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# Generic graph construction ideals and Greene's theorem

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## 1. Introduction

Let $X$ be an $m \times n$ matrix of indeterminates, $m \leq n$, and $T$ a new indeterminate. Consider the polynomial rings $R_0 = K[X]$ and $R = R_0[T]$. For a given positive integer $t \leq m$, consider the ideal $I_t = I_t(X)$ generated by the $t$-minors (i. e. the determinants of the $t \times t$ submatrices) of $X$. Using all these determinantal ideals, we define a new ideal $J$ in $R = R_0[T]$, which we call the *generic graph construction ideal*, as follows:

$$J = (T^m) + I_1 T^{m-1} + \cdots + I_m \,.$$

The object of this paper is to study the Rees algebra $\mathcal{R}(J)$ and in particular to completely describe the primary decompositions of the ideals $J\mathcal{R}(J)$ and $T\mathcal{R}(J)$.

We approach the structure of $\mathcal{R}(J)$ via its initial algebra $\mathrm{in}(\mathcal{R}(J))$ with respect to a suitable term order. The crucial combinatorial argument is Greene's theorem [11] on the invariants of the Knuth–Robinson–Schensted correspondence. It will allow us to find the initial ideals of those ideals of $K[X]$ that occur naturally in the 'expansion' of the powers of $J$.

This problem has its origin in the study of the Grassmannian Graph Construction, introduced by MacPherson in [18]. The Graph Construction is a very important tool in intersection theory. In [18] it was used in the proof of the Deligne–Grothendieck conjecture on the existence of Chern classes of singular algebraic varieties. It also served as the basis of the proof of Baum, Fulton and MacPherson's Singular Riemann–Roch theorem [1] and more recently of the Arithmetic Riemann–Roch theorem by Gillet and Soulé in [10]. For the details of this construction we refer to the above mentioned papers or to [9] or [16]. Below, we just briefly explain the link with the present paper.

The Graph Construction starts with a vector bundle map $d$ above some base scheme $M$ and deforms $M$ to a scheme called $Z_\infty$ (with a canonical projection to $M$) in a way which reflects the degeneracies of $d$. Now, a vector bundle map can be locally represented as a matrix of regular functions. Then, the degeneracy loci are precisely the different zero-schemes of the determinantal ideals of that matrix. It turns out, that the Graph Construction deformation space is isomorphic to the blowup of the ideal defined similarly to $J$ (the only difference being that the entries of the matrix are arbitrary regular functions and not variables). This was first noticed by Roberts in [19] and independently by the second author in [15] (see [16] for an exposition). In particular $Z_\infty$ is isomorphic to the zero-subscheme of $\operatorname{Proj}\mathcal{R}(J)$ given by the ideal $T\mathcal{R}(J)$. This isomorphism commutes with the canonical projections on $M$. The irreducible components of $Z_\infty$ and their multiplicities are of great importance in characteristic classes computations.

It is well known, that for an arbitrary vector bundle map $d$, the $Z_\infty$ scheme has one distinguished irreducible component, which dominates $M$. It is isomorphic to the blowup of the smallest nonvanishing Fitting ideal of the cokernel sheaf of $d$. However, there is no general description of the other components. The only thing being known is that the projection onto $M$ maps them into the singular locus of $d$.

The results of this paper apply to the special case of a vector bundle map given by a matrix of variables. This case is also of geometric interest: it provides a local model for so called generic vector bundle maps [17], where genericity is defined by a natural transversality condition. Thus, we are able to precisely describe the components of $Z_\infty$ in the case when $d$ is a generic map. They are given by the primary decomposition of the ideal $T\mathcal{R}(J)$ in part (d) of Theorem 5. This theorem tells us in particular that $Z_\infty$ is reduced. Apart from the distinguished component, its irreducible components correspond to the different degeneracy loci of $d$, which they dominate by the canonical projection.

A claim similar to the last statement appears without proof in ([9], Example 18.1.6.e). To the best of our knowledge no published proof is available.

## 2. The straightening law

By $\Delta$ we always denote a product $\delta_1 \cdots \delta_w$ of minors, and we assume that the sizes $|\delta_i|$ are descending, $|\delta_1| \geq \cdots \geq |\delta_w|$. We reserve the letter $\Sigma$ for standard monomials of minors, $\Sigma = \delta_1 \cdots \delta_w$ with $\delta_1 \preceq \cdots \preceq \delta_w$. The partial order $\preceq$ for minors which we have just used is defined as follows. We write $[a_1, \dots, a_t \mid b_1, \dots, b_t]$ for the determinant of the submatrix $(X_{a_i b_j} : i = 1, \dots, t,\ j = 1, \dots, t)$. Then

$$[a_1, \dots, a_t \mid b_1, \dots, b_t] \preceq [c_1, \dots, c_u \mid d_1, \dots, d_u] \\ \iff \quad t \geq u \quad \text{and} \quad a_i \leq c_i,\ b_i \leq d_i,\ i = 1, \dots, u.$$

The fundamental straightening law of Doubilet–Rota–Stein says that every element of $R_0$ has a unique presentation as a $K$-linear combination of standard monomials (for example, see Bruns and Vetter [6]).

We now introduce an invariant for products $\Delta = \delta_1 \cdots \delta_w$ of minors (standard or not) that will play a crucial role:

$$\alpha_k(\Delta) = |\delta_1| + |\delta_2| + \cdots + |\delta_k|$$

where $\delta_i = 1$, $|\delta_i| = 0$ if $i > w$. In other words: $\alpha_k(\Delta)$ is the total degree of the product of the first $k$ factors of $\Delta$ (by our convention they have the largest sizes).

The following lemma is an essential observation:

**Lemma 1.** *Let $\Delta = \sum a_\Sigma \Sigma$, $a_\Sigma \in K$, $a_\Sigma \neq 0$, be the representation of $\Delta$ as a linear combination of standard monomials. Then*

*(a)* $\alpha_k(\Delta) \leq \alpha_k(\Sigma)$ *for all $k$ and $\Sigma$;*
*(b)* $\alpha_k(\Delta) = \alpha_k(\Sigma)$ *for at least one $\Sigma$.*

*Proof.* (a) This is an elementary consequence of the straightening procedure by which one repeatedly replaces a non-standard product $\delta_1\delta_2$ by its standard representation $\delta_1\delta_2 = \sum a_i \varepsilon_i \gamma_i$, and in which $|\varepsilon_i| \leq |\delta_1| \leq |\delta_2| \leq |\gamma_i|$ and $|\varepsilon_i| + |\gamma_i| = |\delta_1| + |\delta_2|$. For example see [6], 11.4.

(b) This follows from the fact that at least one standard monomial in the presentation of $\Delta$ must have the same 'shape' as $\Delta$. Again see [6], 11.4.

The following ideals of $R_0$ appear as factors of the powers of $T$ if one expands the powers $J^k$:

$$J(u, d) = \sum I_0^{e_0} I_1^{e_1} \cdots I_m^{e_m}, \\ e_0 + e_1 + \cdots + e_m = u,\ e_1 + 2e_2 + \cdots + me_m = d$$

(where $I_0 = R_0$). Then we have

$$J^k = R\biggl(\sum_{d=0}^{km} J(k, d) T^{km-d}\biggr).$$

The ideals $J(u, d)$ can also be described in terms of the functions $\alpha_k$:

**Lemma 2.** *(a)* $\Delta \in J(u, d) \iff \alpha_u(\Delta) \geq d$.
*(b)* $J(u, d)$ *has a standard monomial $K$-basis of all $\Sigma$ with $\alpha_u(\Sigma) \geq d$.*

*Proof.* We prove the lemma in four steps.

(i) If $\alpha_u(\Delta) \geq d$, then $\Delta$ clearly belongs to $J(u,d)$.

(ii) Every element of $J(u,d)$ is a $K$-linear combination of products $\Delta$ with $\alpha_u(\Delta) \geq d$ (choose the natural generators of $J(u,d)$ times products of the indeterminates). Now apply the straightening procedure to all these products. Then Lemma 1(a) and (i) imply that all the standard monomials needed lie in $J(u,d)$. Thus $J(u,d)$ has a standard monomial basis.

(iii) Choose a standard monomial $\Sigma \in J(u,d)$, write it as a $K$-linear combination of products $\Delta$ with $\alpha_u(\Delta) \geq d$ as in (ii), and apply the straightening procedure to the $\Delta$, which of course just reproduces $\Sigma$. It follows that $\alpha_u(\Sigma) \geq d$.

(iv) The remaining implication $\Rightarrow$ of (a) follows now from Lemma 1(b).

## 3. Gröbner bases and initial ideals

It is our goal to determine Gröbner bases and the initial ideals of the ideals $J(u,d)$ and $J^k$ with respect to a diagonal term order $\tau$ on the polynomial ring $R_0$. (We refer the reader to Eisenbud [8] for Gröbner bases and term orders.) A term order $\tau$ is called diagonal if the initial monomial of every minor of $X$ is the product of its main diagonal entries. We fix such a term order.

For the determination of the initial ideals of the ideals under consideration we will use the Knuth–Robinson–Schensted correspondence. It is a degree preserving bijection between the standard monomials of minors of $X$ (or standard bitableaux) and the ordinary monomials of $K[X]$ (Knuth [14]). This bijection is established by the Robinson-Schensted algorithm of Schensted [20]. Sturmfels [21] used the correspondence for the determinination of a Gröbner basis of $I_t$. Later on, a variant of the correspondence was applied by Herzog and Trung [12]. Throughout this paper we will always refer to the version of the Knuth–Robinson–Schensted correspondence described in [12], and we will denote it by $\mathrm{KRS}$. The application of KRS in this paper was inspired by its use in Bruns and Conca [3] where it was essential for finding the initial ideals of the (symbolic) powers of the ideals $I_t$.

Let $M$ be an ordinary monomial in $R_0$, i. e. a product of powers of the indeterminates. Then we set

$$\alpha_k(M) = \max\{\alpha_k(\Delta)\colon M = \mathrm{in}(\Delta)\}.$$

Here $\mathrm{in}(\Delta)$ is the initial monomial of $\Delta$, a product of minors as above.

The crucial combinatorial argument of this paper is

**Theorem 1.** *For all $\Sigma$ and $k$ one has $\alpha_k(\Sigma) = \alpha_k(\mathrm{KRS}(\Sigma))$.*

The theorem is an analogue of [3, Theorem 2.2] based on Greene's theorem [11] instead of [3, Theorem 2.3]. For the reader's convenience we present the details.

Let $b = b_1, \dots, b_s$ be a sequence of integers. A subsequence $b_{i_1}, \dots, b_{i_k}$, with $i_1 < \dots < i_k$, is *increasing* if $b_{i_1} < \dots < b_{i_k}$. A decomposition $D$ of $b$ into increasing subsequences, an *inc-decomposition* for short, is said to have shape $S = s_1, \dots, s_r$ if the $i$-th subsequence has $s_i$ elements, and $s_i \geq s_{i+1}$ for all $i$. We set

$$\alpha_k(D) = \alpha_k(S) = s_1 + \dots + s_k,$$
$$\alpha_k(b) = \max\bigl\{\alpha_k(D) \colon D \text{ is an inc-decomposition of } b\bigr\}.$$

Let $M = \mathrm{KRS}(\Sigma) = X_{a_1 b_1} \cdots X_{a_s b_s}$, where the sequence of the factors $X_{a_i b_i}$ satisfies the conditions $a_1 \leq a_2 \leq \dots \leq a_s$ and $b_i \geq b_{i+1}$ whenever $a_i = a_{i+1}$. Writing $M$ as $\mathrm{in}_\tau(\Delta)$ with $\Delta$ a product of minors of sizes $s_1, \dots, s_r$ amounts to an inc-decomposition of $b_1, \dots, b_s$ of shape $s_1, \dots, s_r$. Therefore $\alpha_k(M) = \alpha_k(b)$. Since the shape of the standard monomial $\Sigma$ and, hence, $\alpha_k(\Sigma)$ only depends on $b_1, \dots, b_s$, one can assume $a_i = i$ for $i = 1, \dots, s$. Furthermore, exchanging the roles of rows and columns and using [12, 1.1(b)], one can even assume that $\{b_1, \dots, b_s\} = \{1, \dots, s\}$. After these reduction steps, Theorem 1 is a consequence of Greene's theorem [11] (adapted to the version of KRS used in this paper):

**Theorem 2.** *Let $b = b_1, \dots, b_s$ be a sequence of distinct integers and let $P$ be the tableau obtained from $b$ by the Robinson-Schensted algorithm. Denote the shape of $P$ by $S = s_1, \dots, s_r$, and let $k$ be an integer. Then*

$$\alpha_k(b) = \alpha_k(S).$$

The consequences of Theorem 1 for initial ideals are based on [3, Lemma 2.1]; we include it for easier reference. (Its part (a) was observed by Sturmfels [21].) In the following the linear extension of KRS to a $K$-automorphism of $K[X]$ is also denoted by KRS.

**Lemma 3.** *(a) Let $I$ be an ideal of $K[X]$ which has a $K$-basis, say $B$, of standard monomials, and let $S$ be a subset of $I$. Assume that for all $b \in B$ there exists $s \in S$ such that $\mathrm{in}(s) \mid \mathrm{KRS}(b)$. Then $S$ is a Gröbner basis of $I$ and $\mathrm{in}(I) = \mathrm{KRS}(I)$.*

*(b) Let $I$ and $J$ be homogeneous ideals such that $\mathrm{in}(I) = \mathrm{KRS}(I)$ and $\mathrm{in}(J) = \mathrm{KRS}(J)$. Then $\mathrm{in}(I) + \mathrm{in}(J) = \mathrm{in}(I + J) = \mathrm{KRS}(I + J)$ and $\mathrm{in}(I) \cap \mathrm{in}(J) = \mathrm{in}(I \cap J) = \mathrm{KRS}(I \cap J)$.*

Our first result on initial ideals and Gröbner bases is

**Proposition 1.** *With respect to a diagonal term order on $R_0 = K[X]$ the following hold:*

*(a)* $\operatorname{in}(J(u,d)) = \operatorname{KRS}(J(u,d))$*;*
*(b) the products $\Delta = \delta_1 \dots \delta_w$ with $w \le u$ and $\deg \Delta = d$ are a Gröbner basis of $J(u,d)$, and $J(u,d)$ has a minimal system of generators forming a Gröbner basis.*
*(c) a monomial $M$ belongs to $\operatorname{in}(J(u,d))$ if and only if $\alpha_u(M) \ge d$.*

*Proof.* (a) By virtue of Lemma 2 we can apply Lemma 3(a) to $J(u,d)$. Then (a) is an immediate consequence of Theorem 1 which says that $\operatorname{KRS}(J(u,d)) \subset \operatorname{in}(J(u,d))$.

(b) If $\alpha_u(M) > d$ then $M$ is obviously a multiple of a monomial $M'$ with $\alpha_u(M') = d$, and the latter belongs to $\operatorname{in}(J(u,d))$ and is the initial monomial of a product $\Delta$ specified in (b). Of all these we choose a subset whose set of initial monomials has cardinality $\dim_K J(u,d)_d$.

(c) It remains to show the 'only if' part. Suppose that $M \in \operatorname{in}(J(u,d))$. Then (a) implies $M = \operatorname{KRS}(\Sigma))$ for some standard monomial $\Sigma \in J(u,d))$, and $\alpha_u(M) = \alpha_u(\Sigma) \ge d$ by Lemma 2(b).

*Remark 1.* Part (a) of Proposition 1 holds for the individual summands of $J(u,d)$ if the characteristic of $K$ is 0 or $> m$; see [3]. However, at least the first statement of (b) cannot be transferred to the summands; see [3], 3.9.

For the proof of the normality of $\mathcal{R}(J)$ in Sect. 4 we note

**Lemma 4.** *For every monomial $M$ and all $e, k$ one has $\alpha_{ke}(M^e) = e\alpha_k(M)$.*

*Proof.* Suppose that $M = \operatorname{KRS}(\Sigma)$. Then it is not hard to see that $M^e = \operatorname{KRS}(\Sigma^e)$, so $\alpha_{ke}(M^e) = e\alpha_k(M)$ results immediately from Theorem 1.

We now extend our term order from $R_0$ to $R$ by first comparing the total degrees and then the '$X$-factors' of the monomials. Furthermore we extend KRS by setting $\operatorname{KRS}(\Sigma T^w) = \operatorname{KRS}(\Sigma)T^w$. Then Lemma 3 holds analogously, and one obtains

**Proposition 2.** *(a)* $\operatorname{in}(J(u,d)T^w) = \operatorname{KRS}(J(u,d)T^w)$
$= \operatorname{KRS}(J(u,d))T^w$*;*
*(b)* $\operatorname{in}(J^k) = \sum_{d=0}^{km} \operatorname{in}(J(k,d))T^{km-d}$*;*
*(Here all ideals are taken in $R$.)*

The initial ideals of the powers of $J$ are given by the next theorem.

**Theorem 3.** *(a) Let $M \in R_0$ be a monomial. Then $MT^w \in \operatorname{in}(J^k) \iff \alpha_k(M) \ge km - w$.*

*(b)* $\mathrm{in}(J^k) = \mathrm{in}(J)^k$.
*(c)* *The elements* $\delta_1 \cdots \delta_w T^{km-d}$ *with* $\deg \delta_1 \cdots \delta_w = d$ *form a Gröbner basis of* $J^k$*, and* $J^k$ *has a minimal system of generators forming a Gröbner basis.*

*Proof.* (a): If $w \geq km$, then $T^w \in \mathrm{in}(J^k)$. Thus suppose $w < km$. Then $M \in \mathrm{in}(J(k, km - w))$ by Proposition 1, and $MT^w \in \mathrm{in}(J(k, km - w)T^w \subset \mathrm{in}(J^k)$.

As to the converse implication, if $w \geq km$, there is nothing to prove. So let $w < km$. By Proposition 2 we have $MT^w \in \mathrm{in}(J(k, km - l))T^l$ for some $l$. Necessarily $l \leq w$, and $\alpha_k(M) \geq km - l \geq km - w$.

(b) The inclusion $\mathrm{in}(I^k) \supset \mathrm{in}(I)^k$ is true for arbitrary ideals $I$. So choose $MT^w \in \mathrm{in}(J^k)$. Then we can write $M = M_1 \cdots M_k M'$ where $M_i = \mathrm{in}(\delta_i)$ and $\deg M_1 \cdots M_k = \alpha_k(M)$. Set $w_i = m - \deg M_i$ for each $i$. Then

$$w' = w - (w_1 + \cdots + w_k) \geq 0$$

by (a), and $MT^w = (M_1 T^{w_1}) \cdots (M_k T^{w_k}) M' T^{w'}$. Obviously $M_i T^{w_i} \in \mathrm{in}(J)$ for each $i$, and so $MT^w \in \mathrm{in}(J)^k$.

(c) This follows easily if one combines Proposition 1(b) and Proposition 2(b).

## 4. The Rees algebra

It is now very easy to describe the initial algebra of the Rees algebra of the ideal $J$. To this end one must of course first extend the term order on $R$ to the polynomial ring $R[U]$. This is done in a similar way as the extension from $R_0$ to $R$. We first compare the total degree of two monomials in $R[U]$ and then their factors from $R$.

**Theorem 4.** *The initial algebra* $\mathrm{in}(\mathcal{R}(J)) \subset R[U]$ *is a normal semigroup ring finitely generated over* $R$ *by* $\mathrm{in}(J)U$*; in particular,* $\mathrm{in}(\mathcal{R}(J)) = \mathcal{R}(\mathrm{in}(J))$.

*Proof.* That the initial algebra is generated over $R$ by $\mathrm{in}(J)U$ follows immediately from Theorem 3(b). It remains to check the normality. For this it is sufficient (and necessary) that the semigroup of monomials of $\mathrm{in}(\mathcal{R}(J))$ is normal (for example, see Bruns and Herzog [5, Ch. 6]). Suppose that $(MT^wU^k)^e \in \mathrm{in}(\mathcal{R}(J))$. Then $M^eT^{we} \in \mathrm{in}(J^{ke})$, and therefore $\alpha_{ke}(M^e) \geq kem - we$. But then $\alpha_k(M) \geq km - w$ by Lemma 4, and thus $MT^wU^k \in \mathrm{in}(J^k)$. In other words, the semigroup of monomials of $\mathrm{in}(\mathcal{R}(J))$ is normal.

**Corollary 1.** $\mathcal{R}(J)$ *is a normal Cohen–Macaulay domain. It has rational singularities in characteristic* $0$*, and is* $F$*-rational in characteristic* $p$*. The associated graded ring* $\mathrm{gr}_J(R)$ *is Cohen–Macaulay.*

*Proof.* We set $\mathcal{R} = \mathcal{R}(J)$. By the results of Conca, Herzog, and Valla [7], the algebra $\mathcal{R}$ inherits the properties under consideration from its initial algebra. By a theorem of Hochster (for example, see [5, Ch. 6]) normal semigroup rings are Cohen–Macaulay rings. Furthermore it is well-known that (affine) toric varieties have rational singularities. Since every normal semigroup ring $S$ can be embedded into a polynomial ring as a direct $S$-summand, the $F$-rationality in characteristic $p > 0$ follows from a theorem of Hochster and Huneke [13]. The Cohen–Macaulay property of $\operatorname{gr}_J(R)$ follows from standard depth arguments since $\operatorname{gr}_J(R) = \mathcal{R}/J\mathcal{R}$, $R = \mathcal{R}/UJ\mathcal{R}$, and $J\mathcal{R}$ and $UJ\mathcal{R}$ are isomorphic $\mathcal{R}$-modules.

*Remark 2.* There is another method developed in Bruns and Conca [4] by which one could investigate the Rees algebra $\mathcal{R}(J)$. For Corollary 1 it is sufficient that the semigroup of the 'weights' of the elements $\Sigma T^k U^w \in \mathcal{R}(J)$, $\Sigma$ a standard monomial, is normal. This follows easily from the properties of the functions $\alpha_i$. By this approach one can also prove a relative version of Corollary 1 in which $R_0$ is replaced by a residue class ring $R_0/I_{r+1}$ and $J$ by $(T^r) + T^{r-1}I_1 + \cdots + I_r$.

The goal is to find the minimal prime ideals of $J\mathcal{R}(J)$ (equivalently, the minimal prime ideals of $\operatorname{gr}_J(R)$) and those of $T\mathcal{R}(J)$. In the next proposition we identify the first of them, namely the ideal $\mathfrak{m}\mathcal{R}(J)$, the extension of the maximal ideal $\mathfrak{m}$ of $R$ generated by the $X_{ij}$ and $T$.

**Proposition 3.** *(a) One has isomorphisms*

$$\mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J) \cong K[J_m U] \cong K[J_m] \cong G(\tilde{X})$$

*where $J_m$ denotes the degree $m$ component of $J$ in $R$ and $G(\tilde{X})$ is the $K$-algebra generated by the maximal minors of an $(n+m) \times m$ matrix of indeterminates.*

*(b) Furthermore $\mathfrak{m}\mathcal{R}(J)$ is a prime ideal of height* 1.

*Proof.* (a) The first and the second isomorphism are given in [2], (2.2); they result easily from the bigraded structure of the Rees algebra, if the generators of the ideal all have the same degree. For the third isomorphism we map $m \times (m+n)$ $\tilde{X}$ entry by entry to the matrix

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} & 0 & \cdots & \cdots & 0 & T \\ & & & \vdots & & \iddots & \iddots & 0 \\ \vdots & & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ & & & 0 & \iddots & \iddots & & \vdots \\ X_{m1} & \cdots & X_{mn} & T & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

This substitution induces a surjection $K[\tilde{X}] \to K[X]$. Its restriction to $G(\tilde{X})$ maps the latter onto $K[J_m]$ (the $m$-minors of the matrix above are just the generators of $J$), and for reasons of Krull dimension the restriction is an isomorphism. In fact, one has $\dim G(\tilde{X}) = mn + 1$ (for example, see [6]), and also $\dim K[J_m] = mn + 1$, since the field extension $K(J_m) \subset K(X, T)$ is algebraic: $T$ is obviously algebraic over $K(J_m)$, and $K(J_m, T) = K(X, T)$.

(b) Since $\dim \mathcal{R}(J) = mn + 2$ and $\dim \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J) = mn + 1$, the ideal $\mathfrak{m}\mathcal{R}(J)$ has height 1.

Proceeding similarly as in the proof of [2, Theorem 2.3] we can now determine the minimal prime ideals of $J\mathcal{R}(J)$ and the orders by which they divide $J\mathcal{R}(J)$:

**Theorem 5.** *(a) $J\mathcal{R}(J)$ is an unmixed, equivalently: divisorial, ideal.*

*(b) It has exactly $m$ minimal primes $P_1, \dots, P_m$, and up to numbering, $P_i \cap R = (T) + I_i(X)$, $i = 1, \dots, m$.*

*(c) The primary decomposition of $J\mathcal{R}(J)$ is*

$$J\mathcal{R}(J) = \bigcap_{i=1}^{m} P_i^{(m-i+1)}.$$

*(d) The primary decomposition of $T\mathcal{R}(J)$ is*

$$T\mathcal{R}(J) = P_1 \cap \cdots \cap P_m \cap P_{m+1}.$$

*where $P_1, \dots, P_m$ are as in* (b) *and $P_{m+1}$ is a height 1 prime ideal with $P_{m+1} \cap R = (T)$.*

*Proof.* (a) We set $\mathcal{R} = \mathcal{R}(J)$. The ideals $J\mathcal{R}$ and $UJ\mathcal{R}$ are isomorphic $\mathcal{R}$-modules, and the latter is a divisorial prime ideal since $\mathcal{R}/UJ\mathcal{R} \cong R$. This implies (a).

For (b) we use (an easy extension of) the standard induction argument [6, (2.4)] after noting that (b) and (c) are elementary results for $m = 1$. So suppose that $m > 1$. Then $L = \mathcal{R}[X_{uv}^{-1}]$ is a Laurent polynomial extension of a Rees ring $\mathcal{R}(J', K[Y, T])$ where $Y$ is an $(m-1) \times (n-1)$ matrix of indeterminates and

$$J' = (T^{m-1}) + T^{m-2} I_1(Y) + \cdots + I_{m-1}(Y).$$

By induction $J'\mathcal{R}(J', K[Y, T])$ has exactly $m-1$ associated prime ideals $Q_i$, $i = 1, \dots, m-1$ which can be numbered in such a way that $Q_i \cap K[Y, T] = I_i(Y)$. Let $P_{i+1} = Q_i \cap K[X, T]$, and $P_1 = \mathfrak{m}\mathcal{R}$ as defined above.

The extension $K[Y] \subset K[X, X_{uv}^{-1}]$ satisfies the rule

$$I_t(Y)K[X, X_{uv}^{-1}] = I_{t+1}(X)K[X, X_{uv}^{-1}],$$

and since $I_{t+1}(X)K[X, X_{uv}^{-1}] \cap K[X] = I_{t+1}(X)$ it follows that $P_i \cap R = (T) + I_i(X)$ for all $i$. It is now clear that $P_1, \dots, P_m$ are pairwise different minimal prime ideals of $J\mathcal{R}$.

Suppose that $Q$ is a minimal prime ideal of $J\mathcal{R}$ different from $P_1, \dots, P_m$. Then $X_{uv} \notin Q$ for at least one pair $(u, v)$ (otherwise we obviously had $Q = P_1$). By localization as above it would follow that $J'\mathcal{R}(J', K[Y, T])$ had more than $m-1$ minimal prime ideals, which contradicts the induction hypothesis. Thus (a) completes the proof of (b).

(c) Using the inductive argument once more, we only need to determine the $\mathfrak{m}\mathcal{R}$-primary component of $J\mathcal{R}$. Because $\mathfrak{m}\mathcal{R}$ is a divisorial prime, all the primary ideals with radical $\mathfrak{m}\mathcal{R}$ are given by the symbolic powers which by [2, (2.2)(c)] coincide with the ordinary ones. So it suffices to note that $J\mathcal{R} \subset (\mathfrak{m}\mathcal{R})^m$, $J\mathcal{R} \not\subset (\mathfrak{m}\mathcal{R})^{m+1}$.

(d) Since $\mathfrak{m}\mathcal{R}$ is also a minimal prime ideal of $(T)$, we can again use the inductive argument. The same reasoning as in (b) and (c) shows first that $\mathfrak{m}\mathcal{R}$ is the $\mathfrak{m}$-primary component of $T\mathcal{R}$ and second that it is enough to establish the result in the case in which $m = 1$. In this case $\mathcal{R}$ is isomorphic to the $K$-algebra $S = K[T, X_1, \dots, X_m, T', X_1', \dots, X_m']/I_2(W)$ where $W$ is the $2 \times (m+1)$ matrix

$$\begin{pmatrix} T & X_1 & \dots & X_m \\ T' & X_1' & \dots & X_m' \end{pmatrix}$$

and the isomorphism is established by mapping $T, X_1, \dots, X_m$ to themselves and $T', X_1', \dots, X_m'$ to $TU, X_1U, \dots, X_mU$. In fact, this substitution defines a surjective homomorphism $S \to \mathcal{R}$, and since $\dim S = m + 2 = \dim \mathcal{R}$, it is even an isomorphism. From the theory of determinantal rings it is well known that $(T) = (T, X_1, \dots, X_m) \cap (T, T')$ in $S$. Under the isomorphism above $(T, X_1, \dots, X_m)$ is just $P_1 = \mathfrak{m}\mathcal{R}$ and $(T, T')$ is $P_2$.

The theorem of Simis–Trung [22, (1.1)] immediately gives the divisor class group of $\mathcal{R}(J)$:

**Corollary 2.** *One has* $\mathrm{Cl}(\mathcal{R}(J)) \cong \mathbb{Z}^m$*, and the classes of* $P_1, \dots, P_m$ *generate* $\mathrm{Cl}(\mathcal{R}(J))$.

Part (b) of the previous theorem can be generalized to the powers of $J$:

$$J^k\mathcal{R}(J) = \bigcap_{i=1}^{m} P_i^{(k(m-i+1))}.$$

In fact, since $J^k\mathcal{R}(J) \cong J^kU^k\mathcal{R}(J)$ as a $\mathcal{R}(J)$-module and $J^kU^k\mathcal{R}(J)$ is obviously a divisorial ideal, this follows immediately from the formula for $J$ itself by divisorial arithmetic. By retraction to $R$ we can now derive the primary decomposition of $J$ and its powers.

**Corollary 3.** *For all $k \geq 1$ an irredundant primary decomposition of $J^k$ is given by*

$$J^k = \bigcap_{i=1}^{m} ((T) + I_i(X))^{(k(m-i+1))}$$

*Proof.* It is straightforward to verify that $J^k = J^k \mathcal{R}(J) \cap R$. Therefore

$$J^k = \bigcap_{i=1}^{m} P_i^{(k(m-i+1))} \cap R.$$

By the inductive argument in the proof of Theorem 5 one may assume that $P_i^{(k(m-i+1))} \cap R = ((T)+I_i(X))^{(k(m-i+1))}$ for $i = 2, \ldots, m$. So it suffices to show that $\mathfrak{m}\mathcal{R}(J)^{(km)} = \mathfrak{m}^{(km)}$. But this is again a consequence of the fact that the symbolic powers of $\mathfrak{m}\mathcal{R}(J)$ coincide with the ordinary ones [2, (2.2)(c)], and therefore

$$(\mathfrak{m}\mathcal{R}(J))^{(km)} \cap R = (\mathfrak{m}\mathcal{R}(J))^{km} \cap R = \mathfrak{m}^{km} = \mathfrak{m}^{(km)}.$$

That the prime ideals $(T) + I_i$, $i \geq 2$, are associated to $J^k$ can be assumed inductively, and that $\mathfrak{m} = (T) + I_1$ is associated to $J^k$, follows since $\mathfrak{m}^k T^{mk-1} \in J^k$, $T^{mk-1} \notin J^k$, and $\mathfrak{m}$ cannot be contained in any of the other associated prime ideals. Thus the intersection is irredundant.

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