THE CANONICAL MODULE OF A DETERMINANTAL RING

WINFRIED BRUNS

Let R be a commutative noetherian ring. We call an R-module M a *canonical module* of R if for each prime ideal $\underline{p}$ of R the localization $M_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ in the sense of [5]. A *determinantal ring* is a residue class ring of a polynomial ring $B[X_{ij}: 1 \le i \le u, 1 \le j \le v]$ with respect to the ideal $I_{r+1}(X_{ij})$ generated by the determinants of the $(r+1,r+1)$-submatrices of the (u,v)-matrix (X_{ij}) whose entries are the algebraically independent elements X_{ij} over the commutative noetherian ring B.

Determinantal rings play an important role in various geometric and algebraic contexts. They can be considered well-understood since Hochster and Eagon proved their perfection (relative to the polynomial ring $B[X_{ij}]$) in the remarkable article [7] which furthermore contains many useful results on the ideal theory of determinantal rings. We became interested in the canonical modules of determinantal rings when their computation appeared as a rather natural step in our investigation of generic maps of a given rank and the modules associated to these maps [2]. The generic maps of rank r are the maps $\phi: S^u \to S^v$, where $S = \mathbb{Z}[X_{ij}]/I_{r+1}(X_{ij})$ and ϕ is represented by the matrix (x_{ij}) of the residue classes of the indeterminates X_{ij}. It turned out that (in the non-degenerate case where $r \ge 1$) Coker ϕ is perfect if and only if $u \ge v$. This asymmetry in the behaviour of Coker ϕ is caused by the structure of the canonical module of S which represents the asymmetry of the format of a non-square matrix in a ring-theoretic way.

THEOREM. *Let* B *be a normal Gorenstein domain, and* u,v,r *integers such that* $1 \le r \le v \le u$. *Let* X_{ij}, $1 \le i \le u$, $1 \le j \le v$, *be algebraically independent elements over* B, *and* R *the residue class ring* $B[X_{ij}]/I_{r+1}(X_{ij})$. *Let further* $\underline{p}$ *denote the ideal generated by* sequence

$$0 \to x\underline{p}^{(n)} \to \underline{p}^{(n+m)} \to \mathrm{Ext}^1_T(T,\underline{p}^{(n)}) \to 0.$$

The ideal $x\underline{p}^{(n)}$ is divisorial, $\mathrm{cl}(x\underline{p}^{(n)}) = \mathrm{cl}(\underline{p}^{(n+m)} \cap \underline{a})$, $x\underline{p}^{(n)}$ contains $\underline{a}\underline{p}^{(n)}$, and $\underline{p}^{(n+m)}/x\underline{p}^{(n)}$ is isomorphic to a (divisorial) ideal of T. Then necessarily $x\underline{p}^{(n)} = \underline{p}^{(n+m)} \cap \underline{a}$. Finally it is easy to check that conditions (c) and (d) guarantee that

$$\underline{p}^{(n+m)}/(\underline{p}^{(n+m)} \cap \underline{a}) = ((\underline{a}+\underline{p})/\underline{a})^{(n+m)}. \square$$

Determinantal rings R are normal as soon as B is normal [7]. Their divisor class group Cl(R) was computed in [1]: $\mathrm{Cl}(R) = \mathrm{Cl}(B) \oplus \mathbb{Z}$. The second component is generated by the class of the prime ideal $\underline{p}$ of the r-minors of the first r rows of (x_{ij}) and by $-\mathrm{cl}(\underline{p}) = \mathrm{cl}(\underline{q})$, where $\underline{q}$ is the corresponding ideal for the columns. If B is a factorial domain, in particular if $B = \mathbb{Z}$ or B is a field, the natural map $\mathrm{Cl}(R) \to \mathrm{Cl}(R_{\underline{m}})$, $\underline{m}$ generated by the elements x_{ij}, is an isomorphism. We are mainly interested in these cases, and therefore we allow ourselves to speak of *the* canonical module ω_R.

PROPOSITION 1. *Let* B *be a normal Gorenstein domain. Suppose that* $u \geq v$. *Then* $\omega_R \cong \underline{p}^{(u-v)}$.

Proof. We first reduce to the case when r = 1 by a standard localization argument. Let $P = B[X_{ij}]$. Over $P[X_{11}^{-1}]$ we can transform the matrix (X_{ij}) by elementary row and column operations into

$$\begin{bmatrix} X_{11} & 0 & \cdots & 0 \\ 0 & Y_{22} & \cdots & Y_{2v} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & Y_{u2} & \cdots & Y_{uv} \end{bmatrix}$$

where $Y_{ij} = X_{ij} - X_{i1}X_{1j}X_{11}^{-1}$. The elements Y_{ij} are algebraically independent over B, and the elements $X_{11},\ldots,X_{1v},X_{21},\ldots,X_{u1}$ are algebraically independent over $C := B[Y_{ij}]$. Now $R[x_{11}^{-1}]$ can be considered as a flat overring of $C/I_r(Y_{ij})$:

the r*-minors of the first* r *rows of the matrix* (x_{ij}) *of the residue classes* x_{ij} *of* X_{ij}. *Then* $\underline{p}^{u-v}$ *is a canonical module of* R.

We recently learnt from [8; p.500] that our theorem was also noted by Hochster (unpublished). As an easy corollary one obtains the following theorem of Svanes [9; Theorem (5.5.6)].

COROLLARY. R *is a Gorenstein ring if and only if* $u = v$.

In the corollary one can allow B to be an arbitrary Gorenstein ring.

The proof of the theorem consists of two steps. As a first step we compute the divisor class of the canonical module of R, that is, show that it is isomorphic to $\underline{p}^{(u-v)}$, whereas we establish in the second step the equality of the ordinary and symbolic powers of $\underline{p}$. Note that in case R is a normal domain, a canonical module of R is (isomorphic to) a divisorial ideal of R. If the natural homomorphism from the divisor class group of R into the direct product of the divisor class groups of the localizations of R is injective, then M is uniquely determined (up to isomorphism).

An important tool in the computation of the canonical modules of determinantal rings is the following lemma.

LEMMA. *Let* T *be a normal Cohen-Macaulay domain and* $\underline{a}$ *a prime ideal of height* 1 *in* T, *such that* $T/\underline{a}$ *is also normal. Suppose that the following conditions are satisfied:*

(a) *for a prime ideal* $\underline{p}$ *of height* 1, $\underline{p} \neq \underline{a}$, *the symbolic power* $\underline{p}^{(n)}$ *is a canonical module of T;*

(b) $cl(\underline{a}) = -m \cdot cl(\underline{p})$ *with* $m \geq 0$;

(c) $\mathrm{Ann}(\underline{p}^{(n+m)}/\underline{p}^{n+m}) \not\subset \underline{a} + \underline{p}$; *and*

(d) $(\underline{a}+\underline{p})/\underline{a}$ *is a prime ideal of height* 1 *in* $T/\underline{a}$.

Then $((\underline{a}+\underline{p})/\underline{a})^{(n+m)}$ *is a canonical module of* $T/\underline{a}$.

Proof. Consider the exact sequence

$$0 \to \mathrm{Hom}_T(T,\underline{p}^{(n)}) \xrightarrow{\iota} \mathrm{Hom}_T(\underline{a},\underline{p}^{(n)}) \to \mathrm{Ext}^1_T(T,\underline{p}^{(n)}) \to 0.$$

We may identify $\mathrm{Hom}_T(T,\underline{p}^{(n)})$ with $\underline{p}^{(n)}$, $\mathrm{Hom}_T(\underline{a},\underline{p}^{(n)})$ with $\underline{p}^{(n)}:\underline{a}$, and ι with the natural embedding. $\mathrm{Ext}^1_T(T,\underline{p}^{(n)})$ is a canonical module of $T/\underline{a}$. For a suitable element x in the quotient field of T we have $x\cdot(\underline{p}^{(n)}:\underline{a}) = \underline{p}^{(n+m)}$ by condition (b) and thus an exact

$$R[x_{11}^{-1}] = (C/I_r(Y_{ij}))[X_{11},\ldots,X_{1v},X_{21},\ldots,X_{u1}][X_{11}^{-1}].$$

The extension of the ideal $\underline{\tilde{p}}$ of $C/I_r(Y_{ij})$ generated by the $(r-1)$-minors of $(y_{ij}: 2 \le i \le r,\ 2 \le j \le v)$ to $R[x_{11}^{-1}]$ is just $\underline{p}R[x_{11}^{-1}]$.

Suppose that $r > 1$. By [7; Theorem 1], the element x_{11} is a prime element in R. Therefore the natural map $Cl(R) \to Cl(R[x_{11}^{-1}])$ is an isomorphism as is the natural map $Cl(C/I_{r-1}(Y_{ij})) \to Cl(R[x_{11}^{-1}])$. Now it suffices to establish the proposition for $C/I_{r-1}(Y_{ij})$, thereby reducing its proof to the case when $r = 1$.

Let $r = 1$ now and $v = 2$. In case $u = 2$, R is clearly a Gorenstein ring and $\omega_R = R = \underline{p}^{(o)}$. In case $u > 2$ let

$$R_o := B[X_{ij}: 1 \le i \le u-1,\ 1 \le j \le 2]/I_2(X_{ij})$$

and $R_1 := R_o[X_{u1}, X_{u2}]$. Then

$$R = R_1/\underline{a},\quad \underline{a} = \sum_{i=1}^{u-1} R_1(x_{i1}X_{u2} - x_{i2}X_{u1}).$$

By induction on u, and since R_1 is a polynomial extension of R_o, the canonical module of R_1 is $\underline{p}_1^{(u-3)}$, $\underline{p}_1 = R_1x_{11} + R_1x_{12}$. Multiplication by $(x_{11}X_{u2} - x_{12}X_{u1})x_{11}^{-1}$ maps

$$\underline{q}_1 := \sum_{i=1}^{u-1} R_1x_{i1}$$

onto $\underline{a}$, whence $cl(\underline{a}) = cl(\underline{q}_1) = -cl(\underline{p}_1)$. The ideal $(\underline{a}+\underline{p}_1)/\underline{a}$ is a prime ideal of height 1, and the extension of $\underline{p}_1$ in $(R_1)_{\underline{a}+\underline{p}_1}$ is principal. Therefore condition (c) of the lemma is satisfied. An application of the lemma completes the proof for $v = 2$.

Let $v > 2$ now. Because $R[x_{11}^{-1}] = B[X_{11},\ldots,X_{1v},X_{21},\ldots,X_{u1}][X_{11}^{-1}]$, the divisor class of ω_R is a multiple of the divisor class of $\underline{p}$. Dropping the hypothesis that $u \ge v$ momentarily and transposing (X_{ij}) if necessary we may assume that $cl(\omega_R) = t\cdot cl(\underline{p})$ with $t \ge 0$ and $\omega_R \cong \underline{p}^{(t)}$. All the conditions of the lemma are satisfied for

$$\underline{a} = \underline{q} = \sum_{i=1}^{u} Rx_{i1}.$$

By induction on $u+v$ the divisor class of the canonical module of $R/\underline{q}$ is $(u-v+1)cl((\underline{p}+\underline{q})/\underline{q})$, and comparing this with the result of the lemma we conclude that $u - v = t \geq 0$ as desired. (Observe that $cl(\underline{p}+\underline{q})/\underline{p})$ is not a torsion element in $Cl(R/\underline{q})$.) □

Observe that the corollary of the theorem follows already from Proposition 1. The proof of the theorem itself is complete once we have shown that the ordinary and the symbolic powers of $\underline{p}$ coincide. This is stated in [8; Theorem 3.4] (for a field B). For an application in [2] we need a more general result, which could possibly be proved with the methods of [8]. Proposition 2 and the proof we present were found independently.

As above $\underline{q}$ denotes the prime ideal generated by the r-minors of the first r columns of (x_{ij}). For a sequence of integers $H = (i_o,\dots,i_{r-1})$, $0 \leq i_o < \dots < i_{r-1}$, $J(H)$ denotes the ideal of R which is generated by the $(j+1)$-minors of the first i_j rows of (x_{ij}), $j = 0,\dots,r-1$. Note that $R/J(H)$ is a normal domain by [4; Corollary 3] if B is normal.

PROPOSITION 2. *Let* B *be a normal domain and*

$\mathcal{H} := \{(i_o,\dots,i_{r-1}) : 0 \leq i_o < \dots < i_{r-1} < u\}$.

Then, for all $H \in \mathcal{H}$, *we have the following:*

(a) $J(H) + \underline{q}$ *is a prime ideal in* R;

(b) *for all* $s \geq 1$ *the ideal* $J(H) + \underline{q}^s$ *is primary.*

An application of Proposition 2 to the transpose of (X_{ij}) for $H = (0,1,\dots,r-1)$ completes the proof of the theorem. □

Proof of Proposition 2. The ideal in $B[X_{ij}]$ which defines $R/(J(H)+\underline{q})$ as a factor ring of $B[X_{ij}]$ is generated by the $(j+1)$-minors of the first i_j rows of (X_{ij}), $j = 0,\dots,r-1$, the $(r+1)$-minors of the entire matrix (X_{ij}), and, finally, the r-minors of the first r columns of (X_{ij}). Therefore part (a) is a special case of Proposition 3, (b) or (c), below.

In order to specify a minor of the matrix (X_{ij}) we introduce the following notation:

$$\Delta^{j_1\dots j_r}_{i_1\dots i_r}$$

is the minor associated with the row indices $i_1,\dots,i_r$ and the column indices $j_1,\dots,j_r$.

Part (a) is the case in which $s=1$ of the proof of part (b) by induction on s. Assume that $s>1$. We need a total order on the set of r-tuples of integers $(i_1,\dots,i_r)$ such that $0\le i_1<\dots<i_r$:

$$(i_1,\dots,i_r)\le(j_1,\dots,j_r)$$

if and only if $(i_1,\dots,i_r)=(j_1,\dots,j_r)$ or there is a $k\in\{1,\dots,r\}$ such that $i_k<j_k$ and $i_{k+1}\le j_{k+1},\dots,i_r\le j_r$.

To simplify the notation let $\Delta_{j_1\dots j_r}:=\Delta^{1\dots r}_{j_1\dots j_r}$.

For $H\in\mathcal{H}$ let R^* be the residue class ring $R/J(H)$ and $\underline{r}:=\underline{q}R^*$. Since $I_{r+1}(x_{ij})=0$, the r-th exterior power of (x_{ij}) has rank 1, and hence

$$\Delta_{j_1\dots j_r}\Delta^{1\dots r-1\ r+1}_{k_1\dots\ \ k_r}=\Delta_{k_1\dots k_r}\Delta^{1\dots r-1\ r+1}_{j_1\dots\ \ j_r}$$

(minors taken from (x_{ij})). If $\Delta_{j_1\dots j_r}$ is not zero modulo $J(H)$ then $\Delta^{1\dots r-1\ r+1}_{j_1\dots\ \ j_r}$ is not zero modulo $J(H)+\underline{q}$. Thus every minor $\Delta_{j_1\dots j_r}$ which is not zero modulo $J(H)$ generates the extension of $\underline{r}$ in $R^*_{\underline{r}}$: $\underline{r}$ is a prime of height 1 in R^*, and every non-zero minor $\Delta_{j_1\dots j_r}$ is not contained in $\underline{r}^{(2)}$. If for an element $b\in R^*$ the product $b\Delta_{j_1\dots j_r}$ is contained in $\underline{r}^{(s)}$, we can conclude that $b\in\underline{r}^{(s-1)}$, since $R^*_{\underline{r}}$ is a discrete valuation domain.

Assume that

$$b=\sum b_{j_1\dots j_r}\Delta_{j_1\dots j_r}\in\underline{r}^{(s)}$$

and let $(k_1,\dots,k_r)$ denote the maximal r-tuple relative to $\le$ such that $b_{k_1\dots k_r}\Delta_{k_1\dots k_r}\ne 0$. We show that $b\in\underline{r}^s$ by induction on $(k_1,\dots,k_r)$.

If there is only one non-zero term in the representation of b, we have $b_{k_1\dots k_r}\in\underline{r}^{(s-1)}$ by the argument just given, and $b_{k_1\dots k_r}\in\underline{r}^{s-1}$ by induction on s. Suppose that there are at least two non-zero terms in the representation of b and let

$\tilde{H} := (k_1-1,\dots,k_r-1)$.

Then $J(H) \subset J(\tilde{H})$ and $\Delta_{j_1\dots j_r} \in J(\tilde{H})R$ for all $(j_1,\dots,j_r) < (k_1,\dots,k_r)$. This implies that

$$b_{k_1\dots k_r}\Delta_{k_1\dots k_r} \in J(\tilde{H})R^* + \underline{r}^{(s)}.$$

Let $\tilde{R} := R/J(\tilde{H})$ and $\underline{\tilde{r}} := \underline{q}\tilde{R}$. As above the extension of $\underline{\tilde{r}}$ in $\tilde{R}_{\underline{\tilde{r}}}$ is principal, whence

$$\mathrm{Ann}_{R^*}\underline{r}^{(s)}\tilde{R}/\underline{\tilde{r}}^s \not\subset \underline{r} + J(\tilde{H})R^*.$$

Consequently $\pi(b_{k_1\dots k_r})\pi(\Delta_{k_1\dots k_r})$ is an element of $\underline{\tilde{r}}^{(s)}$, where $\pi\colon R^* \to \tilde{R}$ denotes the natural epimorphism. By induction on s it follows that $\pi(b_{k_1\dots k_r}) \in \underline{\tilde{r}}^{(s-1)} = \underline{\tilde{r}}^{s-1}$ and

$$b_{k_1\dots k_r} \in \pi^{-1}(\underline{\tilde{r}}^{s-1}) = J(\tilde{H})R^* + \underline{r}^{s-1}.$$

We may assume that $b_{k_1\dots k_r} \in J(\tilde{H})R^*$. The inclusion

$$J(\tilde{H})\Delta_{k_1\dots k_r} \subset \sum_{(j_1,\dots,j_r)<(k_1,\dots,k_r)} \Delta_{j_1\dots j_r}R^*$$

finally enables us to replace $b_{k_1\dots k_r}\Delta_{k_1\dots k_r}$ by a linear combination of the minors $\Delta_{j_1\dots j_r}$ with $(j_1,\dots,j_r) < (k_1,\dots,k_r)$, and to complete the proof by induction on $(k_1,\dots,k_r)$.

The proof of the crucial inclusion is an exercise in expansion of determinants. Let $\Delta^{v_1\dots v_w}_{u_1\dots u_w} \in J(\tilde{H})$. One expands the (r+1)-minor $\Delta^{v_1\dots v_w t_w\dots t_r}_{u_1\dots u_w k_w\dots k_r} = 0$ with respect to the columns $v_1,\dots,v_w$, the minor $\Delta_{k_1\dots k_r}$ with respect to the rows $k_1,\dots,k_{w-1}$, and combines the two equations obtained. □

In order to prove part (a) of Proposition 2 we introduce a larger class of ideals in $B[X_{ij}]$. Throughout the rest of the article we denote the matrix (X_{ij}) simply by X. Let $H = (u_o,\dots,u_r)$ be a sequence of strictly increasing integers with $0 \le u_o$ and $u_r = u$.

Further, let p and t be integers such that $0 \le t-1 \le p$. We consider the ideals

$$I(H,X) + I_t(X|p)$$

where $I(H,X)$ is the ideal generated by the (j+1)-minors of the first u_j rows of X, $j = 0,\dots,r$, and $I_t(X|p)$ is the ideal generated by the t-minors of the matrix $X|p$ consisting of the first p columns of X.

PROPOSITION 3. (a) $I(H,X) + I_t(X|p)$ *is a (strongly generically) perfect ideal of grade*

$$uv - r(u+v) + \frac{(t-2)\ (t-1)}{2} + \frac{(r-t+2p)\ (r-t+1)}{2} + \sum_{i=0}^{r-1} u_i + r.$$

(b) *If* B *is an integral domain, then* $I(H,X) + I_t(X|p)$ *is a prime ideal*.

(c) *If* B *is a normal integral domain, then*

$$B[X]/(I(H,X) + I_t(X|p))$$

is a normal integral domain.

Proof. In [4] a partial order on the set M of minors of (X_{ij}) is defined by

$$\Delta^{j_1\dots j_k}_{i_1\dots i_k} \le \Delta^{t_1\dots t_l}_{s_1\dots s_l}$$

if and only if $k \ge l$ and $i_1 \le s_1,\dots,i_k \le s_k$, $j_1 \le t_1,\dots,j_k \le t_k$. (The lower indices denote the rows, and the upper indices denote the columns, from which the minor is taken; we require that $i_1,\dots,i_k$ etc. are given in increasing order.) One easily sees that $I(H,X) + I_t(X|p)$ is generated by a poset ideal J in M. Therefore $B[X]/(I(H,X) + I_t(X|p))$ itself is an *algebra with straightening law* on the poset $P := M \setminus J$ [4; Corollary 3.5(3)]; in particular, it is a free B-module. The minor

$$\Delta^{u_0+1\ \dots\ u_{r-1}+1}_{1\ \dots\ r-1\ r+1}$$

is the single minimal element of P. Therefore P is *wonderful* in the sense of [4].

It is not hard to check that one (and, by [4; Lemma 4.3], every) maximal chain in P consists of

$$d := r(u+v) - \frac{(t-2)\ (t-1)}{2} - \frac{(r-t+2p)\ (r-t+1)}{2} - \sum_{i=0}^{r-1} u_i - r$$

elements. Furthermore

$$R := B[X]/(I(H,X) + I_t(X|p))$$

is a Cohen-Macaulay ring, once B is Cohen-Macaulay [4;Corollary 4.2].

The theory of generic perfection [3,6] shows that it suffices to prove part (a) in the case where $B = \mathbb{Z}$ or B is a field. Let $\underline{P}$ and $\underline{p}$ denote the prime ideal generated by the elements X_{ij} in B[X] and its image in R respectively. By [4;Theorem 4.1 and Proposition 3.7] a maximal $R_{\underline{p}}$-sequence in $\underline{p}R_{\underline{p}}$ consists of exactly d elements. R is a Cohen-Macaulay ring. Therefore $I := I(H,X) + I_t(X|p)$ is a perfect ideal and we obtain

$$\text{grade } I = \text{grade } I_{\underline{P}} = u.v - \text{depth } R_{\underline{p}} = u.v - d.$$

(b) As noted above, R is a free B-module. Therefore we may invert $B\backslash\{0\}$ and assume that B is a field. Now (b) is a special case of (c).

(c) We prove the assertion by induction on u+v, starting with the trivial case in which $u = v = 0$. Without restriction we may assume that $u_o = 0$. Let $\underline{a}$ denote the ideal generated by the elements

$$x_{ij},\ 1 \le i \le u_1,\ 1 \le j \le v,$$

in R. The preimage of $\underline{a}$ in B[X] is

$$I(\tilde{H},X) + I_t(X|p)$$

where

$$\tilde{H} = (u_1, u_1 + 1, \max(u_2, u_1 + 2), \ldots, \max(u_{r-1}, u_1 + r-1), u)$$

when $u_1 + r-1 < u$, and

$$\tilde{H} = (u_1, u_1 + 1, \max(u_2, u_1 + 2), \ldots, \max(u_{r-2}, u_1 + r-2), u)$$

when $u_1 + r-1 = u$.

By virtue of (a), $I(\tilde{H},X) + I_t(X|p)$ is a perfect ideal in B[X] and

$$\text{grade } I(\tilde{H},X) + I_t(X|p) \ge 2 + \text{grade } I(H,X) + I_t(X|p).$$

This implies that grade $\underline{a} \ge 2$. If depth $R_{\underline{q}} \le 1$ for a prime ideal $\underline{q}$ of R, then $\underline{a} \not\subset \underline{q}$. Thus it remains for us to prove that the rings

$$R[x_{ij}^{-1}],\ 1 \le i \le u_1,\ 1 \le j \le v,\ x_{ij} \text{ not nilpotent},$$

are normal.

Suppose first that $j \le p$. We again use the localization argument introduced in the proof of Proposition 1. We may assume that $i = 1$. Elementary row and column operations over $B[X][X_{11}^{-1}]$ transform the matrix X into

$$\begin{bmatrix} X_{11} & O & \cdots & O \\ O & Y_{22} & \cdots & Y_{2v} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ O & Y_{u2} & \cdots & Y_{uv} \end{bmatrix}.$$

Let Y be the $(u-1,v-1)$-matrix $(Y_{i+1,j+1})$. Then

$$R[x_{11}^{-1}] \cong (B[Y]/(I(\tilde{H},Y)+I_{t-1}(Y|p-1)))[X_{11},\ldots,X_{1v},X_{21},\ldots,X_{u1}][X_{11}^{-1}]$$

with

$$\tilde{H} = (u_1-1,u_2-1,\ldots,u_r-1).$$

The ring $B[Y]/(I(\tilde{H},Y) + I_{t-1}(Y|p-1))$ is a normal domain by the induction hypothesis.

Suppose now that $p < j$. We may assume that $j = v$. Over $B[X][X_{1v}^{-1}]$ the matrix X can be transformed into

$$W = \begin{bmatrix} X_{11} & \cdots & X_{1v-1} & X_{1v} \\ Y_{21} & \cdots & Y_{1v-1} & O \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ Y_{u1} & \cdots & Y_{uv-1} & O \end{bmatrix}.$$

Clearly $I(H,X) + I_t(X|p) = I(H,W) + I_t(W|p)$ over $B[X][X_{1v}^{-1}]$. Let

$$Z = \begin{bmatrix} Y_{21} & \cdots & Y_{2v-1} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ Y_{u1} & \cdots & Y_{uv-1} \\ X_{11} & \cdots & X_{1v-1} \end{bmatrix}.$$

Then, over $B[X][X_{1v}^{-1}]$,

$$I(H,X) + I_t(X|p) = I(H,W) + I_t(W|p) = I(\tilde{H},Z) + I_t(Z|p)$$

where $\tilde{H} = (u_1-1,u_2-1,\ldots,u_r-1,u)$, and

$$R[X_{11}^{-1}] \cong (B[Z]/(I(\tilde{H},Z) + I_t(Z|p)))[X_{1v},\ldots,X_{uv}][X_{1v}^{-1}].$$

Again the induction hypothesis applies. □

References

1. W. Bruns, "Die Divisorenklassengruppe der Restklassenringe von Polynomenringen nach Determinantenidealen", Rev. Roumaine Math. Pures Appl., 20 (1975), 1109-1111.

2. W. Bruns, "Generic maps and modules", Compositio Math., to appear.

3. J.A. Eagon and D.G. Northcott, "Generically acyclic complexes and generically perfect ideals", Proc. Roy. Soc. London Ser. A, 299 (1967), 147-172.

4. D. Eisenbud, "Introduction to algebras with straightening laws", Ring theory and algebra III. Proceedings of the third Oklahoma Conference (ed. B.R. McDonald, Marcel Dekker, New York and Basel, 1980), pp. 243-267.

5. J. Herzog and E. Kunz, Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Mathematics 238 (Springer, Berlin-Heidelberg-New York, 1971).

6. M. Hochster, "Generically perfect modules are strongly generically perfect", Proc. London Math.Soc. (3), 23 (1971), 477-488.

7. M. Hochster and J.A. Eagon, "Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci", Amer. J. Math., 53 (1971), 1020-1058.

8. C. Huneke, "Powers of ideals generated by weak d-sequences", J. Algebra, 68 (1981), 471-509.

9. T. Svanes, "Coherent cohomology on Schubert subschemes of flag schemes and applications", Adv. in Math., 14 (1974), 369-453.

Fachbereich 3, Naturwissenschaften, Mathematik,
Universität Osnabrück
- Abteilung Vechta -
Driverstrasse 22,
D-2848 Vechta, West Germany.