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The canonical module of an associated graded ring

By

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In the articles [4] and [5] Herzog, Simis and Vasconcelos have computed the canonical module of the Rees ring $S = \bigoplus_{n=0}^{\infty} I^n t \subset R[t]$ of a commutative noetherian Ring R with respect to an ideal I. We want to complement their results by determining the canonical module of the corresponding associated graded ring $S/SI \cong \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ from the canonical module of S. (For the theory of the canonical module we refer to [3].)

Theorem. *Let R be a Cohen-Macaulay ring (locally, always) $I \subset R$ an ideal of height at least 2, S the Rees ring of R with respect to I, and $G = S/SI$ the associated graded ring. Assume that S and G are Cohen-Macaulay rings, and that S has a canonical module ω_S. Then G has a canonical module ω_G, and:*

(i) *If ω_S can be embedded into S such that ω_S (considered as an ideal now) is not contained in a minimal prime ideal of SI or SIt, then $\omega_G \cong (\omega_S + SI)/SI$.*

(ii) *Such an embedding exists if and only if the localizations S_P with respect to the prime ideals $P \subset S$ minimal over 0, SI or SIt are Gorenstein rings.*

Before embarking on the proof one should note that under the Cohen-Macaulay hypothesis on R, S, and G the ideals SI and SIt are unmixed of height 1.

Proof. By virtue of [3], Satz 5.12 the G-module $\mathrm{Ext}_S^1(G, \omega_S)$ is canonical for G.

Since the adjunction of a new indeterminate to R does not affect the validity of (i), we may assume in its proof that the residue class rings $R/R \cap P$ for the prime ideals mentioned and R/Q for the minimal prime ideals $Q \subset R$ of I and 0 have infinitely many elements. Let us write ω for $\omega_S \subset R$. Then the following holds:

(1) *There is an element $x \in I \setminus I^2$ such that x is not a zero-divisor of R (and S) and S/ω and xt is not a zero-divisor of S/ω and G.*

We postpone the proof of (1). From [6], Lemma 1.1 we get for such an element x: (2) $Sx \colon SI = Sx + Sxt$. We further contend: (3) $\omega \cap Sx = \omega x$, $\omega \cap Sxt = \omega xt$, (4) $\omega \cap (Sx + Sxt) = \omega x + \omega xt$, and (5) $SI \cap (Sx + Sxt) = Sx$.

Contention (3) follows directly from (1). For (4) let $ax + \tilde{a}xt \in \omega$, $a, \tilde{a} \in S$. Since x is not a zero-divisor of $R/R \cap \omega$, it is not a zero-divisor of $S/(\omega + SIt) \cong R/R \cap \omega$. Hence

$$a = b + \sum b_i x_i t, \quad b \in \omega,\ b_i \in S,\quad x_i \in I.$$

This gives

$$ax + \tilde{a}xt = bx + (\sum b_i x_i + \tilde{a})\, xt,$$

so $(\sum b_i x_i + \tilde{a})\, xt \in \omega$, and (4) follows from (3). Claim (5) results from similar computations based on the fact that xt is not a zero-divisor of G.

Applying the functor $\operatorname{Hom}_S(G, -)$ to the exact sequence

$$0 \to \omega \xrightarrow{x} \omega \to \omega/\omega x \to 0$$

we obtain an isomorphism

$$\omega_G = \operatorname{Ext}^1_S(G, \omega_S) \cong \operatorname{Hom}_S(G, \omega/\omega x),$$

thus $\omega_G \cong J/\omega x$, where $J = \{z \in \omega;\ Iz \in \omega x\}$. From (2) and (3) above one gets

$$J \subset \omega x + \omega xt,$$

and, on the other hand, $ybxt = (yt)\, bx \in \omega x$ for $b \in \omega$, $y \in I$. So

$$J = \omega x + \omega xt = \omega \cap (Sx + Sxt).$$

(3) and (5) imply

$$\omega_G \cong (\omega \cap (Sx + Sxt))/(\omega \cap (Sx + Sxt)) \cap SI$$

whence

$$\begin{aligned}\omega_G &\cong (\omega \cap (Sx + Sxt) + SI)/SI\\ &\cong (\omega xt + SI)/SI\\ &\cong xt((\omega + SI)/SI)\\ &\cong (\omega + SI)/SI\end{aligned}$$

by virtue of (4) and the fact that xt is not a zero-divisor of G.

For (i) it only remains to prove (1), which we consider as a special case of the following more general statement after having picked a minimal prime ideal $Q \subset R$ of I: *Let $x_1, \ldots, x_n \in I$ such that $\tilde{I} = \sum_{i=1}^{n} Rx_i \not\subset QIR_Q$, $S\tilde{I}$ is not contained in a minimal prime of $0 \subset S$ or ω, and $S\tilde{I}t$ is not contained in a minimal prime of SI or ω; then $\tilde{I}$ contains an element $x \notin I^2$ as desired.* We observe that the hypothesis of this statement is fulfilled for $\tilde{I} = I$. Being obvious for $n = 1$, it is shown by induction. In the inductive step one replaces $x_1, \ldots, x_n$ by

$$y_1 = x_1 + a_n x_n, \quad y_2 = x_2, \ldots, y_{n-1} = x_{n-1}$$

where $a_n \in R$ is chosen such that

$$\begin{aligned}&y_1 \notin QIR_Q, && \text{if } x_n \notin QIR_Q,\\ &y_1 \notin P, && \text{if } x_n \notin P, \quad \text{and}\\ &y_1 t = y_1 t + a_n x_n t \notin P, && \text{if } x_n t \notin P\end{aligned}$$

for each of the critical prime ideals P. It is easy to check that the set of elements excluded from being chosen as a_n is the union of finitely many residue classes with respect to prime ideals of R, for which the residue class rings have infinitely many elements by our initial assumption. Their union cannot fill R, cf. [1], p. 75, Lemma 6.

The necessity of the condition in (ii) is obvious as far as the prime ideals P minimal over SI and SIt are concerned; for the minimal prime ideals of S cf. [3], 6.7. Suppose now that the condition is satisfied. Then $\omega \otimes S_P \cong S_P$ for the prime ideals P under consideration. Since their number is finite there is an $f \in \mathrm{Hom}_S(\omega, S)$ which is basic in $\mathrm{Hom}_S(\omega, S)$ at each of these prime ideals P, i.e. $f \notin P\,\mathrm{Hom}_S(\omega, S)_P$, cf. [2], (2.2). Every element of $\mathrm{Hom}_S(\omega, S)_P = \mathrm{Hom}_{S_P}(\omega \otimes S_P, S_P)$, which is not in $P\,\mathrm{Hom}(\omega, S)_P$, maps $\omega \otimes S_P$ isomorphically onto S_P because of $\omega \otimes S_P \cong S_P$. In particular $f(\omega) \not\subset P$ and $\mathrm{Ker} f = 0$, since $\mathrm{Ker} f \otimes S_P = 0$ for all the prime ideals under consideration, including the associated prime ideals of S, and ω is an S-module without torsion.

Remarks. (a) The hypothesis of (ii) is satisfied whenever S is a normal domain, since the localizations with respect to the prime ideals mentioned are even regular then.

(b) Strictly speaking the contention of (i) should be that $(\omega_S + SI)/SI$ is a canonical module of G, since the canonical module is only unique up to multiplication by an element of the Picard group.

(c) In order to see that the requirement on the height of I is essential one considers the example $R = K[X, Y, W, Z]/(XY - WZ)$, $I = (x, w)$, K a field. Then S is the residue class ring of $K[X, Y, W, T, T_1, T_2]$ with respect to the ideal generated by the 2-minors of the matrix

$$\begin{bmatrix} X & Y & T_1 \\ W & Z & T_2 \end{bmatrix}$$

where the canonical epimorphism from the polynomial ring onto S sends T_1 to xt, T_2 to yt. The ideal $(y, z) \cong (t_1, t_2)$ is a canonical module for S, cf. [5], not contained in a minimal prime of $SI = (x, w)$ or $SIt = (t_1, t_2)$. Since $G(\cong R)$ is Gorenstein again, $((y, z) + SI)/SI = (yG + zG)$ is not a canonical module for G.

(d) The theorem generalizes [6], Proposition 1.2 which covers the case in which S is Gorenstein.

(e) The theorem implies that under its hypotheses ω_G is a quotient of $\omega_S \otimes G$; in general, however, $\omega_G \neq \omega_S \otimes G$, as easily available examples demonstrate.

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