the homomorphism in (6.5).

It would be desirable to have some control over the effect of iterated reflections. The only positive result we can report concerns the alternating syzygies $\alpha' \in \rho_2(\mu^{r-1}(\overset{r-1}{\wedge} M))$ with respect to some $\alpha \in A(w,f_1)$. From (5.4) and (6.3) one finds that the comparison map $\varkappa$ for α' may be chosen such that $\varkappa_k = 0$ for $k \geq 3$.

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