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Canonical modules of Rees algebras

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

We compute the canonical class of certain Rees algebras. Our formula generalizes that of Herzog and Vasconcelos. Its proof relies on the fact that the formation of the canonical module commutes with subintersections in important cases. As an application we treat the classical determinantal ideals and the corresponding algebras of minors.

MSC: 13A20; 13B22; 13C20; 13C40

A considerable part of Wolmer Vasconcelos' work has been devoted to Rees algebras, in particular to their divisorial structure and the computation of the canonical module (see [16–18,23,28]). In this paper we give a generalization of the formula of Herzog and Vasconcelos [18] who have computed the canonical module under more special assumptions. We show that

$$[\omega_{\mathscr{R}}] = [I\mathscr{R}] + \sum_{i=1}^{t} (1 - \operatorname{ht} \mathfrak{p}_i)[P_i]$$

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for ideals I in regular domains R essentially finite over a field for which the Rees algebra $\mathscr{R} = \mathscr{R}(I)$ is normal Cohen–Macaulay and whose powers have a special primary decomposition. In the formula the P_i are the divisorial prime ideals containing $I\mathscr{R}$, and $\mathfrak{p}_i = P_i \cap R$. The condition on the primary decomposition can be expressed equivalently by the requirement that the restriction of the Rees valuation v_{P_i} to R coincides with the $\mathfrak{p}_i$-adic valuation.

It is not difficult to derive a criterion for the Gorenstein property of the Rees algebra, the extended Rees algebra and the associated graded ring from the formula above.

While our hypotheses are far from the most general case, which may very well be intractable, the formula covers many interesting ideals, for example the classical determinantal ideals. As an application we can therefore compute the canonical classes of their Rees algebras, and also those of the algebras generated by minors. The algebras generated by minors can be identified with fiber cones of the determinantal ideals, and therefore are accessible via the Rees algebra.

This paper has been inspired by the work of Bruns and Conca [7] where the case of the determinantal ideals of generic matrices and Hankel matrices has been treated via initial ideals.

The formula above can be proved with localization arguments only if there is no containment relation between the $\mathfrak{p}_i$. It was therefore necessary to investigate the behavior of the canonical module under subintersections, and, as we will show for normal algebras essentially of finite type over a field, its formation does indeed commute with taking a subintersection.

1. Divisor class group and valuations on a Rees algebra

Let R be a normal Noetherian domain, and I an ideal in R for which the Rees algebra

$$\mathscr{R} = \mathscr{R}(I) = \mathscr{R}(R, I) = \bigoplus_{k=0}^{\infty} I^k T^k \subset R[T]$$

is a normal domain. The normality of the Rees algebra is equivalent to the integral closedness of all powers I^k (see [26]). We will assume in the following that ht $I \geqslant 2$.

The divisor class group of $\mathscr{R}$ has been investigated in several articles of which Vasconcelos is a co-author [16–18,23]. The main result, first proved by Simis and Trung [27], describes Cl $(\mathscr{R})$ as follows.

Theorem 1.1. *There is an exact sequence*

$$0 \to \mathbb{Z}^t \to \mathrm{Cl}\,(\mathscr{R}) \to \mathrm{Cl}\,(R) \to 0, \tag{1}$$

where a basis of $\mathbb{Z}^t$ is given by the classes $[P_i]$ of the minimal prime ideals $P_1, \ldots, P_t$ of the divisorial ideal $I\mathscr{R}$.

In the following we will denote the subgroup $\mathbb{Z}[P_1] + \cdots + \mathbb{Z}[P_t]$ simply by $\mathbb{Z}^t$.

The most informative proof of this theorem uses the following lemma, in which Spec ^{1} denotes the set of divisorial prime ideals.

Lemma 1.2. *Let I be an ideal of height at least* 2, *and, with the notation introduced,* $V_i = \mathscr{R}_{P_i}$; *then*

$$\mathscr{R} = R[T] \cap V_1 \cap \cdots \cap V_t \tag{2}$$

and, moreover,

$$R[T] = \bigcap \{\mathscr{R}_P : P \in \operatorname{Spec}^1(\mathscr{R}),\ \ P \neq P_1, \ldots, P_t\}. \tag{3}$$

Proof. Since $\mathscr{R}$ is a Krull domain, it is the intersection of its localizations $\mathscr{R}_P$, where P runs through $\operatorname{Spec}^1(\mathscr{R})$. For (2) we have to show that either $\mathscr{R}_P = R[T]_Q$ for a divisorial prime ideal Q of $R[T]$, or $P = P_i$ for some i.

If $P \supset I\mathscr{R}$, then $P = P_i$ for some i. Otherwise there exists $x \in I$, $x \notin P$. Then $\mathscr{R}_P$ is a localization of $\mathscr{R}[x^{-1}]$, and since $\mathscr{R}[x^{-1}] = R[T, x^{-1}]$, it is also a localization of $R[T]$ with respect to some divisorial prime ideal.

Eq. (3) follows from [7, Lemma 2.1] because the extensions $P_i R[T]$, containing $IR[T]$, have height at least 2. (One should beware of considering (3) as a trivial consequence of (2); for $R = K[X]$, K a field, and $I = RX$, the intersection in (3) is $R[X^{-1}, T]$.) □

Eq. (3) says that $R[T]$ is a subintersection of $\mathscr{R}$ in the terminology of Fossum [15], and so Nagata's theorem [15, Theorem 7.1] yields the exact sequence (1) with $\operatorname{Cl}(R[T])$ in place of $\operatorname{Cl}(R)$, once one has shown that the classes $[P_1], \ldots, [P_t]$ are linearly independent. A proof of the linear independence can be found in Morey and Vasconcelos [23]. Finally one uses that $\operatorname{Cl}(R[T]) = \operatorname{Cl}(R)$.

We want to complement Theorem 1.1 by a description of the Picard group of the Rees algebra. It is the group of isomorphism classes of projective rank 1 modules, and therefore a natural subgroup of the class group. One has $\operatorname{Pic}(R) = \operatorname{Cl}(R)$ if and only if R is locally factorial (i.e. all localizations of R at maximal ideals are factorial).

Proposition 1.3. *The natural map* $\operatorname{Pic}(R) \to \operatorname{Pic}(\mathscr{R})$ *is an isomorphism.*

If R is locally factorial, then the natural map $\operatorname{Cl}(R) = \operatorname{Pic}(R) \to \operatorname{Pic}(\mathscr{R}) \subset \operatorname{Cl}(\mathscr{R})$ *splits the exact sequence* (1). *In particular* $\operatorname{Cl}(\mathscr{R}) = \mathbb{Z}^t \oplus \operatorname{Pic}(\mathscr{R})$.

Proof. Let us first show that $\operatorname{Pic}(\mathscr{R}) = 0$ if R is a local ring with maximal ideal $\mathfrak{m}$. In this case $\mathscr{R}$ is a graded ring whose homogeneous non-units generate the maximal ideal $\mathfrak{M} = \mathfrak{m} \oplus \bigoplus_{k=1}^{\infty} I^k T^k$. Moreover, each divisorial ideal of $\mathscr{R}$, especially every rank 1 projective module, is isomorphic to a graded ideal. A graded projective module is free, since its localization with respect to $\mathfrak{M}$ is free (see [8, 1.5.15]).

We now turn to the general case. The natural map $\operatorname{Pic}(R) \to \operatorname{Pic}(R[T])$, induced by ring extension, is an isomorphism, since R is normal. It factors through $\operatorname{Pic}(\mathscr{R})$. Therefore, the map $\operatorname{Pic}(\mathscr{R}) \to \operatorname{Pic}(R[T])$ is surjective. In order to show that it is injective, we have to verify that $\operatorname{Pic}(\mathscr{R}) \cap \mathbb{Z}^t = 0$. This follows from the fact that each nonzero divisor class $[C]$ in $\mathbb{Z}^t$ survives in at least one Rees algebra $R_{\mathfrak{p}}[I_{\mathfrak{p}}T]$. By the first part of the proof this is impossible for the class of a projective rank one module. In fact, if the coefficient of $[C]$ with respect to $[P_i]$ is nonzero, we choose $\mathfrak{p} = \mathfrak{p}_i$.

The second statement is now obvious. □

The valuations v_{P_i} on the quotient field $Q(\mathscr{R}) = Q(R[T])$ are called the *Rees valuations* of I (cf. [22, Chapter XI]). If v is a valuation on a domain R such that $v(x) \geqslant 0$ for all $x \in R$, then the *center* of v is the prime ideal $\{x : v(x) > 0\}$.

The representation (2) of $\mathscr{R}$ can be translated into a description of the powers of I as an intersection of valuation ideals.

Proposition 1.4. *Let v_i be the Rees valuation on the quotient field $Q(R[T])$ associated with $\mathscr{R}_{P_i}$, $i = 1, \dots, t$, and set $J_i(j) = \{x \in R : v_i(x) \geqslant j\}$. Then*

$$I^k = \bigcap_{i=1}^{t} J_i(kd_i), \quad d_i = -v_i(T). \tag{4}$$

The intersection is irredundant for $k \gg 0$. Moreover,

$$I\mathscr{R} = \bigcap_{i=1}^{t} P_i^{(d_i)}.$$

Proof. We consider Eq. (2) in each T-degree. Then it says

$$I^k T^k = \{aT^k : a \in R,\ v_i(a) \geqslant -v_i(T^k),\ i = 1, \dots, t\}$$

and this is evidently equivalent to Eq. (4).

To see that the intersection is irredundant for $k \gg 0$, we use that the representation (2) is irredundant: there exists $y_i \in R$ and $n_i \in \mathbb{N}$ such that $v_i(y_i T^{n_i}) < 0$, but $v_j(y_i T^{n_i}) \geqslant 0$ for $j \neq i$. Moreover, $IT\mathscr{R} = \mathscr{R} \cap TR[T]$ is a divisorial prime ideal different from P_i for all i. Thus we can find $x_i \in I$ such that $v_i(x_i T) = 0$. It follows that $v_i(x_i^m y_i T^{n_i+m}) < 0$ for all $m \geqslant 0$, but $v_j(x_i^m y_i T^{n_i+m}) \geqslant 0$ for $j \neq i$. To sum up: the representation is irredundant for $n \geqslant \max_i n_i$.

For the second formula we note that $IT\mathscr{R}$ is a prime ideal different from $P_1, \dots, P_t$, the divisorial prime ideals containing $I\mathscr{R}$. So $0 = v_i(IT\mathscr{R}) = v_i(T) + v_i(I\mathscr{R})$. Therefore $v_i(I\mathscr{R}) = -v_i(T)$. $\square$

Let $\mathfrak{p}_i = P_i \cap R = J_i(1)$. Then one sees immediately that $J_i(j)$ is $\mathfrak{p}_i$-primary for all j, and thus (4) yields a primary decomposition of I^k for all k. But even if the intersection is irredundant, it need not be an irredundant primary decomposition in the usual sense, since it may very well happen that $\mathfrak{p}_i = \mathfrak{p}_j$ for $i \neq j$.

In the next section we want to compute the class of the canonical module of $\mathscr{R}$ in certain cases, in which we can identify the ideals $J_i(j)$.

Suppose that R is a regular domain. Then each prime ideal $\mathfrak{p}$ of R defines a discrete valuation on $Q(R)$ as follows. First we replace R by $R_\mathfrak{p}$, and may assume that R is local with maximal ideal $\mathfrak{p}$. Now we set $v_\mathfrak{p}(x) = \max\{i : x \in \mathfrak{p}^{(i)}\}$ for each $x \in R$, $x \neq 0$, and extend this function naturally to $Q(R)$ (with $v_\mathfrak{p}(0) = \infty$). That $v_\mathfrak{p}$ is indeed a valuation follows from the fact that the associated graded ring of the filtration $(\mathfrak{p}^i)$ is a polynomial ring over the field $R/\mathfrak{p}$ and therefore an integral domain. It guarantees that $v_\mathfrak{p}(xy) = v_\mathfrak{p}(x) + v_\mathfrak{p}(y)$. A similar argument shows that the symbolic powers of prime ideals in regular domains are integrally closed.

One says that a valuation on the polynomial ring $R[T]$ (where R may be an arbitrary domain) is *graded* if

$$v(f) = \min_i v(a_i T^i)$$

for all polynomials $f = \sum a_i T^i$, $a_i \in R$. Every valuation on R can be extended to a graded valuation on $R[T]$. One can freely choose $v(T)$ and then use the previous equation to define the extension of v (see [4, Chapter VI, Section 10, no. 1, Lemme 1]).

The Rees valuations of I on $R[T]$ are graded. In fact, if S is a normal graded subalgebra of $R[T]$ with $Q(S) = Q(R[T])$, then the valuations associated with graded divisorial ideals of S are graded, since all symbolic powers of graded prime ideals are graded as well. Moreover, the associated prime ideals of the graded ideal $I\mathscr{R}$ are graded.

Proposition 1.5. *Let R be a regular ring and I an ideal of height* $\geqslant 2$.

(a) *Then the following are equivalent*:

  (i) *$\mathscr{R}(I)$ is normal, and for each minimal prime ideal P of $I\mathscr{R}$ the Rees valuation v_P restricts on R to the valuation $v_\mathfrak{p}$, $\mathfrak{p} = P \cap R$;*
  (ii) *there exist prime ideals $\mathfrak{p}_i, \ldots, \mathfrak{p}_u$ in R and $d_1, \ldots, d_u \in \mathbb{N}$ such that $I^k = \bigcap_{i=1}^u \mathfrak{p}_i^{(d_i k)}$ for all k.*

(b) *Moreover, if* (i) *holds, and $P_1, \ldots, P_t$ are the minimal prime ideals of $I\mathscr{R}$, then one can choose $\mathfrak{p}_i = P_i \cap R$, $d_i = -v_i(T)$, and the intersection in* (ii) *is irredundant for $k \gg 0$.*

(c) *Conversely, if there exists k_i for each $i = 1, \ldots, u$ such that $\mathfrak{p}_i^{(d_i k)}$ cannot be omitted in the representation of I^{k_i} in* (ii), *then the graded extensions of the v_i to $R[T]$ with $v_i(T) = -d_i$ are the Rees valuations of I on $Q(R[T])$.*

Proof. (a) The implication (i)⇒(ii) has been proved above, together with the description of the $\mathfrak{p}_i$ in terms of the prime ideals P_i.

Suppose now that (ii) holds. Then all powers of I are integrally closed, since the symbolic powers of prime ideals in regular rings are integrally closed. Let v_i be the valuation on $R[T]$ that we obtain as the graded extension of $v_{\mathfrak{p}_i}$ to $R[T]$ with $v_i(T) = -d_i$. The representation of I^k, $k \in \mathbb{N}$, can immediately be translated into the description of $\mathscr{R}(I)$ as the intersection of $R[T]$ with the discrete valuation rings V_i associated with the valuations v_i

$$\mathscr{R} = R[T] \cap V_1 \cap \cdots \cap V_u. \tag{5}$$

Let P_i be the center of v_i in $\mathscr{R}$. Then $I\mathscr{R} = \bigcap P_i^{(d_i)}$. If this representation is not irredundant, we can shorten it to an irredundant primary decomposition, which is unique since $I\mathscr{R}$ is a divisorial ideal. As seen above, we can shorten the representation (5) accordingly. Thus we may assume that $P_1, \ldots, P_u$ are the minimal primes of $I\mathscr{R}$. That the associated valuations satisfy the condition in (i), follows from their construction.

(b) is only a restatement of Proposition 1.4 under the special hypothesis made in (i).

(c) The condition guarantees that none of the V_i can be omitted in (5), and the rest has been proved above. □

Remark 1.6. With the appropriate modifications, the results in this section remain true if one considers an arbitrary ideal of height $\geqslant 2$ in a normal (or regular) ring and replaces the (in general non-normal) Rees algebra by its integral closure

$$\overline{\mathscr{R}} = \bigoplus_{k=0}^{\infty} \overline{I^k} T^k,$$

where $\overline{I^k}$ is the integral closure of I^k. (The ideal $I\mathscr{R}$ must be replaced by $\bigoplus \overline{I^{k+1}} T^k$.)

2. The canonical class

As in the previous section, we assume that I is an ideal in the normal domain R with a normal Rees algebra $\mathscr{R} = \mathscr{R}(I)$. Suppose further that R is a Cohen–Macaulay residue class ring of a Gorenstein ring, and $\mathscr{R}$ is also Cohen–Macaulay. Then $\mathscr{R}$ has a canonical module $\omega_{\mathscr{R}}$ (see [8]). The canonical module is (isomorphic to) a divisorial ideal, and this allows us to find $\omega_{\mathscr{R}}$ by divisorial computations.

The canonical module is unique only up to tensor product with a projective rank 1 module. In other words, only its residue class modulo the Picard group $\operatorname{Pic}(R) \subset \operatorname{Cl}(R)$ is unique.

Let us assume for the moment (and without essential restriction for the theorem to be proved) that R is factorial. Because of the exact sequence (1) and since R (and, along with it, $R[T]$) is factorial, the class of $\omega_{\mathscr{R}}$ is a linear combination of the classes $[P_i]$,

$$[\omega_{\mathscr{R}}] = w_1[P_1] + \cdots + w_t[P_t].$$

We have to determine the coefficients w_i.

Since the behavior of the class group and the canonical module under localization is easily controlled, we can first replace R by $R_{\mathfrak{p}_i}$ and I by $IR_{\mathfrak{p}_i}$ in order to compute w_i (as above, $\mathfrak{p}_i = P_i \cap R$). However, in general this localization does not strip off all the other components $\mathbb{Z}[P_j]$: those with $\mathfrak{p}_j \subset \mathfrak{p}_i$ survive.

Therefore, we need a finer instrument to isolate w_i: we pass to the subintersection $R[T] \cap V_i$ (after the localization). Then we must

(i) determine the structure of $R[T] \cap V_i$ and find its canonical class, and
(ii) show that the canonical module is preserved under subintersection.

Let us first turn to problem (ii). We cannot present a solution in complete generality. However, the next theorem should cover many interesting applications.

Theorem 2.1. *Let K be a field, R a normal Cohen–Macaulay K-algebra essentially of finite type over K, and $Y \subset \operatorname{Spec}^1(R)$. Suppose that the subintersection $S = \bigcap_{\mathfrak{p} \in Y} R_{\mathfrak{p}}$ is again essentially of finite type over K and Cohen–Macaulay.*

Then the canonical module of S is $(\omega_R \otimes_R S)^{\dagger\dagger}$, where † denotes the functor $\operatorname{Hom}_S(\_, S)$. *In other words, the canonical class of S is the image of ω_R under the natural map* $\operatorname{Cl}(R) \to \operatorname{Cl}(S)$.

Proof. For technical simplicity let us first assume that K is a perfect field, and let $\Omega_{R/K}$ be the module of Kähler differentials of R. It has been proved by Kunz [20] that the canonical module is given by the regular differential r-forms $\mathrm{R}^r_K(R)$, $r = \operatorname{rank} \Omega_{R/K}$, and Platte and Storch [24] have noticed that $\mathrm{R}^r_K(R)$ can be identified with the R-bidual $(\bigwedge^r \Omega_{R/K})^{**}$. The same applies to S. (A proof will be given below.)

The extension $\phi : R \to S$ gives rise to an R-linear map $d\phi : \Omega_{R/K} \to \Omega_{S/K}$, and $d\phi$ induces a natural S-linear map

$$\psi : \left(\bigwedge^r \Omega_{R/K}\right) \otimes_R S \to \bigwedge^r \Omega_{S/K}.$$

Let $\mathfrak{q}$ be a height 1 prime ideal of S. Then $S_\mathfrak{q} = R_{\mathfrak{q} \cap R}$, and therefore $\psi \otimes_S S_\mathfrak{q}$ is an isomorphism. It follows that the S-bidual extension $\psi^{\dagger\dagger}$ is an isomorphism at all height 1 prime ideals $\mathfrak{q}$ of S. Since the S-biduals are reflexive, $\psi^{\dagger\dagger}$ is an isomorphism itself.

The second statement about the divisor classes follows immediately, since $(I \otimes S)^{\dagger\dagger}$ is exactly the divisorial ideal of S to which a divisorial ideal I of R extends; see [15].

If K is not perfect (this can happen only in characteristic $p > 0$), one replaces K by a subfield K_0 with $[K : K_0] < \infty$ that is admissible in the sense of [21, 6.23] for R, S and regular K-algebras A and B essentially of finite type for which there exist presentations $R = A/I$ and $S = B/J$. All the algebras involved are then essentially of finite type over K_0, too.

Let us now show that $(\bigwedge^r \Omega_{R/K_0})^{**}$ is the canonical module of R. To this end we let $[M]$ denote the divisor class of a finitely generated R-module: $[M]$ is the isomorphism class of $(\bigwedge^n M)^{**}$ where $n = \operatorname{rank} M$.

We use the complex

$$0 \to I/I^2 \to \Omega_{A/K_0} \otimes_A R \to \Omega_{R/K_0} \to 0$$

that is exact at the right and in the middle and exact in I/I^2 at all prime ideals $\mathfrak{p}$ for which $R_\mathfrak{p}$ is regular. Moreover, I/I^2 is free at such $\mathfrak{p}$ of rank $c = \operatorname{ht} I$ (see [21]). By divisorial calculation,

$$[I/I^2] = -[\Omega_{R/K_0}],$$

since $\Omega_{A/K_0} \otimes_A R$ is a free module, and so has class 0. Finally, by Herzog and Vasconcelos [18, Lemma], $[\omega_R] = -[I/I^2]$. □

Before we state and prove the main result, let us single out a very special case.

Proposition 2.2. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$. Then the Rees algebra $\mathscr{R}_k = \mathscr{R}(\mathfrak{m}^k)$ is normal and Cohen–Macaulay. Its canonical module is unique up to isomorphism and has class $(k - \dim R + 1)[P_k] = [\mathfrak{m}^k \mathscr{R}_k] - (\dim R - 1)[P_k]$, where $P_k = \mathfrak{m}\mathscr{R}_k$ is the only divisorial prime ideal of $\mathscr{R}_k$ containing $\mathfrak{m}^k \mathscr{R}_k$.*

Proof. By Proposition 1.3 one has $\operatorname{Pic}(\mathscr{R}_k) = 0$, hence the uniqueness of the canonical module.

For the equation $P_k = \mathfrak{m}\mathscr{R}_k$ it is enough to show that $\mathfrak{m}\mathscr{R}_k$ is a prime ideal of height 1. This follows immediately from the fact that $\mathscr{R}_k/\mathfrak{m}\mathscr{R}_k$ is the kth Veronese subring of the associated graded ring of R with respect to $\mathfrak{m}$, a polynomial ring over the field $R/\mathfrak{m}$.

Let $x_1, \ldots, x_r$, $r = \dim R$, be a regular system of parameters and set

$$J_k = (x_1 \cdots x_r T R[T]) \cap P_k.$$

We know that $[P_k]$ generates $\operatorname{Cl}(\mathscr{R}_k)$, and $TR[T] \cap \mathscr{R}_k = \mathfrak{m}^k T\mathscr{R}_k \cong \mathfrak{m}^k \mathscr{R}_k$. Furthermore, $(x_i R[T]) \cap \mathscr{R}_k$ has divisor class $-[P_k]$. In fact $x_i\mathscr{R}_k = x_i R[T] \cap P_k$, since $x_i R[T] \cap \mathscr{R}_k$ and P_k are the only divisorial prime ideals of $\mathscr{R}_k$ containing the prime element x_i of $R[T]$ (see Lemma 1.2); moreover, x_i obviously has value 1 under the corresponding valuations.

This shows that J_k has the class given in the theorem, and it only remains to prove that it is the canonical module. In the case $k = 1$ this follows immediately from Herzog and Vasconcelos [18]. Before going to general k, we want to convince ourselves that J_1 is the graded canonical module of $\mathscr{R}_1$ in the sense of [8, 3.6]. In fact, there is an exact sequence

$$0 \to \operatorname{Hom}(\mathscr{R}_1, J_1) \to \operatorname{Hom}(\mathfrak{m}T\mathscr{R}_1, J_1) \to \operatorname{Ext}^{\mathscr{R}}_1(R, J_1) \to 0.$$

The graded canonical module of $\mathscr{R}_1$ is of the form $J_1(-s)$ for some integer s. Exactly for the right choice of s, $\operatorname{Ext}^{\mathscr{R}}_1(R, J_1(-s))$ is the graded canonical module $\omega_R = R$ of R (with the trivial grading $R_0 = R$). We have only to check that this holds with $s = 0$. But $\operatorname{Hom}(\mathscr{R}_1, J_1)$ has only components of positive degree, and the degree 0 component of $\operatorname{Hom}(\mathfrak{m}T\mathscr{R}_1, J_1)$ is $x_1 \cdots x_r R \cong R$, as desired.

For general k we have

$$\omega_{\mathscr{R}_k} = \omega_{\mathscr{R}(\mathfrak{m})} \cap \mathscr{R}_k = J_1 \cap \mathscr{R}_k = J_k$$

by Goto and Watanabe [8, 3.6.21] since $\mathscr{R}_k$ is the kth Veronese subalgebra of $\mathscr{R}_1$ with respect to the T-grading. □

We can now prove the main result.

Theorem 2.3. *Let K be a field, R be a regular domain essentially of finite type over K, and I an ideal with a Cohen–Macaulay normal Rees algebra $\mathscr{R} = \mathscr{R}(I)$. Let $P_1, \ldots, P_t$ be the divisorial prime ideals of $I\mathscr{R}$, $I\mathscr{R} = \bigcap_{i=1}^t P_i^{(d_i)}$, and suppose that $v_{P_i}|R = v_{\mathfrak{p}_i}$ for $i = 1, \ldots, t$, with $\mathfrak{p}_i = P_i \cap R$. Then a module of class*

$$\sum_{i=1}^{t} (d_i + 1 - \operatorname{ht}\mathfrak{p}_i)[P_i] = [I\mathscr{R}] + \sum_{i=1}^{t} (1 - \operatorname{ht}\mathfrak{p}_i)[P_i]$$

is a canonical module $\omega_{\mathscr{R}}$ of $\mathscr{R}$. Moreover, $\mathscr{R}$ is Gorenstein if and only if $d_i = \operatorname{ht}\mathfrak{p}_i - 1$ for all $i = 1, \ldots, t$.

Proof. Let C be a module of the class given in the theorem. It is enough to show that each of its localizations $C_{\mathfrak{M}}$ with respect to maximal ideals $\mathfrak{M}$ of $\mathscr{R}$ is a canonical module of $\mathscr{R}_{\mathfrak{M}}$. Such a localization $\mathscr{R}_{\mathfrak{M}}$ is a localization of R $_{\mathfrak{m}}$ with $\mathfrak{m} = R \cap \mathfrak{M}$. Since the definition of $[C]$ commutes with localization in R (in fact, primary decomposition commutes with such localizations), we may assume that R is factorial. Then $\mathrm{Cl}(\mathscr{R}) = \mathbb{Z}^t$, $\mathrm{Pic}(\mathscr{R}) = 0$ (by Proposition 1.3), and we have a unique isomorphism class

$$[\omega_{\mathscr{R}}] = w_1[P_1] + \cdots + w_t[P_t]$$

for the canonical module of $\mathscr{R}$. It is enough to determine, say, w_1. We localize R with respect to $\mathfrak{p}_1$, and may then assume that R is regular local with maximal ideal $\mathfrak{m} = \mathfrak{p}_1$. In the next step we pass to the subintersection $S = R[T] \cap V_1$. But this subintersection is exactly $\mathscr{R}(\mathfrak{m}^{d_1})$, as follows from Propositions 1.4 and 1.5. (Since $\mathfrak{m}$ is a maximal ideal, its ordinary and symbolic powers coincide.)

According to Theorem 2.1 the formation of the canonical class commutes with subintersection, and so w_1 is the coefficient of the canonical module of $\mathscr{R}(\mathfrak{m}^{d_1})$ with respect to the extension of P_1. By Proposition 2.2 this coefficient is $d_1 + 1 - \dim R$, as desired.

Because of the splitting $\mathrm{Cl}(\mathscr{R}) = \mathbb{Z}^t \oplus \mathrm{Pic}(\mathscr{R})$, the class given in the theorem is the $\mathbb{Z}^t$-component of any canonical module of $\mathscr{R}$. Therefore, $\mathscr{R}$ is Gorenstein if and only if the $\mathbb{Z}^t$-component vanishes. □

Often it is useful to know the graded canonical module of $\mathscr{R}$, as we have seen in the proof of Proposition 2.2.

Corollary 2.4. *With the hypotheses of Theorem* 2.3 *suppose we can find an element $x \in R$ such that $v_{P_i}(x) = \mathrm{ht}\,\mathfrak{p}_i$ for all i.*

(a) *Then*

$$\omega_{\mathscr{R}} = xTR[T] \cap P_1 \cap \cdots \cap P_t$$

is the graded canonical module of $\mathscr{R}$ (with respect to the grading by T).

(b) *Suppose that K is infinite, R is the polynomial ring over K in n variables, graded by total degree, I is graded and x is homogeneous, and choose $x' \in R$ such that $\deg x' = n - \deg x$ and $x' \notin \mathfrak{p}_i$ for $i = 1, \dots, t$. Then $xx'TR[T] \cap P_1 \cap \cdots \cap P_t$ is the bigraded canonical module with respect to the natural bigrading on $\mathscr{R}$.*

This follows as in the special case considered in Proposition 2.2. Note that the element x' needed for (b) can always be found by prime avoidance. (If the ideal $(X_1, \dots, X_n)$ is among the $\mathfrak{p}_i$, we choose $x' = 1$.)

An analogous statement as in (b) holds for monomial ideals. Then we can choose $x = X_1, \dots, X_n$ and obtain the multigraded canonical module of $\mathscr{R}$. This can be proved directly from the theorem of Danilov and Stanley describing the canonical module of a normal semigroup ring (see [18, 6.3.5]).

Example 2.5. The following example shows that the condition on the Rees valuations in Theorem 2.3 is crucial. Let I be the integral closure of the ideal $(X^2, Y^3, Z^5) \subset K[X, Y, Z]$, K a field. It has been noticed by Reid, Roberts, and Vitulli [25] (and can easily be checked by `normaliz` [9]) that $\mathscr{R} = \mathscr{R}(I)$ is normal. A K-basis of $\mathscr{R}$ is given by all monomials $X^aY^bZ^cT^d$ where $15a + 10b + 6c - 30d \geqslant 0$. In other words, $\mathscr{R} = R[T] \cap V_1$, where the valuation defining V_1 is the multigraded extension of the function that takes the values

$v_1(X)=15$, $v_1(Y)=10$, $v_1(Z)=6$ and $v_1(T)=-30$. It follows that $[I\mathscr{R}]=30[P_1]$. By the theorem of Danilov and Stanley one has $\omega_{\mathscr{R}}=XYZTR[T]\cap P_1$. Arguing as in the proof of Proposition 2.2 one obtains $[\omega_{\mathscr{R}}]=(-15-10-6+30)[P_1]+[P_1]=0$. So $\mathscr{R}$ is a Gorenstein ring. Its canonical module is the principal ideal generated by *XYZT*.

Remark 2.6. (a) The hypotheses of the theorem can be weakened. If we *define* the canonical module via Kähler differentials (the description used in the proof of Theorem 2.1), then the hypothesis that the Rees algebra is Cohen–Macaulay is no longer necessary. The proof of the theorem shows that the canonical module has class $[\Omega_K(R)\otimes\mathscr{R}]+\sum_{i=1}^{t}(d_i+1-\operatorname{ht}\mathfrak{p}_i)[P_i]$.

(b) Instead of requiring that $\mathscr{R}(I)$ is normal, one could consider the normalization of $\mathscr{R}(I)$. As indicated in Remark 1.6, $I\mathscr{R}$ has then to be replaced by $\bigoplus \overline{I^{k+1}}T^k$.

(c) One can generalize Theorem 2.3 in such a way that Example 2.5 is covered. The first part of its proof, namely the isolation of each w_i, does not use the hypothesis on the Rees valuations. Therefore, as soon as one can compute the canonical module of $R[T]\cap V_i$ for each i, a generalization is possible. A suitable hypothesis generalizing the condition $v_{P_i}|R=v_{\mathfrak{p}_i}$ is the following: there exists a regular system of parameters $x_1,\dots,x_m$ of $R_{\mathfrak{p}_i}$ such that each of the ideals $\{x\in R_{\mathfrak{p}_i} : v_{P_i}(x)\geqslant k\}$ is generated by monomials in $x_1,\dots,x_m$. Then one can replace ht $\mathfrak{p}_i$ in the theorem by $v_{P_i}(x_1,\dots,x_m)=\sum_{j=1}^{m}v_{P_i}(x_j)$. However, there exist valuations that do not allow such a "monomialization". A counterexample was communicated by Cutkosky.

In view of Theorem 2.3 it is not difficult to decide when the extended Rees algebra or the associated graded ring are Gorenstein. (We are grateful to S. Goto for suggesting the inclusion of the corollary.)

Corollary 2.7. *Suppose that R and I satisfy the hypothesis of Theorem* 2.3. *Then the extended Rees algebra* $\widehat{\mathscr{R}}=\mathscr{R}[T^{-1}]$ *or, equivalently, the associated ring* $\operatorname{gr}_I(R)=\mathscr{R}/I\mathscr{R}$, *is Gorenstein if and only if there exist* $c_i\in\mathbb{N}$, $i=1,\dots,t$, *such that*

(i) $c_id_i=\operatorname{ht}\mathfrak{p}_i-1$ *for all* $i=1,\dots,t$, *and*
(ii) $c_i=c_j$ *whenever there exists a maximal ideal* $\mathfrak{m}$ *of R with* $\mathfrak{p}_i,\mathfrak{p}_j\subset\mathfrak{m}$.

Proof. The Cohen–Macaulay property of $\mathscr{R}$ is inherited by $\widehat{\mathscr{R}}$ and $\operatorname{gr}_I(R)$ (if R is Cohen–Macaulay), as is well-known.

The Gorenstein property of $\widehat{\mathscr{R}}$ is local with respect to Spec R, and the same holds for the associated graded ring. We can therefore assume that R is local with maximal ideal $\mathfrak{m}$. Then $\operatorname{Pic}(\widehat{\mathscr{R}})=0$, as follows by the same argument as in the proof of Proposition 1.3. Moreover, $\widehat{\mathscr{R}}$ is Gorenstein if and only if $\operatorname{gr}_I(R)$ is so, since the latter is the residue class ring by the homogeneous regular element T^{-1}.

The extended Rees algebra $\widehat{\mathscr{R}}$ is a subintersection of $\mathscr{R}$, namely

$$\widehat{\mathscr{R}}=\bigcap\{\mathscr{R}_Q : Q\in\operatorname{Spec}^1(\mathscr{R}),\ Q\neq IT\mathscr{R}\}$$

and we can apply Theorem 2.1. (Note that $IT\mathscr{R}=TR[T]\cap\mathscr{R}$.) Thus its divisor class group is $\operatorname{Cl}(\mathscr{R})/\mathbb{Z}[IT\mathscr{R}]$ (this was noticed in [17]). Since $[IT\mathscr{R}]=[I\mathscr{R}]$, the canonical class of

$\widehat{\mathscr{R}}$ vanishes if and only if $[\omega_{\mathscr{R}}]$ is a multiple of $[I\mathscr{R}]$. In view of the theorem this is clearly equivalent to (i) and (ii) (where in (ii) we now have $c_i = c_j$ for all i and j). □

One should note that the extended Rees algebra (and the associated graded ring) can have non-trivial projective rank 1 modules, even if $\operatorname{Pic}(R) = 0$. (Therefore, it is not possible to replace condition (ii) by the requirement that $c_i = c_j$ for *all* i and j.) For example, let I be the intersection of two maximal ideals in $R = K[X_1, X_2]$. Then it is easy to check by localization at the maximal ideals of R that the extended Rees algebra is locally factorial. On the other hand, it has divisor class group isomorphic to $\mathbb{Z}$, and all its divisorial ideals are projective modules.

Another interesting algebra that can be accessed through the Rees algebra is the *fiber cone* $\mathscr{R}/\mathfrak{m}\mathscr{R}$, especially in the situation in which $R = K[X_1, \dots, X_n]$ and $\mathfrak{m} = (X_1, \dots, X_n)$. If I has a system of generators $f_1, \dots, f_m$ of constant degree, then one has a natural embedding $K[f_1, \dots, f_m]$ into $\mathscr{R}$. Moreover, $K[f_1, \dots, f_m]$ is isomorphic to $\mathscr{R}/\mathfrak{m}\mathscr{R}$, and therefore a retract of $\mathscr{R}$ (see [5, (2.2)]). We refer the reader to the next section where some interesting examples will be discussed.

3. Applications

As mentioned in the introduction, Theorem 2.3 was inspired by the its special case derived in [7]. We will now use it in order to extend the results of [7] to other classes of determinantal ideals, and in particular to algebras generated by minors.

Let X be one of the following types of matrices over a field K:

(G) an $m \times n$ matrix of indeterminates;
(S) an $n \times n$ symmetric matrix of indeterminates;
(A) an $n \times n$ alternating matrix of indeterminates.

By M_t we denote the set of t-minors of X in the cases (G) and (S) and the set of $2t$-pfaffians in case (A) (the $2t$-pfaffians are also elements of degree t). In view of the very detailed analysis of case (G) in [7] we restrict ourselves to an outline containing all the main steps.

There seems to be no single source providing simultaneously all the details of the cases (G), (S), and (A). For (G) they can be found in [7] and [12], for (S) in Abeasis [1], and for (G) in Abeasis and Del Fra [2] and De Negri [14]. We will freely use these sources.

Let $R = K[X]$, I_t be the ideal in R generated by M_t, and A_t the K-subalgebra generated by M_t. Set $\mathscr{R}_t = \mathscr{R}(I_t)$. Since the elements of I_t have all the same degree, one has a retract

$$A_t \to \mathscr{R}_t \to A_t,$$

where the embedding $A_t \to \mathscr{R}_t$ is induced by the assignment $f \mapsto fT$, $f \in M_t$, and the kernel of $\mathscr{R}_t \to A_t$ is $\mathfrak{m}R$, with $\mathfrak{m}$ denoting the irrelevant maximal ideal of R. In fact, the bigrading on $\mathscr{R}_t$ induces a splitting

$$\mathscr{R}_t = A_t \oplus \mathfrak{m}\mathscr{R}_t$$

(see [5, (2.2)]). It follows that $\mathfrak{m}\mathscr{R}_t$ is a prime ideal.

We assume that the characteristic of K is *non-exceptional*, i.e. char $K = 0$ or char $K > \min(t, m-t, n-t)$ in case (G), char $K > \min(t, n-t)$ in case (S), and char $K > \min(2t, n-2t)$ in case (A). Then

$$I_t^k = \bigcap_{i=1}^{t} I_i^{(t-i+1)}. \tag{6}$$

Moreover, if $t < \min(m, n)$, $t < n$, or $2t < n-1$, respectively, the intersection is irredundant for $k \gg 0$; it follows that the irrelevant maximal ideal is the center of a Rees valuation, and in particular $\dim A_t = \dim \mathscr{R}_t/\mathfrak{m}\mathscr{R}_t = \dim R$. In the other cases the symbolic and the ordinary powers of I_t coincide (and the canonical module of $\mathscr{R}_t$ has been discussed in Bruns et al. [11]).

That $\mathscr{R}_t$ and A_t are Cohen–Macaulay in characteristic 0 has been shown for all three types in [5]. In positive non-exceptional characteristic one finds this result for (G) in [6], for (A) in Baeţica [3], and for (S) in Bruns et al. [10].

As soon as ht $I_t > 1$, and this is equivalent to I_t being non-principal, Theorem 2.3 yields the canonical class of $\mathscr{R}_t$. The hypothesis on the Rees valuations is satisfied in view of Proposition 1.5.

In certain cases the structure of A_t is very easily determined or classically known:

(1) If $t = 1$, then $A_t = R$.
(2) If $t = m-1 = n-1$ in case (G), $t = n-1$ in case (S), or $2t = n-2$ in case (A), then A_t is isomorphic to a polynomial ring over K. This is easily shown by comparing the Krull dimension with the number of generators.
(3) If $t = n$ in case (S) or $2t = n$ in case (G), then A_t is isomorphic to a polynomial ring over K for trivial reasons.
(4) If $t = \min(m, n)$ in case (G), then A_t is the homogeneous coordinate ring of a Grassmannian, a factorial Cohen–Macaulay domain.
(5) If $2t = n-1$ in case (A), then A_t is isomorphic to a polynomial ring over K (this was observed by De Negri and follows from a theorem of Huneke [19]).

In the following we exclude all these cases, in which the canonical class is well-known (*and trivial*).

The Veronese subring $R^{(t)}$ can be embedded into $R[T]$ in the same way as A_t into $\mathscr{R}_t$, namely by the assignment $f \mapsto fT$ for all monomials of degree t. Then, inside $\mathscr{R}_t$, one obviously has

$$A_t = \mathscr{R}_t \cap R^{(t)}.$$

Lemma 1.2 immediately yields a representation of A_t as an intersection of $R^{(t)}$ with discrete valuations rings. As in the case (G) treated in [7], one always has the somewhat surprising equation

$$A_t = R^{(t)} \cap V_2;$$

furthermore, $R^{(t)}$ is the subintersection of A_t obtained by omitting the discrete valuation ring V_2. This yields an exact sequence

$$0 \to \mathbb{Z}[\mathfrak{p}] \to \mathrm{Cl}\,(A_t) \to \mathrm{Cl}\,(R^{(t)}) \to 0$$

in which $\mathfrak{p} = P_2 \cap A_t$. The group $\mathrm{Cl}\,(R^{(t)})$ is cyclic of order t. It is not hard to show that $\mathrm{Cl}\,(A_t)$ is cyclic (of rank 1), generated by the class $[\mathfrak{q}]$ of the prime ideal $f S[T] \cap A_t$, f an arbitrary element of M_{t+1}. Moreover, $[\mathfrak{p}] = -t[\mathfrak{q}]$.

We define an element $\mathscr{D} \in R$ as follows:

(G) $\mathscr{D}$ is the product of all minors of X whose main diagonals are the parallels to $X_{11}, \dots, X_{mm}$. (Such a parallel starts in each of the X_{i1} and X_{1j}.)
(S) $\mathscr{D}$ is the product of all minors of X, whose main diagonal is the parallel to $X_{11}, \dots, X_{nn}$ starting in one of the entries X_{i1}.
(A) $\mathscr{D}$ is the product of all pfaffians whose anti-diagonal (in the lower triangular part of X) is a parallel to the anti-diagonal of X.

The valuation associated with I_j is always the (extension of) the function γ_j, as defined in De Concini et al. [13], [1], or [2], respectively. For an element δ of M_i one has

$$\gamma_j(\delta) = \begin{cases} 0, & i < j, \\ i - j + 1, & i \geqslant j. \end{cases}$$

It is now an easy combinatorial exercise to compute $\gamma_j(\mathscr{D})$ in all cases, and it turns out that

$$\gamma_j(\mathscr{D}) = \mathrm{ht}\, I_j = \begin{cases} (m-j+1)(n-j+1), & (\mathrm{G}), \\ \binom{n-j+1}{2}, & (\mathrm{S}), \\ \binom{n-j}{2}, & (\mathrm{A}). \end{cases}$$

Thus we have found elements satisfying the condition discussed in Corollary 2.4, and

$$\omega_{\mathscr{R}_t} = \mathscr{D}T\,R[T] \cap P_1 \cap \cdots \cap P_t.$$

We have the presentation $A_t = \mathscr{R}_t/\mathfrak{m}\mathscr{R}_t$. Now, since $\mathfrak{m}\mathscr{R}_t$ is contained in P_1, it follows that $P_1 = \mathfrak{m}\mathscr{R}_t$. The (graded) canonical module of A_t is given by the exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{R}_t}(\mathscr{R}_t, \omega_{\mathscr{R}_t}) \to \mathrm{Hom}_{\mathscr{R}_t}(P_1, \omega_{\mathscr{R}_t}) \to \mathrm{Ext}_1^{\mathscr{R}_t}(A_t, \omega_{\mathscr{R}_t}) \to 0.$$

Let J be the middle term. By divisorial calculation one has

$$J = \omega_{\mathscr{R}_t} : P_1 = \mathscr{D}T\,R[T] \cap P_2 \cap \cdots \cap P_t$$

and the image of $\mathrm{Hom}_{\mathscr{R}_t}(\mathscr{R}_t, \omega_{\mathscr{R}_t}) = \omega_{\mathscr{R}_t}$ is just $J \cap P_1$. So $\omega_{A_t} = J/J \cap P_1$. In other words, ω_{A_t} is the image of J under the epimorphism $\mathscr{R}_t \to A_t$ with kernel P_1. Since all our ideals are bigraded with respect to the ordinary total degree in R and degree in T, we can replace the image with the intersection:

$$\begin{aligned} \omega_{A_t} &= \mathscr{D}T\,R[T] \cap P_2 \cap \cdots \cap P_t \cap A_t, \\ &= \mathscr{D}T\,R[T] \cap P_2 \cap A_t. \end{aligned}$$

The second equation follows since $P_3, \ldots, P_t$ meet A_t in prime ideals of height > 1. They are superfluous in the representation of a divisorial ideal.

As remarked above, $P_2 \cap A_t = \mathfrak{p}$ has class $-t[\mathfrak{q}]$, and one can also compute the other term, splitting $\mathscr{D}T$ into its factors. The intersection $TR[T] \cap A_t$ is the irrelevant maximal ideal of A_t and can be omitted, but $fR[T] \cap A_t$, $f \in M_j$, has class $(j-t)[\mathfrak{q}]$. For the case (G) this has been computed in [7, 5.3], and the other cases are completely analogous. A careful count (for (A) one should distinguish the cases n odd and n even) yields:

Theorem 3.1. $\omega_{A_t} = w[\mathfrak{q}]$ *with*

$$w = \begin{cases} mn - t(m+n), & \text{(G)}, \\ \binom{n+1}{2} - t(n+1), & \text{(S)}, \\ \binom{n-1}{2} - 2t(n-1), & \text{(A)}. \end{cases}$$

Corollary 3.2. *Set* $m = n$ *in the cases* (S) *and* (A), $u = t$ *in the cases* (G) *and* (S), *and* $u = 2t$ *in case* (A). *Then* A_t *is Gorenstein if and only if*

$$\frac{1}{u} = \frac{1}{m} + \frac{1}{n}.$$

The reader should note that we have excluded cases (1)–(5) above, in the theorem as well as in the corollary. In case (G) we have only reproduced the results of [7], and our derivation of Theorem 3.1 from Theorem 2.3 has been indicated in [7, 5.6]. The case of Hankel matrices contained in [7] can be treated in the same manner.

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