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On multigraded resolutions

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Introduction

This paper was initiated by a question of Eisenbud who asked whether the entries of the matrices in a minimal free resolution of a monomial ideal (which, after a suitable choice of bases, are monomials) divide the least common multiple of the generators of the ideal. We will see that this is indeed the case, and prove it by lifting the multigraded resolution of an ideal, or more generally of a multigraded module, keeping track of how the shifts 'deform' in such a lifting; see Theorem 2·1 and Corollary 2·2.

For perfect multigraded modules an even stronger statement is possible: if the vectors $a_{ij} \in \mathbb{N}^n$, $j = 1, \ldots, b_i$, denote the shifts in the ith position of a minimal multigraded resolution of a multigraded module M defined over a polynomial ring, then the monomials $x^{a_{ij}}$ with exponent a_{ij} satisfy the condition

$$\operatorname{lcm}(x^{a_{i1}}, \ldots, x^{a_{ib_i}}) = \operatorname{lcm}(x^{a_{11}}, \ldots, x^{a_{1b_1}})$$

for all $i \geqslant 1$; see Theorem 3·1. In a similar, but less strict, way one can bound the shifts of a multigraded module defined over a ring $K[x_1, \ldots, x_n]/I$ where I is an ideal generated by monomials; see Theorem 3·4 which complements similar results of Aramova, Barcanescu, and Herzog [**1**], Backelin [**3**] and Eisenbud, Reeves, and Totaro [**9**].

Other restraints on the shifts in the resolution of a monomial ideal follow from the multiplicative structure of $\operatorname{Tor}^R(K, R/I)$. In particular, for a monomial Gorenstein ideal we deduce in Theorem 4·2 that any variable occurs with the same multiplicity in those generators of the ideal that it divides, or, equivalently, that any monomial Gorenstein ideal arises from a squarefree such ideal by a substitution $x_k \mapsto x_k^{c(k)}$ of the variables.

For monomial Gorenstein ideals of height three we refine this result, and succeed in proving a 'structure theorem' for this class of ideals. It asserts (see Theorem 6·1) that any such ideal with n generators (n odd) is obtained from the ideal generated by the monomials

$$x_i x_{i+1} \cdots x_{i+m-1}, \quad i = 1, \ldots, n, \quad m = (n-1)/2$$

via a substitution replacing the variables by a regular sequence of monomials. (Here $x_j = x_{j-n}$ for $j > n$.) In a combinatorial formulation, our theorem says that every simplicial homology sphere with n vertices and of dimension $n-4$ arises from a cyclic polytope by a certain inflation process in which one repeatedly replaces one vertex by two.

For the proof we need some general information about the second syzygy module of an ideal generated by monomials. We show in Proposition 5·1 that this syzygy module is generated by what we call 'cyclic syzygies'. In particular we conclude in Corollary 5·3 that, quite contrary to the behaviour of the higher Betti numbers, the second Betti number of a monomial ideal is independent of the field over which it is defined; this answers a question posed to the authors by Hibi (see also [**10**]).

1. *Generalities on liftings of modules and resolutions*

For the reader's convenience we recall a few basic facts on liftings of graded modules. Let R be a Noetherian graded ring, and $\mathbf{x} = x_1, \dots, x_n$ a homogeneous R-sequence contained in the graded Jacobson radical $\operatorname{Rad} R$, which is the intersection of the graded maximal ideals of R. A graded $R/\mathbf{x}R$-module M is called liftable (to R) if there exists a graded R-module N for which $\mathbf{x}$ is an N-sequence and such that $M \cong N/\mathbf{x}N$. It is clear that M is liftable to R if and only if it is liftable to $R/(x_1, \dots, x_{n-1})R$, and then to $R/(x_1, \dots, x_{n-2})R$, and so on. Thus the crucial case is that of a length 1 regular sequence. For such one uses the following lemma whose easy proof we leave to the reader.

LEMMA 1·1. *Let R be a Noetherian graded ring, $x \in \operatorname{Rad} R$ a homogeneous R-regular element,*

$$F : F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

an exact sequence of graded R/xR-modules, and

$$G : G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$

a complex of graded R-modules such that $F = G \otimes R/xR$. If x is G_0-regular and G_1 is a finite R-module, then G is exact.

Now we prove the following well-known liftability criterion.

PROPOSITION 1·2. *Let R and x be as in* 1·1, *and assume further that M is a graded finite R/xR-module with a graded resolution*

$$F : \cdots \rightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \rightarrow M \rightarrow 0$$

by graded finite free R/xR-modules F_i. If there exists a graded complex

$$G : G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$

of graded free R-modules such that $\phi_i = \psi_i \otimes R/xR$ for $i = 1, 2$, then $N = \operatorname{Coker} \psi_1$ is a lifting of M. Moreover, if

$$G : \cdots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0 \rightarrow 0$$

is a graded complex of finite free R-modules such that $F = G \otimes R/xR$, then G is a graded free R-resolution of N.

Proof. First observe that $M \cong N/xN$, by the right exactness of the tensor product. Then use 1·1 to see that G is exact. Therefore $\mathrm{Tor}_1^R(N, R/xR) \cong H_1(F) = 0$; in other words, x is N-regular. One applies 1·1 again to see that G is a graded free R-resolution of N.

2. *Liftings of multigraded modules*

Let K be a field and $R = K[x_1, \dots, x_n]$ the polynomial ring with its natural multigrading: $f \in R$ is *homogeneous of degree* $a \in \mathbb{Z}^n$ if $f = \lambda x^a$, where $\lambda \in K$ and $x^a = x_1^{a_1} \cdots x_n^{a_n}$ for $a = (a_1, \dots, a_n)$.

Let M be a finite multigraded R-module. Without loss of generality we may assume that the degrees of the homogeneous generators of M all belong to $\mathbb{N}^n$. Then M has a minimal multigraded resolution

$$F: \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to 0$$

with
$$F_i = \bigoplus_{j=1}^{b_i} R(-a_{ij}), \quad a_{ij} \in \mathbb{Z}^n.$$

The vectors a_{ij} are called the *ith shifts* of F (or of M).

We call a_{ij} *squarefree* if $x^{a_{ij}}$ is a squarefree monomial. Note that the entries of the ϕ_i are monomials if we choose the natural homogeneous bases of the F_i. Indeed, the matrix of ϕ_i is given by

$$\phi_i = (\lambda_{jk} x^{a_{ij} - a_{i-1k}})_{jk}, \quad \lambda_{jk} \in K, \quad \text{with} \quad \lambda_{jk} = 0 \quad \text{if} \quad a_{ij} - a_{i-1k} \notin \mathbb{N}^n.$$

THEOREM 2·1. *The multigraded R-module M can be lifted to a multigraded S-module N, S a polynomial ring over R, such that all shifts in the minimal multigraded free S-resolution of N are squarefree.*

Proof. If all shifts of F are squarefree, then no lifting is required. Thus we may assume that there is at least one shift whose last entry, say, is > 1.

We define a map $\mathbb{N}^n \to \mathbb{N}^{n+1}$, $a \mapsto a'$. For $a = (a_1, \dots, a_n)$ we set

$$a' = \begin{cases} (a_1, \dots, a_n, 0) & \text{if} \quad a_n \leqslant 1, \\ (a_1, \dots, a_n - 1, 1) & \text{if} \quad a_n > 1; \end{cases}$$

then we introduce a new variable x_{n+1}, choose $S' = K[x_1, \dots, x_{n+1}]$, and consider the complex

$$G': \cdots \xrightarrow{\psi_{i+1}} \bigoplus_j S'(-a'_{ij}) \xrightarrow{\psi_i} \cdots,$$

where, for all i, ψ_i is defined by the matrix

$$(\lambda_{jk} x^{a'_{ij} - a'_{i-1k}})_{jk}.$$

Note that $x^{a'_{ij} - a'_{i-1k}} \in S'$ if $\lambda_{jk} \neq 0$, since $a'_{ij} - a'_{i-1k} \in \mathbb{N}^{n+1}$ if $a_{ij} - a_{i-1k} \in \mathbb{N}^n$. It is clear that the ψ_i are homogeneous, and that $\psi_{i-1} \circ \psi_i = 0$ for all i. But then, by Proposition 1·2, $N' = \mathrm{Coker}\ \psi_1$ is a lifting of M $(= N'/(x_{n+1} - x_n)N')$, and G is a multigraded free S'-resolution of N'.

After a finite number of such liftings we achieve the desired lifting N of M. ∎

For later applications it is important to understand how the shifts of M are related to those of N. The proof of Theorem 2·1 shows the following. Let $a_{ij}(k)$ be the kth component of the shift a_{ij} of M, and for $k = 1, \dots, n$ set

$$d_k = \max_{i,j} \{a_{ij}(k)\}.$$

Then N is a module over the polynomial ring S over K in the variables $y_{11}, \dots, y_{1d_1}, \dots, y_{n1}, \dots, y_{nd_n}$, and we have

COROLLARY 2·2. *If b_{ij} is the shift of the lifted module N corresponding to the shift a_{ij} of M, then*

$$y^{b_{ij}} = \prod_{k=1}^{n} \prod_{l=1}^{a_{ij}(k)} y_{kl}.$$

3. *On the shifts in multigraded resolutions*

As in the previous section we consider a multigraded module M over the polynomial ring $R = K[x_1, \dots, x_n]$ with minimal multigraded resolution

$$0 \to \bigoplus_{j=1}^{b_p} R(-a_{pj}) \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_2} \bigoplus_{j=1}^{b_1} R(-a_{1j}) \xrightarrow{\phi_1} \bigoplus_{j=1}^{b_0} R(-a_{0j}) \to M \to 0,$$

and assume for simplicity, and without loss of generality, that all a_{0j} (and *a fortiori* all a_{ij}) belong to $\mathbb{N}^n$.

THEOREM 3·1. (*a*) *For all i and all $j = 1, \dots, b_i$ we have*

$$x^{a_{ij}} \mid \operatorname{lcm}(x^{a_{11}}, \dots, x^{a_{1b_1}}).$$

(*b*) *If M is perfect then for all $i \geqslant 1$,*

$$\operatorname{lcm}(x^{a_{i1}}, \dots, x^{a_{ib_i}}) = \operatorname{lcm}(x^{a_{11}} \dots, x^{a_{1b_1}}).$$

Proof. (*a*) If the assertion is not true then there exist integers i, j, and k such that $a_{ij}(k) > a_{1h}(k)$ for all $h = 1, \dots, b_1$. Let N be the lifting of M whose multigraded resolution has squarefree shifts b_{ij}, as described in Theorem 2·1 and Corollary 2·2. Then

$$y^{b_{ij}(k)} = \prod_{l=1}^{a_{ij}(k)} y_{kl} \quad \text{and} \quad y^{b_{1h}(k)} = \prod_{l=1}^{a_{1h}(k)} y_{kl}$$

for $h = 1, \dots, b_1$.

Let $c = \max\{a_{11}(k), \dots, a_{1b_1}(k)\}$; then there must exist an entry in the matrix of one of the maps ψ_r, $r = 2, \dots, i$, which is divisible by a non-trivial monomial in the variables $y_{kc+1}, \dots, y_{ka_{ij}(k)}$, while in the first relation matrix of N no such monomial occurs. This clearly contradicts the minimality of the resolution.

(*b*) It follows from (*a*) that

$$\operatorname{lcm}(x^{a_{i1}}, \dots, x^{a_{ib_i}}) \mid \operatorname{lcm}(x^{a_{11}}, \dots, x^{a_{1b_1}}).$$

Thus it remains to show that for all $i = 1, \dots, p-1$ each $x^{a_{ij}}$ divides $x^{a_{i+1l}}$ for some l. Suppose this is not the case. Then the jth row of the matrix of ϕ_{i+1} must be zero. Therefore in the R-dual of F there appears a matrix with a zero column, contradicting the fact that the R-dual of F is, by the perfection of M, a minimal free resolution of the R-dual of Coker ϕ_p. ∎

Since each entry of ϕ_i in the multigraded resolution of M divides one of the $x^{a_{ij}}$, our Theorem 3·1 has the following immediate consequence:

COROLLARY 3·2. *If λx^a, $\lambda \in K$, is the entry of some ϕ_i in the multigraded resolution of M, then*

$$x^a \mid \operatorname{lcm}(x^{a_{11}}, \dots, x^{a_{1b_1}}).$$

In particular, the entries in the multigraded resolution of a monomial ideal I divide the least common multiple of the generators of I.

One might expect an even stronger result, namely, that any monomial entry in the multigraded resolution of a monomial ideal I divides some generator of I. However, in general one cannot even avoid an entry whose degree exceeds that of every generator. We choose $I = (abc, def, cgh, fgh)$ and let R be a polynomial ring in the variables $a, \dots, h$. Then R/I has a free resolution

$$0 \to R \xrightarrow{\phi_3} R^4 \xrightarrow{\phi_2} R^4 \xrightarrow{\phi_1} R \to R/I \to 0,$$

where ϕ_3 is given by the matrix $(abde, -abc, def, gh)$. Since the entries of ϕ_3 form a minimal monomial system of generators of $\operatorname{Ext}_R^2(R/I, R)$, they occur in every minimal multigraded resolution (up to constant factors).

For a perfect multigraded module we have a result dual to 3.1(b):

COROLLARY 3·3 *With the hypothesis of* 3·1(*b*) *one has*

$$\gcd(x^{a_{i1}}, \dots, x^{a_{ib_i}}) = \gcd(x^{a_{01}}, \dots, x^{a_{0b_0}})$$

for all $i \leqslant p-1$. In particular, if $M = R/I$ where I is a perfect monomial ideal, then gcd $(x^{a_{i1}}, \dots, x^{a_{ib_i}}) = 1$ *for $i \leqslant p-1$.*

Proof. Let $x^{a_p} = \operatorname{lcm}(x^{a_{p1}}, \dots, x^{a_{pb_p}})$; then, after a shift by a_p, the cokernel C of the R-dual of ϕ_p has the following multigraded resolution

$$0 \to \bigoplus_{j=1}^{b_0} R(-(a_p - a_{0j})) \to \dots \to \bigoplus_{j=1}^{b_p} R(-(a_p - a_{pj})) \to C(-a_p) \to 0.$$

By 3·1(*b*) *we get*

$$\frac{x^{a_p}}{\gcd(x^{a_{i1}}, \dots, x^{a_{ib_i}})} = \operatorname{lcm}\left(\frac{x^{a_p}}{x^{a_{i1}}}, \dots, \frac{x^{a_p}}{x^{a_{ib_i}}}\right) = \operatorname{lcm}\left(\frac{x^{a_p}}{x^{a_{01}}}, \dots, \frac{x^{a_p}}{x^{a_{0b_0}}}\right)$$

$$= \frac{x^{a_p}}{\gcd(x^{a_{01}}, \dots, x^{a_{0b_0}})}$$

for all $i \leqslant p-1$. This implies the assertion. ∎

Theorem 3·1 has some consequences for the resolution of a multigraded module M defined over the ring $R = S/I$ where $S = K[x_1, \dots, x_n]$ is a polynomial ring, and $I \subset S$ is an ideal generated by monomials. We denote by a_{ij} the ith shifts of M as an R-module, by $a_1, \dots, a_r$ its first shifts as an S-module, and for $u \in \mathbb{R}$ we set $\lfloor u \rfloor$ to denote the largest integer $\leqslant u$.

THEOREM 3·4. *Let $x^{b_1}, \dots, x^{b_m}$ be a minimal set of generators of I, and let $x^b = \operatorname{lcm}(x^{b_1}, \dots, x^{b_m})$ and $x^a = \operatorname{lcm}(x^{a_1}, \dots, x^{a_r})$. Then*

$$x^{a_{ij}} \mid x^a (x^b)^{\lfloor i/2 \rfloor}$$

for all $i \geqslant 1$ and all j.

Proof. For the multigraded R-module M, the K-vector spaces $\operatorname{Tor}_i^R(M,K)$ are multigraded with homogeneous generators of degrees a_{ij}, $j = 1, \dots$. One defines the multigraded Poincaré series in the variables $s_1, \dots, s_n$ and t by

$$P_M^R(s,t) = \sum_{i \geqslant 0} \left(\sum_{a \in \mathbb{Z}^n} \dim_K \operatorname{Tor}_i^R(M,K)_a \, s^a \right) t^i.$$

Note that each coefficient $\sum_{a \in \mathbb{Z}^n} \dim_K \operatorname{Tor}_i^R(M,K)_a \, s^a$ is an element of $\mathbb{Z}[s_1, \dots, s_n]$.

The following observation is crucial for the proof: there are natural isomorphisms of multigraded vector spaces

$$\operatorname{Ext}_R^i(M,K) \cong \operatorname{Tor}_i^R(M,K),$$

and a standard change of rings spectral sequence (see for example Cartan and Eilenberg [7])

$$\operatorname{Ext}_R^p(M, \operatorname{Ext}_S^q(R,K)) \Rightarrow \operatorname{Ext}_S^{p+q}(M,K)$$

respecting the internal gradings of the Ext-groups. This provides the coefficientwise inequality

$$P_M^R \leqslant P_M^S (1 + t - t P_R^S)^{-1} \tag{1}$$

of formal power series. Write $P_R^S = \sum_{i \geqslant 0} r_i(s) t^i$ with $r_i(s) \in \mathbb{Z}[s_1, \dots, s_n]$, $r_0(s) = 1$. Then

$$1 + t - t P_R^S = 1 - \sum_{i \geqslant 2} \tilde{q}_i(s) t^i$$

with $\tilde{q}_i(s) = r_{i-1}(s)$ for all i, and we get

$$(1 + t - t P_R^S)^{-1} = 1 + \sum_{i \geqslant 2} q_i(s) t^i$$

with $q_i(s) = \sum_{k \geqslant 1} \sum_{i_1 + \dots + i_k = i} \tilde{q}_{i_1}(s) \cdots \tilde{q}_{i_k}(s)$. Hence if we set $P_M^S = \sum_{i \geqslant 0} p_i(s) t^i$, then

$$P_M^S (1 + t - t P_R^S)^{-1} = p_0(s) + p_1(s) t + \sum_{i \geqslant 2} \left(\sum_{j=2}^{i} p_{i-j}(s) q_j(s) + p_i(s) \right) t^i. \tag{2}$$

For a polynomial $h(x) \in \mathbb{Z}[x]$ ($= \mathbb{Z}[x_1, \dots, x_n]$) we set $h(x^{-1}) = h(x_1^{-1}, \dots, x_n^{-1})$. Then Theorem 3·1 implies that $x^a p_i(x^{-1})$ and $x^b \tilde{q}_i(x^{-1})$ belong to $\mathbb{Z}[x]$ for all $i \geqslant 1$. We may as well assume that $x^a p_0(x^{-1}) \in \mathbb{Z}[x]$. Because if M has no free S-summand, as we may assume, and if c is a 0-th shift of M, then x^c divides some x^{a_i}. In other words, all monomials in $p_0(x)$ divide x^a.

For a given i, let k be the maximum of the numbers such that

$$i_1 + \dots + i_k \leqslant i, \quad \text{with all} \quad i_l \geqslant 2.$$

Then it is clear that $k = \lfloor i/2 \rfloor$. Hence by the definition of the $q_i(z)$ we have $(x^b)^{\lfloor i/2 \rfloor} q_i(x^{-1}) \in \mathbb{Z}[x]$ for all i. Therefore, if $P_M^S(1 + t - t P_R^S)^{-1} = \sum_{i \geqslant 0} c_i(s) t^i$, formula (2) shows that

$$x^a (x^b)^{\lfloor i/2 \rfloor} c_i(x^{-1}) \in \mathbb{Z}[x] \quad \text{for all} \quad i,$$

which is equivalent to saying that all monomials of $c_i(x)$ divide $x^a(x^b)^{\lfloor i/2\rfloor}$. By the inequality of power series (1) the monomials $x^{a_{ij}}$ form a subset of the monomials in $c_i(x)$. This implies our assertion. ∎

For a $\mathbb{Z}$-graded module N, let p_N denote the maximal degree of an element in a minimal set of homogeneous generators of N. Since any multigraded module is naturally $\mathbb{Z}$-graded, we may define $t_i(M) = p_{F_i}$ where F is a minimal multigraded resolution of M. Let b and a be defined as in Theorem 3·4, and set $\alpha = \deg_{\mathbb{Z}} x^a$ and $\beta = \deg_{\mathbb{Z}} x^b$. Then

COROLLARY 3·5. *$t_i(M) \leqslant \beta\lfloor i/2\rfloor + \alpha$ for all $i \geqslant 1$.*

4. *Monomial Gorenstein ideals*

In this section we discuss a constraint on the shifts of a multigraded resolution of R/I, where $R = K[x_1, \dots, x_n]$ is the polynomial ring, and I is a monomial ideal. The Koszul homology $H(\mathbf{x}; R/I)$ ($\mathbf{x} = x_1, \dots, x_n$) has a multigraded structure as a K-algebra, and the natural homomorphism

$$\mathrm{Tor}^R(K, R/I) \cong H(x; R/I)$$

is an isomorphism of graded K-vector spaces. Hence if R/I has the shifts a_{ij}, then $H_i(\mathbf{x}; R/I)$ has a homogeneous K-basis whose elements are of degrees a_{ij}.

Let c, $d \in H(\mathbf{x}; R/I)$ be homogeneous elements. We call (c, d) a *non-trivial pair* if $cd \neq 0$.

PROPOSITION 4·1. *With the notation introduced, suppose (c,d) is a non-trivial pair with* $\deg c = a$ *and* $\deg d = b$. *Then* $\gcd(x^a, x^b) = 1$.

Proof. Assume $\gcd(x^a, x^b) \neq 1$; then we have $h = \min\{a(t), b(t)\} \neq 0$ for some t. We lift the module R/I to S/J with squarefree shifts as described in Theorem 2·1 and Corollary 2·2. Say R/I has the shifts a_{ij}. Then we have $a = a_{kl}$ and $b = a_{rs}$ for some numbers k, l, r and s.

Next we observe that the natural epimorphism $S/J \to R/I$ induces an isomorphism of K-algebras

$$H(\mathbf{y}; S/J) \cong H(\mathbf{x}; R/I)$$

compatible with the multigradings. Hence if we denote by u and v the images of c and d in $H(\mathbf{y}; S/J)$, then $\deg u = b_{kl}$ and $\deg v = b_{rs}$ where the b_{ij} arise from the a_{ij} as described in Corollary 2·2. Moreover since (u, v) is again a non-trivial pair of $H(\mathbf{y}; S/J)$ it follows that $b_{kl} + b_{rs} = \deg uv$ is a shift of S/J, and hence is squarefree.

On the other hand, $y^{b_{kl}}$ and $y^{b_{rs}}$ contain the non-trivial common factor $\prod_{j=1}^{h} y_{tj}$, a contradiction. ∎

Now let us consider the particular case when R/I is Gorenstein. Then, by a theorem of Avramov and Golod [2] (see also [4]), $H(\mathbf{x}; R/I)$ is a Poincaré algebra. In particular, if $H_p(\mathbf{x}; R/I)$ is the last non-vanishing homology of the Koszul complex, then $H_p(\mathbf{x}; R/I)$ is a 1-dimensional K-vector space, generated by a homogeneous element, say e, and there exist homogeneous bases $c_1, \dots, c_m$ of $H_1(\mathbf{x}; R/I)$ and $d_1, \dots, d_m$ of $H_{p-1}(\mathbf{x}; R/I)$ such that $c_i d_i = e$ for $i = 1, \dots, m$. Therefore the (c_i, d_i) are non-trivial pairs; hence we have $\gcd(x^{\deg c_i}, x^{\deg d_i}) = 1$ by Proposition 4·1, and of course

$x^{\deg c_i} x^{\deg d_i} = x^{\deg e}$. Since the degrees of the c_i are the degrees of the generators of I, and since $x^{\deg e}$ is the least common multiple of the generators of I (see Section 3·1 (*b*)), we can summarize these observations as follows:

THEOREM 4·2. *Let I be a monomial Gorenstein ideal in the polynomial ring $R = K[x_1, \dots, x_n]$ which is minimally generated by the monomials $x^{a_1}, \dots, x^{a_m}$, and let $\prod_{k=1}^n x_k^{c(k)} = \operatorname{lcm}(x^{a_1}, \dots, x^{a_m})$. Then for $i = 1, \dots, m$ there exists a subset J_i of $\{1, \dots, n\}$ such that*

$$x^{a_i} = \prod_{k \in J_i} x_k^{c(k)}.$$

In particular, for all k the variable x_k occurs with the same multiplicity in all x^{a_i} that it divides.

The theorem provides a simple method to show that certain monomial ideals cannot be Gorenstein. For example, the ideal generated by x^2yz, wyz, uvw, zuw, xwv is not Gorenstein, since x has multiplicity 2 in the first generator but multiplicity 1 in the last generator.

There are two immediate consequences of Theorem 4·2. First we have:

COROLLARY 4·3. *Any monomial Gorenstein ideal $I \subset K[x_1, \dots, x_n]$ is obtained from a squarefree monomial Gorenstein ideal $J \subset K[x_1, \dots, x_n]$ by a substitution $x_k \mapsto x_k^{c(k)}$, $k = 1, \dots, n$, for some positive integers $c(k)$.*

The other consequence is the following well-known fact (Stückrad [**15**]):

COROLLARY 4·4. *Let I be a monomial Gorenstein ideal of the polynomial ring $R = K[x_1, \dots, x_n]$ such that $\dim R/I \leqslant 1$. Then R/I is a complete intersection.*

Proof. If $\dim R/I = 0$, the ideal I must contain pure monomials $x_1^{a_1}, \dots, x_n^{a_n}$ among its minimal generators. By Theorem 4·2, if x_k divides a generator of I, then $x_k^{a_k}$ divides it too. Therefore, $I = (x_1^{a_1}, \dots, x_n^{a_n})$.

Now suppose that $\dim R/I = 1$, and let J be the ideal determined by Corollary 4·3. The residue class ring R/J is the Stanley-Reisner ring of a simplicial complex of dimension 0. Thus the h-vector (see [4]) is given by (1) or (1, 1). Hence $J = (x_1, \dots, x_{n-1})$ or $J = (x_1, \dots, x_{n-2}, x_{n-1}x_n)$, up to a permutation of the x_i.

5. *The second syzygy module of a monomial ideal*

In this section we describe the second syzygy module of a monomial ideal I in $R = \mathbb{Z}[x_1, \dots, x_n]$. As a consequence it will turn out that the second Betti number of a monomial ideal is independent of the field of coefficients.

Let I be generated by the monomials $u_1, \dots, u_m$ and consider the start

$$\bigwedge^2 R^m \xrightarrow{\phi_2} R^m \xrightarrow{\phi_1} R \to R/I \to 0$$

of the Taylor resolution (see [**16**] or Eisenbud [**8**]). We denote by $e_1, \dots, e_m$ the canonical basis of R^m. Then $\phi_1(e_i) = u_i$ for all i and

$$\phi_2(e_i \wedge e_j) = \frac{\operatorname{lcm}(u_i, u_j)}{u_i} e_i - \frac{\operatorname{lcm}(u_i, u_j)}{u_j} e_j \quad \text{for all} \quad i < j.$$

The Taylor resolution is multigraded with $\deg e_i = \deg u_i$, $\deg e_i \wedge e_j = \deg \operatorname{lcm}(u_i, u_j)$.

We choose $i_1, \dots, i_l \in \{1, \dots, m\}$, and set

$$s(i_1, \dots, i_l) = \sum_{k=1}^{l} \frac{\operatorname{lcm}(u_{i_1}, \dots, u_{i_l})}{\operatorname{lcm}(u_{i_k}, u_{i_{k+1}})} e_{i_k} \wedge e_{i_{k+1}} \quad (i_{l+1} = i_1).$$

Then $\phi_2(s(i_1, \dots, i_l)) = 0$. We call the elements $s(i_1, \dots, i_l)$ *cyclic syzygies.*

PROPOSITION 5·1. *Let U be a subset of $\{(i,j): 1 \leqslant i < j \leqslant m\}$ and set $F = \sum_{(i,j) \in U} Re_i \wedge e_j$. Then* $\operatorname{Ker}(\phi_2 | F)$ *is generated by cyclic syzygies.*

Proof. We use induction on the number of elements in U. If U is empty there is nothing to show. So let us assume that $|U| > 0$, and that the assertion is proved for all U with fewer elements. Let $x \in F \cap \operatorname{Ker} \phi_2$ be a homogeneous element (in the multigraded sense),

$$x = \sum_{(i,j) \in U'} a_{ij} e_i \wedge e_j, \quad a_{ij} \neq 0.$$

We may assume that $U' = U$, because otherwise we may apply the induction hypothesis.

Consider the (non-directed) graph Γ with edges $(i,j) \in U$. If Γ contains no cycles, then the elements $\phi_2(e_i \wedge e_j)$, $(i,j) \in U$, are linearly independent over R. Otherwise there is a cycle in Γ, say $(i_1, i_2), (i_2, i_3), \dots, (i_{l-1}, i_l), (i_1, i_l)$.

Write

$$x = \sum_{k=1}^{l} \alpha_k v_k e_{i_k} \wedge e_{i_{k+1}} + r \, (i_{l+1} = i_1),$$

where $\alpha_k \in \mathbb{Z}$ and v_k is a monomial for all k, and where r is a linear combination in the remaining basis elements $e_i \wedge e_j$. Since x is homogeneous we must have

$$v_1 \operatorname{lcm}(u_{i_1}, u_{i_2}) = \dots = v_l \operatorname{lcm}(u_{i_l}, u_{i_1}).$$

Set $v = v_1 \operatorname{lcm}(u_{i_1}, u_{i_2})$. Then there exists a monomial $\tilde{v}$ such that $v = v_k \operatorname{lcm}(u_{i_k}, u_{i_{k+1}}) = \tilde{v} \operatorname{lcm}(u_{i_1}, \dots, u_{i_l})$ for all k. Hence we see that $v_k = \tilde{v} \operatorname{lcm}(u_{i_1}, \dots, u_{i_l}) / \operatorname{lcm}(u_{i_k}, u_{i_{k+1}})$ for all k, and it follows that the new cycle

$$x' = x - \alpha_1 \tilde{v} s(i_1, \dots, i_l)$$

is a linear combination of fewer basis elements than x. Therefore, and by our induction hypothesis, the proof is complete. ∎

COROLLARY 5·2. *There exists a multigraded exact sequence*

$$F \xrightarrow{\psi_2} R^m \xrightarrow{\phi_1} R \to R/I \to 0$$

with a free R-module F such that $\operatorname{Ker} \psi_2 \subset (x) F$.

Proof. We choose a subset $U \subset \{(i,j): 1 \leqslant i < j \leqslant m\}$ such that the elements $\phi_2(e_i \wedge e_j)$ generate $\operatorname{Ker} \phi_1$ minimally, that is, none of them can be left out. Set $F = \sum_{(i,j) \in U} Re_i \wedge e_j$, and $\psi_2 = \phi_2 | F$. Suppose $\operatorname{Ker} \psi_2 \not\subset (x) F$; then there exists a cyclic syzygy $s(i_1, \dots, i_l) \supseteq F \setminus (x) F$. Such a cyclic syzygy contains a coefficient ± 1, a contradiction to the minimality of U.

COROLLARY 5·3. *Let K be a field, and J a monomial ideal in $S = K[x_1, \dots, x_n]$. Then the multigraded Hilbert series of* $\mathrm{Tor}_2^S(S/J, K)$ *is independent of K.*

Proof. We let I be the ideal in $\mathbb{Z}[x_1, \dots, x_n]$ generated by the same monomials as J, and tensor the complex of Corollary 5·2 with K (over $\mathbb{Z}$). This yields the start of a minimal multigraded resolution of S/J. ∎

The next corollary is an immediate consequence of Corollary 5·3 and Hochster's formula [11, (5·1)].

COROLLARY 5·4. *Let Δ be a simplicial complex, and K a field. Then the K-dimension of the reduced simplicial homology $\tilde{H}_{|\Delta|-3}(\Delta, K)$ is independent of K.*

6. *Monomial Gorenstein ideals of height* 3

Let K a field, $R = K[x_1, \dots, x_n]$ the polynomial ring in n indeterminates over K where $n \geqslant 3$ is odd. We set $m = (n-1)/2$, and consider the ideal I generated by the monomials

$$u_i = x_i x_{i+1} \cdots x_{i+m-1}, \quad i = 1, \dots, n,$$

where $x_j = x_{j-n}$ for $j > n$.

In this section we prove the following classification of monomial Gorenstein ideals of height 3 which refines the structure theorem of Buchsbaum and Eisenbud [5].

THEOREM 6·1. (*a*) *I is a Gorenstein ideal of height* 3.

(*b*) *Let $J \subset S = K[y_1, \dots, y_r]$ be a Gorenstein ideal of height 3 which is minimally generated by the monomials $v_1, \dots, v_n$. Then there exist pairwise coprime monomials $p_1, \dots, p_n \in S$ such that if $\phi: R \to S$ is the K-algebra homomorphism with $\phi(x_i) = p_i$ for $i = 1, \dots, n$, then $v_j = \phi(u_j)$ for $j = 1, \dots, n$, after a suitable renumbering of the monomials v_j. In other words, up to a substitution of the regular sequence $p_1, \dots, p_n$ for $x_1, \dots, x_n$, the ideal J equals I.*

Proof. (*a*) *The n* rows of the following matrix are obvious Taylor relations for the monomials $u_1, \dots, u_n$:

$$\begin{pmatrix} -x_{m+1} & x_1 & 0 & \cdots & 0 \\ 0 & -x_{m+2} & x_2 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -x_{m-1} & x_{n-1} \\ x_n & 0 & \cdots & 0 & -x_m \end{pmatrix}.$$

A cyclic permutation of the rows moving the first row into position $m+2$ yields a skew-symmetric matrix whose $(n-1)$-order pfaffians are the generators of I, up to sign. For example, for $n = 5$ this new matrix is

$$\begin{pmatrix} 0 & 0 & -x_5 & x_3 & 0 \\ 0 & 0 & 0 & -x_1 & x_4 \\ x_5 & 0 & 0 & 0 & -x_2 \\ -x_3 & x_1 & 0 & 0 & 0 \\ 0 & -x_4 & x_2 & 0 & 0 \end{pmatrix}.$$

Note that the ideal I is invariant under the action of the cyclic group whose generator sends x_i to x_{i+1} ($x_{n+1} = x_1$). Since each monomial u_i is 'shorter' than half the length of the cycle, it is impossible to choose indices j_1 and j_2 such that each u_i is divisible by x_{j_1} or x_{j_2}. On the other hand one can easily find j_1, j_2, j_3 for which $I \subset (x_{j_1}, x_{j_2}, x_{j_3})$. Therefore height $I = 3$, and the Buchsbaum–Eisenbud structure theorem [5] yields that R is Gorenstein. Moreover

$$0 \to R \xrightarrow{\phi_3} R^n \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \to R/I \to 0$$

is a minimal multigraded free resolution with $\phi_1(e_i) = u_i$ for $i = 1, \dots, n$, ϕ_2 given by the matrix above, and with ϕ_3 given by $(u_{m+2}, \dots, u_n, u_1, \dots, u_{m+1})$.

(*b*) Since, by assumption, J is a Gorenstein ideal of height 3, S/J has the multigraded resolution

$$0 \to S \xrightarrow{\psi_3} S^n \xrightarrow{\psi_2} S^n \xrightarrow{\psi_1} S \to S/J \to 0,$$

where $\operatorname{Im} \psi_2$ is generated by Taylor relations. According to Proposition 5·1 these relations form a unique cycle, and hence we may assume that ψ_3 is described by the following matrix

$$\begin{pmatrix} -v_{12} & v_{21} & 0 & \cdots & 0 \\ 0 & -v_{23} & v_{32} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -v_{n-1n} & v_{nn-1} \\ v_{1n} & 0 & \cdots & 0 & -v_{n1} \end{pmatrix},$$

where $v_{ij} = \operatorname{lcm}(v_i, v_j)/v_j$ for $1 \leqslant i < j \leqslant n$.

By Corollary 4·3 we may further assume that the generators v_i of J are squarefree. Now for each variable y_i we define the 'indicator' σ_i of y_i by

$$\sigma_i = \{j : y_i \mid v_j\},$$

and claim that the set σ_i is a segment of length $m < n$ in $\{1, \dots, r\}$ for each i, that is, a subset of $\{1, \dots, r\}$ of the form $\{a, a+1, \dots, a+m-1\}$ or $\{a, a+1, \dots, r, 1, 2, \dots, a+m-r-1\}$ if $a+m-1 > r$. The segment represented by this set is said to end with $a+m-1$ in the first case, and with $a+m-r-1$ in the second case.

This claim implies our assertion. Indeed, let p_j be the product of the y_i for which σ_i ends with j. Then the p_j are pairwise disjoint, and $v_i = \phi(u_i)$ for $i = 1, \dots, n$ as desired.

To prove the claim, let $(q_1, \dots, q_n)$ be the matrix representing ψ_3. By the duality of the resolution the q_i are, up to signs, just the v_j (in some different order).

Let Δ be the determinant of the matrix obtained from ψ_2 by deleting the first column and last row. According to the Buchsbaum–Eisenbud factorization theorem [6] we have

$$q_n v_1 = \Delta = \frac{\operatorname{lcm}(v_1, v_2)}{v_2} \cdots \frac{\operatorname{lcm}(v_{n-1}, v_n)}{v_n}.$$

A cyclic permutation of the indices gives further $n-1$ relations of this kind, which, when multiplied with each other, yield

$$q_1 \cdots q_n v_1^n \cdots v_n^n = \operatorname{lcm}(v_1, v_2)^{n-1} \cdots \operatorname{lcm}(v_{n-1}, v_n)^{n-1} \operatorname{lcm}(v_n, v_1)^{n-1}.$$

Observe that $q_1 \cdots q_n = \pm v_1 \cdots v_n$, so that

$$v_1^{n+1} \cdots v_n^{n+1} = \pm \operatorname{lcm}(v_1, v_2)^{n-1} \cdots \operatorname{lcm}(v_{n-1}, v_n)^{n-1} \operatorname{lcm}(v_n, v_1)^{n-1}.$$

On the left-hand side of this equation y_i appears with multiplicity $(n+1)|\sigma_i|$, and on the right-hand side with multiplicity $(n-1)(|\sigma_i|+\epsilon_i)$, where ϵ_i is the number of segments constituting σ_i. Thus we see that $2|\sigma_i| = (n-1)\epsilon_i$. Note that $|\sigma_i| \leqslant n-2$ since otherwise we would have height $J \leqslant 2$. But then the last equation implies that $\epsilon_i = 1$ and $|\sigma_i| = (n-1)/2$. ∎

The ideal $I = (u_1, \dots, u_n)$ defined above is well known in combinatorics: it is the defining ideal of the Stanley–Reisner ring of the boundary complex of the cyclic polytope $C(n, n-3)$; see for example [**4**]. If the ideal J of Theorem 6·1 is generated by square-free monomials, then the substitution that leads from I to J can be decomposed into a series of substitutions each of which replaces an indeterminate by the product of two new indeterminates. Such a substitution transforms the boundary complex of a convex polytope $P \subset \mathbb{R}^q$ with vertices $v_1, \dots, v_p$ into the boundary complex of the convex polytope $P' \subset \mathbb{R}^{q+1}$ whose vertices $v'_0, \dots, v'_p$ are given by

$$v'_0 = (v_1, -1), \quad v'_1 = (v_1, 1), \quad v'_i = (v_i, 0), \quad i = 2, \dots, p$$

if the indeterminate corresponding to v_1 is sent to a product of two new ones. Let us call this transformation a *1-vertex inflation*.

COROLLARY 6·2. *Let Δ be a simplicial complex with r vertices. If Δ is a homology sphere of dimension $r-4$, then it is the boundary complex of convex polytope that arises from a cyclic polytope $C(n, n-3)$ (n odd) by a series of 1-vertex inflations.*

In fact, Δ is a homology sphere if and only if $\Delta = \operatorname{core} \Delta$ and the Stanley–Reisner ring $K[\Delta]$ is Gorenstein; see Stanley [**14**, p. 74] or [**4**, 5.5.1]. Furthermore $\dim K[\Delta] = \dim \Delta + 1$ so that the defining ideal of $K[\Delta]$ has height 3.

Corollary 6·2 refines a result of P. Mani [**13**]. P. Kleinschmidt [**12**] proved a similar theorem for non-simplicial spheres. We are grateful to B. Sturmfels for these references.

Remark 6·3. It is not difficult to give a classification of perfect monomial ideals I of grade 2 that is similar to Theorem 6·1. For such an ideal a minimal multigraded free resolution of R/I has the form

$$0 \to R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \to R/I \to 0,$$

where the relations of the generators of I given by ϕ_2 are Taylor relations and R^{n-1} has a basis of the form $e_i \wedge e_j$, $(i,j) \in U$. Let Γ be the graph associated with U as described in the proof of Proposition 5·1; Γ is a tree with n vertices. We choose $2(n-1)$ new variables, and place them into an $(n-1) \times n$ matrix ψ_2 such that the non-zero entries of ψ_2 and ϕ_2 are in the same positions. As the reader may check, the $(n-1)$-minors of ψ_2 are indeed monomials (up to sign). Furthermore I arises from the ideal $I_{n-1}(\psi_2)$ by the substitution that sends each entry of ψ_2 to the corresponding entry of ϕ_2.

Therefore the 'generic' types of perfect monomial ideals $I = (u_1, \dots, u_n)$ of grade 2 are in bijective correspondence to the trees with n vertices. However, note that in general Γ is not uniquely determined by I, and that the entries of ϕ_2 need not form a regular sequence.

Acknowledgement. In an earlier version we stated Theorem 3.4 in a stronger form. There we defined a by the equation $x^a = \text{lcm}(x^{a_{11}}, \ldots, x^{a_{1\beta_1}})$. We are grateful to Srikanth Iyengar who informed us that our proof of Theorem 3·4 as it was originally stated was incomplete. In the meantime Iyenkar succeeded in proving the stronger version of the theorem by different methods.

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