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Cohen–Macaulay Partially Ordered Sets with Pure Resolutions

WINFRIED BRUNS AND TAKAYUKI HIBI[†]

We give a combinatorial classification of Cohen–Macaulay partially ordered sets P for which a minimal free resolution of the Stanley–Reisner ring $k[\Delta(P)]$ of the order complex $\Delta(P)$ of P is pure.

1. BACKGROUND

1.1. Let $V = \{x_1, x_2, \ldots, x_v\}$ be a finite set, called the *vertex set*, and Δ a *simplicial complex* on V. Thus Δ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \le i \le v$ and (ii) $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. Set $d = \max\{\#(\sigma); \sigma \in \Delta\}$. Here $\#(\sigma)$ is the cardinality of σ as a finite set. Then the *dimension* of Δ is defined by $\dim \Delta = d - 1$. A maximal face of Δ is also called a *facet* of Δ. We say that Δ is *pure* if $\#(\sigma) = d$ for every facet σ of Δ.

Given a subset W of V, we write Δ_W for the subcomplex of Δ defined by

$$\Delta_W = \{\sigma \in \Delta; \sigma \subset W\}.$$

Thus, in particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. On the other hand, if k is a field, then let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with coefficients k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \ne \{\emptyset\}$, and $\tilde{H}_{-1}(\{\emptyset\}; k) \cong k$, $\tilde{H}_i(\{\emptyset\}; k) = 0$ for each $i \ge 0$.

1.2. Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v variables over k. We identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by those square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \le i_1 < i_2 < \cdots < i_r \le v$, with $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. The quotient algebra $k[\Delta] := A/I_\Delta$ is called the *Stanley–Reisner ring* of Δ over k. In this paper, we consider A to be the graded ring $A = \oplus_{n \ge 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \oplus_{n \ge 0} (k[\Delta])_n$ as a graded module over A with the quotient grading.

Let $A(j)$, $j \in \mathbb{Z}$, denote the graded module $A(j) = \oplus_{n \in \mathbb{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Here $\mathbb{Z}$ is the set of integers.

1.3. A graded *finite free resolution* of $k[\Delta]$ over A is an exact sequence

$$0 \to \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \to 0 \tag{1}$$

of graded modules over A, where each $\oplus_{j \in \mathbb{Z}} A(-j)^{\beta_{i_j}}$, $1 \le i \le h$, is a graded free module of rank $\sum_{j \in \mathbb{Z}} \beta_{i_j} < \infty$, and where each φ_i is degree-preserving. The *homological dimension* $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over A is the minimal h possible in (1). It is known that $v - d \le h \le v$. The second inequality is Hilbert's syzygy theorem (see e.g., [1, (2.2.14)]), and the first is

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a consequence of the Auslander–Buchsbaum formula (see, e.g., [1, (1.3.3)]). A finite free resolution (1) is called *minimal* if each $\sum_{j\in\mathbb{Z}} \beta_{i_j}$ is smallest possible. There exists a 'unique' minimal free resolution of $k[\Delta]$ over A. See [1, p. 35].

Suppose that (1) is a minimal free resolution of $k[\Delta]$ over A; then in particular $h = \mathrm{hd}_A(k[\Delta])$. Hochster's formula [4, Theorem 5.1] guarantees that

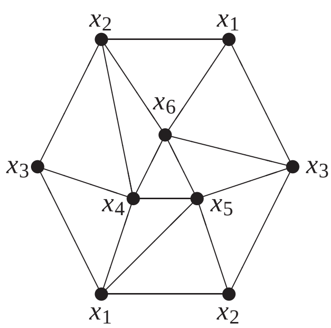
$$\beta_{i_j} = \sum_{W\subset V, \#(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \tag{2}$$

We say that $\beta_i^A = \beta_i^A(k[\Delta]) := \sum_{j\in\mathbb{Z}} \beta_{i_j}$, is the i-th *Betti number* of $k[\Delta]$ over A. A minimal free resolution (1) is called *pure* if, for each $1 \le i \le h$, there exists $c_i \in \mathbb{Z}$ such that $\beta_{i_j} = 0$ if $j \ne c_i$. Thus, (1) can be written as

$$0 \to A(-c_h)^{\beta_h} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} A(-c_1)^{\beta_1} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \to 0. \tag{3}$$

The *type* of a pure resolution (3) is the sequence $(c_1, c_2, \ldots, c_h) \in \mathbb{Z}^h$. Note that $2 \le c_1 < c_2 < \cdots < c_h$. On the other hand, a pure resolution (3) is called *m-linear* if $c_i = (m-1)+i$ for every $1 \le i \le h$. We say that $k[\Delta]$ has a pure (resp. m-linear) resolution if a graded minimal free resolution of $k[\Delta]$ over A is pure (resp. m-linear). Thus, in particular, if $k[\Delta]$ has a pure resolution, then I_Δ is generated by square-free monomials of the same degree $(= c_1)$. Moreover, if $k[\Delta]$ is Cohen–Macaulay and has a pure resolution, then some relations (Herzog–Kuehl and Huneke–Miller formulas) between the type $(c_1, c_2, \ldots, c_h)$ and the Betti number sequence $(\beta_1, \beta_2, \ldots, \beta_h)$ are known, see [1, (4.1.17)].

1.4. We say that a simplicial complex Δ has a pure resolution (resp. m-linear) resolution over k if $k[\Delta]$ has a pure (resp. m-linear) resolution. Note that this concept depends on the base field k. For example, if Δ is the simplicial complex (cf. [5]) drawn below (the facets of Δ are the triangles of the figure), then Δ has a 3-linear resolution if $\mathrm{char}(k) \ne 2$, while a minimal free resolution of $k[\Delta]$ is not pure if $\mathrm{char}(k) = 2$. See, e.g., [7] for related results.



The final goal of our project is to find a combinatorial classification of simplicial complexes with pure resolutions. In the present paper, we give a characterization of Cohen–Macaulay partially ordered sets P for which the order complex $\Delta(P)$has a pure resolution, see Theorem 3.6

2. Simplicial Complexes of Dimension One

We first classify all the simplicial complexes of dimension one with pure resolutions. A simplicial complex Δ of dimension one on the vertex set V may be considered as a (simple) graph on V. We employ some standard terminologies (e.g., tree, forest, cycle) in graph theory.

EXAMPLE 2.1. (a) Let Δ be a complete graph on the vertex set V. Then the subgraph Δ_W on $W, \emptyset \neq W \subset V$, is also complete. Hence $\tilde{H}_{\#(W)-i-1}(\Delta_W; k) = 0$ if $i \neq \#(W) - 2$, and $\dim_k \tilde{H}_{\#(W)-(\#(W)-2)-1}(\Delta_W; k) = \binom{\#(W)-1}{2}$. Thus $k[\Delta]$ has a 3-linear resolution with $\mathrm{hd}_A(k[\Delta]) = \#(V) - 2$ and $\beta_i^A(k[\Delta]) = \binom{i+1}{2}\binom{\#(V)}{i+2}$ for $1 \leq i \leq \#(V) - 2$.

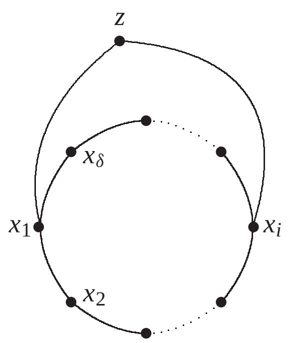
(b) Let Δ be a forest on V. If $\emptyset \neq W \subset V$, then Δ_W is a forest. Thus $\tilde{H}_{\#(W)-i-1}(\Delta_W; k) = 0$ if $i \neq \#(W)-1$, and $\dim_k \tilde{H}_{\#(W)-(\#(W)-1)-1}(\Delta_W; k)$ is one less than the number of connected components of Δ_W. Hence $k[\Delta]$ has a 2-linear resolution. If Δ is not connected, then $\mathrm{hd}_A(k[\Delta]) = \#(V) - 1$. On the other hand, if Δ is connected, i.e., Δ is a tree on V, then $\mathrm{hd}_A(k[\Delta]) = \#(V) - 2$ and $\beta_i^A(k[\Delta]) = i\binom{\#(V)-1}{i+1}$ for $1 \leq i \leq \#(V) - 2$.

(c) Let Δ be a cycle on V. Then $\tilde{H}_{\#(V)-i-1}(\Delta; k) = 0$ if $i \neq \#(V) - 2$ and $\tilde{H}_{\#(V)-(\#(V)-2)-1}(\Delta; k) \cong k$. If $\emptyset \neq W \subset V, W \neq V$, then Δ_W is a forest. Thus $\mathrm{hd}_A(k[\Delta]) = \#(V) - 2$ and $k[\Delta]$ has a pure resolution of type $(2, 3, \ldots, \#(V) - 3, \#(V) - 2, \#(V))$. Note that $\beta_i^A(k[\Delta]) = \frac{i(\#(V)-i-2)}{\#(V)-1}\binom{\#(V)}{i+1}$ for $1 \leq i \leq \#(V)-3$ and $\beta^A_{\#(V)-2}(k[\Delta]) = 1$.

We now show that a simplicial complex Δ of dimension one with a pure resolution is, in fact, one of the above (a), (b) and (c) in Example 2.1.

PROPOSITION 2.2. *Let Δ be a simplicial complex of dimension one. Then the Stanley–Reisner ring $k[\Delta]$ has a pure resolution if and only if Δ is one of the following: (i) complete graph; (ii) forest; (iii) cycle.*

PROOF. Let Δ be a simplicial complex on the vertex set V with $\dim \Delta = 1$ and suppose that $k[\Delta]$ has a pure resolution. If we assume that Δ is neither a forest nor a cycle, then we can show that Δ is a complete graph in what follows. If $\{x, y\} \notin \Delta$ for some $x, y \in V$, then I_Δ is generated by square-free monomials of degree two (as $k[\Delta]$ has a pure resolution). Hence Δ contains no cycle of length three. On the other hand, as Δ is not a forest, there exists a cycle in Δ. Thus a shortest cycle Γ in Δ is of length $\delta \geq 4$. Moreover, if $\delta = \#(V)$ then $\Delta = \Gamma$ by the minimality of the length δ of Γ, hence $\delta < \#(V)$ because Δ is not a cycle. Thus, there exists a vertex $z \in V$ which does not belong to Γ. If U is the vertex set of Γ, then $\tilde{H}_{\delta-(\delta-2)-1}(\Delta_U; k) \neq 0$. Hence, by Eqn. (2), if $W \subset V$ and $\#(W) = \delta - 1$, then $\tilde{H}_{\#(W)-(\delta-2)-1}(\Delta_W; k) = 0$, i.e., Δ_W is connected. If $\delta = 4$ and $\Gamma = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}\}$, then both $\Delta_{\{x_1,x_3,z\}}$ and $\Delta_{\{x_2,x_4,z\}}$ are connected, while $\{x_1, x_3\} \notin \Delta$ and $\{x_2, x_4\} \notin \Delta$, thus $\{x_3, z\} \in \Delta$ and $\{x_2, z\} \in \Delta$, hence $\Delta_{\{x_2,x_3,z\}}$ is a cycle of length three, a contradiction. On the other hand, if $\delta \geq 5$ and $\Gamma = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_\delta, x_1\}\}$, then Δ_W with $W = \{z, x_1, \ldots, x_{\delta-2}\}$ is connected, say $\{z, x_1\} \in \Delta$. Moreover, as Δ_W with $W = \{z, x_2, \ldots, x_{\delta-1}\}$ is also connected, we have $\{z, x_i\} \in \Delta$ for some $2 \leq i \leq \delta - 1$:



This contradicts the minimality of the length δ of Γ. □

3. Cohen–Macaulay Partially Ordered Sets

In this section we study a combinatorial classification of Cohen–Macaulay partially ordered sets with pure resolutions. Let P be a finite partially ordered set (*poset*, for short) and $\Delta(P)$ the *order complex* of P. Thus, $\Delta(P)$ is a simplicial complex on the vertex set P such that $C \subset P$ is a face of $\Delta(P)$ if and only if C is a *chain* (i.e., a totally ordered subset) of P. The *rank* of P is defined to be the dimension of $\Delta(P)$. We say that P is *pure* if the order complex $\Delta(P)$ is pure. If P is pure of rank $d-1$, then P is a disjoint union $P = P_0 \cup P_1 \cup \cdots \cup P_{d-1}$, called the *rank decomposition* of P, such that every maximal chain C of P is of the form $C : x_0 < x_1 < \cdots < x_{d-1}$ with each $x_i \in P_i$. On the other hand, we say that P has a pure resolution if the order complex $\Delta(P)$ has a pure resolution.

Lemma 3.1. *Let a poset P be pure of rank $d-1$ with $d \geq 3$ and $P = P_0 \cup P_1 \cup \cdots \cup P_{d-1}$ the rank decomposition of P. Suppose that*

(i) the subposet $P_i \cup P_{i+1}$ of rank one is connected from every $0 \leq i \leq d-2$;
(ii) P does not contain the poset  *as a subposet;*
(iii) $k[\Delta(P)]$ has a pure resolution.

Then the number of maximal chains of P is equal to $\#(P) - d + 1$.

Proof. Set $Q_i = P_i \cup P_{i+1}$ for $0 \leq i \leq d-2$. First, we show that Q_i is a tree. As $k[\Delta(P)]$ has a pure resolution, thanks to Eqn. (2), $k[\Delta(Q_i)]$ has a pure resolution. Thus, by Proposition 2.2, Q_i is either a tree or a cycle because Q_i is connected. As $d \geq 3$, we have either $i > 0$ or $i + 1 < d - 1$. Let us assume $i > 0$. The case $i + 1 < d - 1$ can be done in a similar fashion. As Q_{i-1} is connected, there exists $x \in P_i$ and $y, z \in P_{i-1}$ with $y \neq z$ such that $y < x$ and $z < x$. Hence, if Q_i is a cycle, then  is contained in $P_{i-1} \cup P_i \cup P_{i+1}$, thus  is contained in P as a subposet, a contradiction.

Now, let Q denote the subposet $P_0 \cup P_1 \cup \cdots \cup P_{d-2}$ of P. Then, by induction on d, the number of maximal chains of Q is $\#(Q) - d + 2$. (If $d = 3$, then $Q = P_0 \cup P_1$ is a tree, hence the number of edges of Q is $\#(Q) - 1$.) Let $P_{d-2} = \{y_1, y_2, \ldots, y_s\}$ and, for each $1 \leq j \leq s$, write p_j for the number of maximal chains C of Q with $y_j \in C$, and write q_j for the number of elements $x \in P_{d-1}$ with $y_j < x$. By (ii), if $p_j \geq 2$, then $q_j = 1$. Hence, the number of maximal chains of P is

$$\begin{aligned}
\sum_{j=1}^{s} p_j q_j &= \sum_{j=1}^{s} p_j + \sum_{j=1}^{s} (q_j - 1) \\
&= \sum_{j=1}^{s} p_j + \sum_{j=1}^{s} q_j - \#(P_{d-2}) \\
&= \sum_{j=1}^{s} p_j + \{\#(P_{d-2}) + \#(P_{d-1}) - 1\} - \#(P_{d-2}) \qquad \text{(because } Q_{d-2} \text{ is a tree)} \\
&= \#(Q) - d + 2 + \#(P_{d-1}) - 1 \\
&= \#(P) - d + 1
\end{aligned}$$

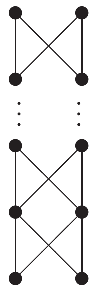
as desired. □

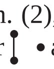
Lemma 3.2. *Let a poset P be pure of rank $d-1$ with $d \geq 2$ and $P = P_0 \cup P_1 \cup \cdots \cup P_{d-1}$ the rank decomposition of P. Suppose that*

(i) each $\#(P_i) \geq 2$;

(ii) P contains the poset  *as a subposet;*
(iii) $k[\Delta(P)]$ has a pure resolution.

Then P is the poset drawn below.



PROOF. As $\dim_k \tilde{H}_{4-2-1}(\Delta($$); k) = 1$, by (iii) and Eqn. (2), every three-element subposet of P must be connected, i.e., P contains neither ••• nor  as a subposet. Hence each $\#(P_i) = 2$ and $P_i \cup P_{i+1} =$  as required. □

We here recall some fundamental material on Cohen–Macaulay partially ordered sets. Let Δ be a simplicial complex of dimension $d-1$ and $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ, i.e., f_i is the number of faces σ of Δ with $\#(\sigma) = i+1$. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$$

with $f_{-1} := 1$. The *Hilbert series* of $k[\Delta] = \oplus_{n\geq 0}(k[\Delta])_n$ is the formal power series $F(k[\Delta], \lambda) = \sum_{n=0}^{\infty}(\dim_k(k[\Delta])_n)\lambda^n$. Then $F(k[\Delta], \lambda) = (h_0 + h_1 + \cdots + h_d\lambda^d)/(1-\lambda)^d$. We say that Δ is *Cohen–Macaulay* over k if $k[\Delta]$ is Cohen–Macaulay, i.e., $\mathrm{hd}_A(k[\Delta]) = v - d$ (with the notation in (1.3)). (See, e.g., [1, 3, 6] for the 'familiar' definition of Cohen–Macaulay rings with systems of parameters and regular sequences.) The h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of a Cohen–Macaulay complex Δ is non-negative, i.e., each $h_i \geq 0$. A one-dimensional simplicial complex Δ is Cohen–Macaulay if and only if Δ is connected. Every Cohen–Macaulay complex is pure. A finite poset P is called Cohen–Macaulay if the order complex $\Delta(P)$ is Cohen–Macaulay. Let P be a Cohen–Macaulay poset of rank $d-1$ with the rank decomposition $P = P_0 \cup P_1 \cup \cdots \cup P_{d-1}$. Then every rank-selected subposet $P_{i_0} \cup P_{i_1} \cup \cdots \cup P_{i_j}$, $0 \leq i_0 < i_1 < \cdots < i_j \leq d-1$, is Cohen–Macaulay. Consult [1, 3, 4, 6] for further information.

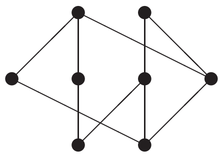
PROPOSITION 3.3. *Suppose that P is a Cohen–Macaulay poset of rank $d-1$ such that the number of maximal chains of P is equal to $\#(P) - d + 1$. Then $k[\Delta(P)]$ has a pure resolution.*

PROOF. Let $f(\Delta(P)) = (f_0, f_1, \ldots, f_{d-1})$ be the f-vector of $\Delta(P)$ and $h(\Delta(P)) = (h_0, h_1, \ldots, h_d)$ the h-vector of $\Delta(P)$. Recall that $h_0 = 1$, $h_1 = f_0 - d$ and $f_{d-1} = h_0 + h_1 + \cdots + h_d$. Thus, it follows from $f_{d-1} = \#(P) - d + 1$ (and $f_0 = \#(P)$) that $h_2 + \cdots + h_d = 0$, while each $h_i \geq 0$ because P is Cohen–Macaulay. Hence $h_2 = \cdots = h_d = 0$. Thus, the Hilbert Series $F(k[\Delta(P)], \lambda)$ of $k[\Delta(P)]$ is $(1 + (\#(P) - d)\lambda)/(1-\lambda)^d$.

It is known, see, e.g., [1, Exercise 4.1.7] (and, in fact, not difficult to prove), that if an ideal I of $A = k[x_1, x_2, \ldots, x_v]$ is generated by homogeneous polynomials of degree m (≥ 2)

and if $R = A/I = \oplus_{n \geq 0} R_n$ is Cohen–Macaulay, then a minimal free resolution of R over A is m-linear if and only if the Hilbert Series $F(R, \lambda) = \sum_{n=0}^{\infty} (\dim_k R_n)\lambda^n$ is of the form $(h_0 + h_1\lambda + \cdots + h_{m-1}\lambda^{m-1})/(1-\lambda)^d$ with each $0 < h_i \in \mathbb{Z}$ for some $d \geq 0$. By virtue of this ring-theoretical result, a minimal free resolution of our $k[\Delta(P)]$ turns out to be 2-linear as desired. □

EXAMPLE 3.4. Let P be the pure poset drawn below. Then the h-vector of $\Delta(P)$ is $(1, 5, -1, 1)$, thus P is not Cohen–Macaulay, while the number of maximal chains of P is $\#(P) - 3 + 1$. As P contains, •••, , and , as subposets, a minimal free resolution of $k[\Delta(P)]$ is not pure.



EXAMPLE 3.5. Let P be the poset of rank $d-1$ discussed in Lemma 3.2. Then P is Cohen-Macaulay and $k[\Delta(P)]$ has a pure resolution of type $(2, 4, \ldots, 2d)$ with Betti numbers $\beta_i^A(k[\Delta(P)]) = \binom{d}{i}$, $1 \leq i \leq d$ $(= \mathrm{hd}_A(k[\Delta]))$.

We now come to the main result of this paper. See also [2].

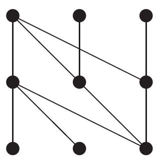
THEOREM 3.6. *Suppose that P is a Cohen–Macaulay partially ordered set of rank $d-1$ (with $d \geq 2$) which possesses the rank decomposition $P = P_0 \cup P_1 \cup \cdots \cup P_{d-1}$ with each $\#(P_i) \geq 2$. Then the Stanley–Reisner ring $k[\Delta(P)]$ of the order complex $\Delta(P)$ of P has a pure resolution if and only if we have one of the following:*

(i) $d = 2$ and P is a cycle;
(ii) $d \geq 3$ and P is the poset drawn in Lemma 3.2;
(iii) the number of maximal chains of P is $\#(P) - d + 1$.

PROOF. First, by Example 2.1, Example 3.5 and Proposition 3.3, if P is one of the (i), (ii), and (iii), then $k[\Delta(P)]$ has a pure resolution. On the other hand, suppose that $k[\Delta(P)]$ has a pure resolution. If $d = 2$, then thanks to Proposition 2.2, P is either (i) a cycle or (iii) a tree, because P is connected. Let us assume $d > 2$. Note that every subposet $P_i \cup P_{i+1}$ of rank one is connected since $P_i \cup P_{i+1}$ is Cohen–Macaulay. Thus, by Lemma 3.1 together with Lemma 3.2, P is either (ii) or (iii) as required. □

REMARK 3.7. (a) Every simplicial complex of dimension zero has a 2-linear resolution.
(b) If $\#(P_i) = 1$ in the above Theorem 3.6, then P has a pure resolution if and only if the subposet $P_0 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_{d-1}$ of P has a pure resolution.

EXAMPLE 3.8. The Cohen–Macaulay poset P drawn below is an example of (iii) in Theorem 3.6.



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W. BRUNS
FB Mathematik/Informatik,
Universität Osnabrück,
49069 Osnabrück,
Germany
E-mail: winfried@dido.mathematik.uni-osnabrueck.de

T. HIBI
Department of Mathematics,
Graduate School of Science,
Osaka University,
Toyonaka, Osaka 560-0043,
Japan
E-mail: hibi@math.sci.osaka-u.ac.jp