Castelnuovo–Mumford Regularity and Powers

Winfried Bruns, Aldo Conca, and Matteo Varbaro

1 Castelnuovo–Mumford Regularity Over General Base Rings

Castelnuovo–Mumford regularity was introduced in the early eighties of the twentieth century by Eisenbud and Goto in [12] and by Ooishi [18] as an algebraic counterpart of the notion of regularity for coherent sheaves on projective spaces discussed by Mumford in [19].

One of the most important features of Castelnuovo–Mumford regularity is that it can be equivalently defined in terms of (and hence it bounds) the vanishing of local cohomology modules, the vanishing of Koszul homology modules and the vanishing of syzygies.

This triple nature of Castelnuovo–Mumford regularity is usually stated for graded rings over base fields, but indeed it holds in general as we will show in this section.

Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a $\mathbb{N}$-graded ring with R_0 commutative and Noetherian. We assume that R is standard graded, i.e., it is generated as an R_0-algebra by finitely many elements $x_1, \ldots, x_n$ of degree 1. Let $S = R_0[X_1, \ldots, X_n]$ with $\mathbb{N}$-graded structure induced by the assignment $\deg X_i = 1$. The R_0-algebra map $S \to R$ sending X_i to x_i induces an S-module structure on R and hence on every R-module.

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W. Bruns
Universität Osnabrück, Institut für Mathematik, Osnabrück, Germany
e-mail: wbruns@uos.de

A. Conca · M. Varbaro (✉)
Dipartimento di Matematica, Universitá di Genova, Genova, Italy
e-mail: conca@dima.unige.it; varbaro@dima.unige.it

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Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module. Given $a \in \mathbb{Z}$ we will denote by $M(a)$ the module that it is obtained from M by shifting the degrees by a, i.e. $M(a)_i = M_{i+a}$.

The Castelnuovo–Mumford regularity of M is defined in terms of local cohomology modules $H^i_{Q_R}(M)$ with support on

$$Q_R = R_+ = (x_1, \dots, x_n).$$

For general properties of local cohomology modules we refer the readers to [2, 6, 13]. In our setting the module $H^i_{Q_R}(M)$ is $\mathbb{Z}$-graded and its homogeneous component $H^i_{Q_R}(M)_j$ of degree $j \in \mathbb{Z}$ vanishes for large j. The Castelnuovo–Mumford regularity of M or, simply, the regularity of M is defined as

$$\operatorname{reg}(M) = \max\{i + j : H^i_{Q_R}(M)_j \neq 0\}.$$

We may as well consider M as an S-module by means of the map $S \to R$ and local cohomology supported on

$$Q_S = (X_1, \dots, X_n).$$

Since $H^i_{Q_S}(M) = H^i_{Q_R}(M)$ the resulting regularity is the same.

Here we list some simple properties of regularity that we will freely use.

(1) $\operatorname{reg}(M(-a)) = \operatorname{reg}(M) + a$.
(2) $\operatorname{reg}(S) = 0$ because $H^n_{Q_S}(S) = (X_1 \cdots X_n)^{-1} R_0[X_1^{-1}, \dots, X_n^{-1}]$ and $H^i_{Q_S} = 0$ for all $i \neq n$.
(3) If $0 \to N \to M \to L \to 0$ is a short exact sequence of finitely generated graded R-modules with maps of degree 0 then :

$$\begin{aligned}\operatorname{reg}(N) &\le \max\{\operatorname{reg}(M), \operatorname{reg}(L) + 1\},\\ \operatorname{reg}(M) &\le \max\{\operatorname{reg}(L), \operatorname{reg}(N)\},\\ \operatorname{reg}(L) &\le \max\{\operatorname{reg}(M), \operatorname{reg}(N) - 1\}.\end{aligned}$$

A minimal set of generators of M is, by definition, a set of generators that is minimal with respect to inclusion. The number of elements in a minimal set of generators is not uniquely determined, but the set of the degrees of the elements in a minimal set of homogeneous generators of M is uniquely determined because it coincides with the set of $i \in \mathbb{Z}$ such that $[M/Q_R M]_i \neq 0$. So we have a well defined notion of largest degree of a minimal generator of M that we denote by $t_0(M)$, that is,

$$t_0(M) = \max\{i \in \mathbb{Z} : [M/Q_R M]_i \neq 0\}$$

if $M \neq 0$. We use t_0 because $M/Q_R M \simeq \operatorname{Tor}^R_0(M, R_0) = \operatorname{Tor}^S_0(M, R_0)$.

The following result establishes the crucial link between the regularity and the degree of generators of a module. It appears in [18, Thm.2], where it is attributed to Mumford, and it appears also in [2, Thm.16.3.1].

Lemma 1.1 $t_0(M) \le \operatorname{reg}(M)$.

Proof Let $v = t_0(M)$. Then the R_0-module $[M/Q_S M]_v$ is non-zero. Therefore there is a prime ideal P of R_0 such that $[M/Q_S M]_v$ localized at P is non-zero. In other words, the localization M' of M at the multiplicative set $R_0 \setminus P$ is a graded module over $(R_0)_P[X_1, \dots, X_n]$ with $t_0(M') = t_0(M)$. Since $\operatorname{reg}(M') \le \operatorname{reg}(M)$ we may assume right away that R_0 is local with maximal ideal, say, $\mathfrak{m}$. Similarly we may also assume that the residue field of R_0 is infinite. If $M = H^0_{Q_S}(M)$, the assertion is obvious. If $M \neq H^0_{Q_S}(M)$ then set $M' = M/H^0_{Q_S}(M)$. Clearly $t_0(H^0_{Q_S}(M)) \le \operatorname{reg}(M)$ and $\operatorname{reg}(M') \le \operatorname{reg}(M)$. Since $t_0(M) \le \max\{t_0(M'), t_0(H^0_{Q_S}(M))\}$ it is enough to prove the statement for M'. That is to say, we may assume that $\operatorname{grade}(Q_S, M) > 0$. Because the residue field of R_0 is infinite, there exists $L \in S_1 \setminus \mathfrak{m}S_1$ such that L is a non-zero-divisor on M. By a change of coordinates we may assume that $L = X_n$. The short exact sequence

$$0 \to M(-1) \to M \to \overline{M} = M/(X_n)M \to 0$$

implies that $\operatorname{reg}(\overline{M}) \le \operatorname{reg}(M)$ (it is actually equal but we do not need it). As $\overline{M}$ is a finitely generated graded module over $R_0[X_1, \dots, X_{n-1}]$, we may assume, by induction on the number of variables, that it is generated in degree $\le \operatorname{reg}(M)$. But then it follows easily that also M is generated in degree $\le \operatorname{reg}(M)$. □

Next we consider the (graded) Koszul homology $H(Q_R, M) = H(Q_S, M)$ and set:

$$\operatorname{reg}_1(M) = \max\{j - i : H_i(Q_R, M)_j \neq 0\}.$$

In this case, since $H_0(Q_R, M) \cong M/Q_R M$, the assertion

$$t_0(M) \le \operatorname{reg}_1(M)$$

is obvious. Now, let

$$\mathbb{F} : \cdots \to F_c \to F_{c-1} \to \cdots \to F_0 \to 0$$

be a graded S-free resolution of M, i.e., each F_i is a graded and S-free of finite rank, the maps have degree 0 and $H_i(F) = 0$ for all i with the exception of $H_0(\mathbb{F}) \simeq M$. We say that $\mathbb{F}$ is minimal if a basis of F_0 maps to a minimal set of homogeneous generators of M, a basis of F_1 maps to a minimal set of homogeneous generators of the kernel of $F_0 \to M$ and for $i \ge 2$ a basis of F_i maps to a minimal set of homogeneous generators of the kernel of $F_{i-1} \to F_{i-2}$.

If R_0 is a field then a (finite) minimal S-free resolution always exists and it is unique up to an isomorphism of complexes. For general R_0, it is still true that every module has a minimal free graded resolution but it is, in general, not finite and furthermore it is not unique up to an isomorphism of complexes.

Given a minimal graded S-free resolution $\mathbb{F}$ of M we set:

$$\operatorname{reg}_2(\mathbb{F}) = \max\{t_0(F_i) - i \ : \ i = 0, \ldots, n - \operatorname{grade}(Q_S, M)\}$$

and

$$\operatorname{reg}_3(\mathbb{F}) = \max\{t_0(F_i) - i : i \in \mathbb{N}\}.$$

Obviously we have $t_0(M) \le \operatorname{reg}_2(\mathbb{F}) \le \operatorname{reg}_3(\mathbb{F})$. We are ready to establish the following fundamental result:

Theorem 1.2 *With the notation above and for every minimal S-free resolution $\mathbb{F}$ of M, we have:*

$$\operatorname{reg}(M) = \operatorname{reg}_1(M) = \operatorname{reg}_2(\mathbb{F}) = \operatorname{reg}_3(\mathbb{F}).$$

Proof Set $Q = Q_S$ and $g = \operatorname{grade}(Q, M) = \min\{i : H^i_Q(M) \neq 0\}$.

We first prove that $\operatorname{reg}(M) \le \operatorname{reg}_1(M)$. We prove the statement by decreasing induction on g. Suppose $g = n$. The induced map $H^n_Q(F_0) \to H^n_Q(M)$ is surjective. Hence we have

$$\operatorname{reg}(M) \le \operatorname{reg}(F_0) = t_0(F_0) = t_0(M) = \max\{j : H_0(Q, M)_j \neq 0\} = \operatorname{reg}_1(M).$$

Now assume that $g < n$ and consider

$$0 \to M_1 \to F_0 \to M \to 0.$$

We have $\operatorname{grade}(Q, M_1) = g + 1$ and

$$\operatorname{reg}(M) \le \max\{\operatorname{reg}(F_0), \operatorname{reg}(M_1) - 1\}.$$

By induction $\operatorname{reg}(M_1) \le \operatorname{reg}_1(M_1)$. Since $H_i(Q, M_1) = H_{i+1}(Q, M)$ for $i > 0$ and

$$0 \to H_1(Q, M) \to H_0(Q, M_1) \to H_0(Q, F_0) \to H_0(Q, M) \to 0$$

is an exact sequence, we have

$$\operatorname{reg}_1(M_1) = \max\{j - i : H_i(Q, M_1)_j \neq 0\} = \max\{a, b\}$$

with $a = \max\{j : H_0(Q, M_1)_j \neq 0\}$ and $b = \max\{j - i : H_{i+1}(Q, M)_j \neq 0 \text{ and } i > 0\}$. So $b \leq \operatorname{reg}_1(M) + 1$ and, since $a \leq \max\{t_0(F_0), \max\{j : H_1(Q, M)_j \neq 0\}\}$, we have that $a \leq \operatorname{reg}_1(M) + 1$ as well. Hence

$$\operatorname{reg}_1(M_1) \leq \operatorname{reg}_1(M) + 1$$

and it follows that $\operatorname{reg}(M) \leq \operatorname{reg}_1(M)$.

Secondly we prove that $\operatorname{reg}_1(M) \leq \operatorname{reg}_2(\mathbb{F})$. Since

$$H_i(Q, M) = \operatorname{Tor}_i^S(M, R_0) = H_i(\mathbb{F} \otimes R_0)$$

we have that $H_i(Q, M)$ is a subquotient of $F_i \otimes R_0$ and hence

$$\max\{j : H_i(Q, M)_j \neq 0\} \leq t_0(F_i).$$

Furthermore, $H_i(Q, M) = 0$ if $i > n - g$. Therefore $\operatorname{reg}_1(M) \leq \operatorname{reg}_2(\mathbb{F})$.

That $\operatorname{reg}_2(\mathbb{F}) \leq \operatorname{reg}_3(\mathbb{F})$ is obvious by definition, so it remains to prove that $\operatorname{reg}_3(\mathbb{F}) \leq \operatorname{reg}(M)$. Set $M_0 = M$ and consider the exact sequence

$$0 \to M_{i+1} \to F_i \to M_i \to 0.$$

By the minimality of $\mathbb{F}$ we have $t_0(F_i) = t_0(M_i) \leq \operatorname{reg}(M_i)$. Hence

$$\operatorname{reg}(M_{i+1}) \leq \max\{t_0(F_i), \operatorname{reg}(M_i) + 1\} = \operatorname{reg}(M_i) + 1$$

for all $i \geq 0$. It follows that

$$t_0(F_i) = t_0(M_i) \leq \operatorname{reg}(M_i) \leq \operatorname{reg}(M) + i$$

for every i, that is,

$$t_0(F_i) - i \leq \operatorname{reg}(M),$$

in other words,

$$\operatorname{reg}_3(\mathbb{F}) \leq \operatorname{reg}(M).$$ □

Remark 1.3 Let $T \to R_0$ be any surjective homomorpism of unitary rings. It extends uniquely to $S' = T[X_1, \ldots, X_n] \to S = R_0[X_1, \ldots, X_n]$. Therefore a finitely generated graded R-module M can be regarded as a finitely generated graded S'-module. Hence the regularity of M can be computed also using a graded minimal free resolution as S'-module.

2 Bigraded Castelnuovo–Mumford Regularity

Assume now $R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$ is $\mathbb{N}^2$-graded with $R_{(0,0)}$ commutative and Noetherian and that R is generated as an $R_{(0,0)}$-algebra by elements $x_1,\dots,x_n,y_1,\dots,y_m$ with the x_i homogeneous of degree $(1,0)$ and the y_j homogeneous of degree (0, 1)

We will denote by $R^{(*,0)}$ the subalgebra $\bigoplus_i R_{(i,0)}$ of R and by $Q_{(1,0)}$ the ideal of $R^{(*,0)}$ generated by $R_{(1,0)}$ i.e., by $x_1,\dots,x_n$. Similarly $R^{(0,*)}$ is the subalgebra $\bigoplus_j R_{(0,j)}$ of R and $Q_{(0,1)}$ the ideal of $R^{(0,*)}$ generated by $R_{(0,1)}$ i.e., by $y_1,\dots,y_m$. We have (at least) three ways of getting an $\mathbb{N}$-graded structure out of the $\mathbb{N}^2$-graded structure:

(1) $(1,0)$-graded structure: the homogeneous component of degree $i\in\mathbb{N}$ is given by $R^{(i,*)} = \bigoplus_j R_{(i,j)}$. The degree 0 part is $R^{(0,*)}$ and the ideal of the homogeneous elements of positive degree is $Q_{(1,0)}R = (x_1,\dots,x_n)$.
(2) $(0,1)$-graded structure: the homogeneous component of degree $j\in\mathbb{N}$ is given by $R^{(*,j)} = \bigoplus_i R_{(i,j)}$. The degree 0 part is $R^{(*,0)}$ and the ideal of the homogeneous elements of positive degree is $Q_{(0,1)}R = (y_1,\dots,y_m)$.
(3) total degree: the homogeneous component of degree $u\in\mathbb{N}$ is $\bigoplus_{i+j=u} R_{(i,j)}$. The degree 0 part is $R_{(0,0)}$ and the ideal of the homogeneous elements of positive degree is $(x_1,\dots,x_n,y_1,\dots,y_m)$.

In the same way, any $\mathbb{Z}^2$-graded R-module $M = \bigoplus M_{(i,j)}$ can be turned into a $\mathbb{Z}$-graded module by regrading it with respect to the $(1,0)$-grading or with respect to the $(0,1)$-grading or with respect to the total degree.

We may hence consider the Castelnuovo–Mumford regularity of M with respect to any of these different graded structures. To distinguish them we will denote by $\operatorname{reg}_{(1,0)} M$ the regularity of M with respect to the $(1,0)$-graded structure and by $\operatorname{reg}_{(0,1)} M$ the regularity of M with respect to the $(0,1)$-graded structure.

Given $i,j\in\mathbb{Z}$ we set $M^{(i,*)} = \bigoplus_v M_{(i,v)}$ and $M^{(*,j)} = \bigoplus_v M_{(v,j)}$. Clearly $M = \bigoplus_i M^{(i,*)}$ as a $R^{(0,*)}$-graded module and $M = \bigoplus_j M^{(*,j)}$ as an $R^{(*,0)}$-graded module. Also, it is simple to check that, if M is a finitely generated $\mathbb{Z}^2$-graded module, then $M^{(i,*)}$ is a finitely generated $R^{(0,*)}$-graded module for all $i\in\mathbb{Z}$ and $M^{(*,j)}$ is a finitely generated $R^{(*,0)}$-graded module for all $j\in\mathbb{Z}$.

Let $S = R_{(0,0)}[X_1,\dots,X_n,Y_1,\dots,Y_m]$ with the $\mathbb{N}^2$-graded structure induced by the assignment $\deg X_i = (1,0)$ and $\deg Y_j = (0,1)$. We have:

Proposition 2.1 *Let M be a finitely generated $\mathbb{Z}^2$-graded R-module. Let $\mathbb{F}$ be a bigraded S-free minimal resolution of M. Let v_i be the largest integer v such that F_i has a minimal generator in degree $(v,*)$ and w_i be the largest integer w such that F_i has a minimal generator in degree $(*,w)$ Then we have*

$$\max\{\operatorname{reg} M^{(*,j)} : j\in\mathbb{Z}\} = \operatorname{reg}_{(1,0)} M = \max\{v_i - i : i = 0,\dots,n\},$$

$$\max\{\operatorname{reg} M^{(i,*)} : i\in\mathbb{Z}\} = \operatorname{reg}_{(0,1)} M = \max\{w_i - i : i = 0,\dots,m\},$$

where $\operatorname{reg} M^{(*,j)}$ *is the regularity as an* $R^{(*,0)}$*-graded module and* $\operatorname{reg} M^{(i,*)}$ *is the regularity as an* $R^{(0,*)}$*-graded module.*

Proof Set $Q = Q_{(1,0)}$, i.e. Q is the ideal of $R^{(*,0)}$ generated by $R_{(1,0)}$. The $(1,0)$-regularity of M is defined by means of the local cohomology $H^*_{QR}(M)$. We may regard M as an $R^{(*,0)}$-module, so that $H^c_{QR}(M) = H^c_Q(M) = \bigoplus_j H^c_Q(M^{(*,j)})$ for all c. This explains the first equality. For the second equality, by Theorem 1.2 $\operatorname{reg}_{(1,0)} M$ can be computed from any graded minimal free resolution of M as an $R^{(0,1)}[X_1, \dots, X_n]$-module but we have observed in Remark 1.3 that it can be as well computed from any minimal free resolution of M as an S-module. So a minimal bigraded resolution of M as S-modules serves to compute both the $(1,0)$ and the $(0,1)$ regularity. □

3 A Non-standard $\mathbb{Z}^2$-Grading

For later applications we will consider in this section a polynomial ring

$$A = A_0[Y_1, \dots, Y_g]$$

over a ring A_0 with a (non-standard) $\mathbb{Z}^2$-graded structure given by

$$\deg Y_j = (d_j, 1)$$

where $d_1, \dots, d_g \in \mathbb{N}$.

For every $\mathbb{Z}^2$-graded A-module $N = \bigoplus N_{(i,v)}$ and for every $v \in \mathbb{Z}$ we set

$$\rho_N(v) = \sup\{i \in \mathbb{Z} : N_{(i,v)} \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}.$$

We will study the behaviour of $\rho_N(v)$ as a function of v. We start with two general facts.

Lemma 3.1 *Given an chain of submodules* $0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_p = N$ *of* $\mathbb{Z}^2$*-graded A modules one has* $\rho_N(v) = \max\{\rho_{N_i/N_{i-1}}(v) : i = 1, \dots, p\}$ *for all* v.

Proof The function $\rho_N(v)$ behaves well on short exact sequences with maps of degree 0. Then the statement follows by induction on p using the short exact sequences associated to the chain of submodules. □

Let F be a finitely generated $\mathbb{Z}^2$-graded free A-module with basis $e_1, \dots, e_p$ and let $<$ be a monomial order on F. For every $\mathbb{Z}^2$-graded A-submodule U of F we denote by $\operatorname{in}_<(U)$ the A_0-submodule of F generated by leading monomials (with coefficients!) of the non-zero elements in U. Since U is an A-submodule of F, it turns out that $\operatorname{in}_<(U)$ is an A-submodule of F as well. Furthermore for every monomial $aY^\alpha e_i$ in $\mathrm{in}_<(U)$ there exists an element $u \in U$ such that $\mathrm{in}_<(u) = aY^\alpha e_i$. One has:

Lemma 3.2 $\rho_{F/U}(v) = \rho_{F/\mathrm{in}_<(U)}(v)$ *for all* v.

Proof It is enough to prove that, given (i, v), one has $U_{(i,v)} = F_{(i,v)}$ if and only if $\mathrm{in}_<(U_{(i,v)}) = F_{(i,v)}$ The "only if" implication is obvious. For the "if" implication, we argue by contradiction. Suppose $\mathrm{in}_<(U_{(i,v)}) = F_{(i,v)}$ and $U_{(i,v)} \neq F_{(i,v)}$. Let $Y^\alpha e_i$ be the smallest (with respect to the monomial order) monomial of degree (i, v) which is not in $U_{(i,v)}$. Since $Y^\alpha e_i \in \mathrm{in}_<(U_{(i,v)})$ there exists $u \in U$ such that $\mathrm{in}_<(u) = Y^\alpha e_i$. We may assume that u is homogeneous of degree (i, v). If not, we simply replace u with the homogeneous component of u of degree (i, v) which is in U since U is graded. So we have $u = Y^\alpha e_i + u_1$ where u_1 is a A_0-linear combination of monomials of degree (i, v) that are $< Y^\alpha e_i$. Hence, by assumption, $u_1 \in U_{(i,v)}$. It follows that $Y^\alpha e_i = u - u_1 \in U_{(i,v)}$, a contradiction. □

The fact that A has no elements of degree $(i, 0) \in \mathbb{Z}^2$ with $i \neq 0$ has an important consequence.

Lemma 3.3 *Let N be a $\mathbb{Z}^2$-graded and finitely generated A-module. Then $\rho_N(v)$ is eventually either a linear function of v with leading coefficient in $\{d_1, \dots, d_g\}$ or $-\infty$.*

Proof First we observe that if n is a generator of N of degree, say, $(\alpha, \beta) \in \mathbb{Z}^2$, then $Y_1^{\alpha_1} \cdots Y_g^{\alpha_g} n$ has degree $(\sum_j \alpha_j d_j + \alpha, \sum_j \alpha_j + \beta)$. Hence $N_{(i,v)}$ is non-zero only if $(i, v) = (\sum_j \alpha_j d_j + \alpha, \sum_j \alpha_j + \beta)$ for some $(\alpha_1, \dots, \alpha_g) \in \mathbb{N}^g$ and some (α, β) degree of a minimal generator of N. If we set $D = \max\{d_1, \dots, d_g\}$, then $N_{(i,v)} \neq 0$ implies $\alpha \le i \le (v - \beta)D + \alpha$ for some degree (α, β) of a minimal generator of N. As the module N is finitely generated, it follows that $\{i \in \mathbb{Z} : N_{(i,v)} \neq 0\}$ is finite for every $v \in \mathbb{Z}$. To prove that $\rho_N(v)$ is either eventually linear in v or $-\infty$, we present N as F/U where F is a finitely generated A-free bigraded module and U is a bigraded A-submodule of F. Let $<$ be a monomial order on F. Then $\rho_{F/U}(v) = \rho_{F/\mathrm{in}_<(U)}(v)$. Hence we may assume right away that U is generated by monomials (with coefficients). We can consider a bigraded chain of submodules

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_p = N$$

with cyclic quotients $C_i = N_i/N_{i-1}$ annihilated by a monomial prime ideal, i.e., an ideal of the form $pA + J$ where p is a prime ideal of A_0 and J is an ideal generated by a subset of the variables $Y_1, \dots, Y_g$. It follows that

$$\rho_N(v) = \max\{\rho_{C_i}(v) : i = 1, \dots, p\}.$$

Since the maximum of finitely many eventually linear functions in one variable is an eventually linear function, it is enough to prove the statement for each C_i. That is, we may assume that, up to a shift $(-w_1, -w_2) \in \mathbb{Z}^2$, the module N has the form A/P with $P = pA + J$ where p is a prime ideal of A_0 and J is generated by a subset of the variables. With $G = \{i : Y_i \notin P\}$, we have

$$\rho_N(v) = \begin{cases} \max\{d_i : i \in G\}(v - w_2) + w_1 & \text{if } G \neq \emptyset \text{ and } v \geq w_2, \\ -\infty & \text{if } G = \emptyset \text{ and } v > w_2. \end{cases}$$

□

4 Regularity and Powers

We return to the notation of Sect. 1. For a finitely generated graded R-module M and a homogeneous ideal I of R we will study the behaviour of $\operatorname{reg}(I^v M)$ as a function of $v \in \mathbb{N}$. For simplicity we will assume throughout that $I^v M \neq 0$ for every v. Let us consider the Rees algebra $\operatorname{Rees}(I)$ of I:

$$\operatorname{Rees}(I) = \bigoplus_{v \in \mathbb{N}} I^v$$

with its natural bigraded structure given by

$$\operatorname{Rees}(I)_{(i,v)} = (I^v)_i.$$

The Rees module of the pair I, M

$$\operatorname{Rees}(I, M) = \bigoplus_{v \in \mathbb{N}} I^v M$$

is clearly a finitely generated $\operatorname{Rees}(I)$-module naturally bigraded by

$$\operatorname{Rees}(I, M)_{(i,v)} = (I^v M)_i.$$

Let $f_1, \ldots, f_g$ be a set of minimal homogeneous generators of I of degrees, say, $d_1, \ldots, d_g \in \mathbb{N}$. We may present $\operatorname{Rees}(I)$ as a quotient of

$$B = R[Y_1, \ldots, Y_g]$$

via the map

$$\psi : B \to \operatorname{Rees}(I), \quad Y_i \to f_i \in I_{d_i} = \operatorname{Rees}(I)_{(d_i,1)}.$$

Actually B is naturally bigraded if we assign bidegree $(i, 0)$ to $x \in R_i$ as an element of B and by set $\deg Y_j = (d_j, 1)$.

Consider the extension $Q_R B$ of Q_R to B and the Koszul homology

$$H(Q_R B, \text{Rees}(I, M)) = H(Q_R, \text{Rees}(I, M)) = \bigoplus_{v \in \mathbb{N}} H(Q_R, I^v M).$$

Since $Q_R H(Q_R, \text{Rees}(I, M)) = 0$ the module $H(Q_R, \text{Rees}(I, M))$ acquires naturally the structure of finitely generated $\mathbb{Z}^2$-graded $B/Q_R B$-module. Here

$$B/Q_R B = R_0[Y_1, \dots, Y_g]$$

has a bigraded structure defined in Sect. 3. Now for $i = 0, \dots, n$ we let

$$t_i(M) = \sup\{j : H_i(Q_R, M)_j \neq 0\}.$$

We have:

Theorem 4.1 *Let I be a homogeneous ideal of R minimally generated by homogeneous elements of degree $d_1, \dots, d_g$ and M be a finitely generated graded R-module. Then there exist $\delta \in \{d_1, \dots, d_g\}$ and $c \in \mathbb{Z}$ such that*

$$\text{reg}(I^v M) = \delta v + c \textit{ for } v \gg 0.$$

Proof For $i = 0, \dots, n$ consider the i-th Koszul homology module:

$$H_i = H_i(Q_R, \text{Rees}(I, M)) = \bigoplus_{v \in \mathbb{N}} H_i(Q_R, I^v M).$$

As already observed H_i is a finitely generated $\mathbb{Z}^2$-graded $B/Q_R B$-module. Furthermore $\rho_{H_i}(v) = t_i(I^v M)$. Therefore we may apply Lemma 3.3 and have that either $H_i(Q_R, I^v M) = 0$ for large v or $t_i(I^v M)$ is a linear function of v for large v with leading coefficient in $\{d_1, \dots, d_g\}$. As $\text{reg}(I^v M) = \max\{t_i(I^v M) - i : i = 0, \dots, n\}$ we may conclude that $\text{reg}(I^v M)$ is eventually a linear function in v with leading coefficient in $\{d_1, \dots, d_g\}$. □

Theorem 4.1 has been proved in [11] and [17] when R is a polynomial ring over a field and in [21] for general base rings. Our proof is a modification (and a slight simplification) of the one given in [11]. Here and also in Sect. 2 our work was largely inspired by the papers of Chardin on the subject, in particular by [7–10]. The δ appearing in Theorem 4.1 can be characterized in terms of minimal reductions, see [17, 21] for details. The nature of the others invariants arising from Theorem 4.1, i.e., the constant term c and the least v_0 such that the formula holds for each $v \geq v_0$, have been deeply investigated in [1, 8, 10, 14, 15] and are relatively well understood in small dimension but remain largely unknown in general.

5 Linear Powers

Assume now that the minimal generators of I have all degree d and that the minimal generators of M have all degree d_0. Hence $I^v M$ is generated by elements of degree $vd + d_0$ and therefore $\operatorname{reg}(I^v M) \geq vd + d_0$ for every v.

Definition 5.1 We say that I has linear powers with respect to M if $\operatorname{reg}(I^v M) = vd + d_0$ for every v.

When R_0 is a field, I has linear powers with respect to M if and only if for every v the matrices representing the maps in the minimal S-free resolution of $I^v M$ have entries of degree 1.

We will give a characterization of linear powers in terms of the homological properties of the Rees module $\operatorname{Rees}(I, M)$. Note that, under the current assumptions, $\operatorname{Rees}(I)$ and $\operatorname{Rees}(I, M)$ can be given a compatible and "normalized" $\mathbb{Z}^2$-graded structure:

$$\operatorname{Rees}(I)_{(i,v)} = (I^v)_{vd+i},$$
$$\operatorname{Rees}(I, M)_{(i,v)} = (I^v M)_{vd+d_0+i}.$$

From the presentation point of view, this amounts to set $\deg Y_i = (0, 1)$ so that $B = R[Y_1, \ldots, Y_g]$ is a $\mathbb{Z}^2$-graded R_0-algebra with generators in degree $(1, 0)$, the elements of R_1, and in degree $(0, 1)$, the Y_i's. With the notations introduced in Sect. 2, we have that $\operatorname{Rees}(I, M)^{(*,v)} = (I^v M)(vd+d_0)$. So, applying Proposition 2.1:

$$\operatorname{reg}_{(1,0)} \operatorname{Rees}(I, M) = \max\{\operatorname{reg} \operatorname{Rees}(I, M)^{(*,v)} : v \in \mathbb{N}\} = \max\{\operatorname{reg} I^v M - vd - d_0 : v \in \mathbb{N}\}.$$

Summing up we have:

Theorem 5.2

(1) $\operatorname{reg} I^v M \leq vd + d_0 + \operatorname{reg}_{(1,0)} \operatorname{Rees}(I, M)$ *for all v and the equality holds for at least one v.*
(2) *I has linear powers with respect to M if and only if* $\operatorname{reg}_{(1,0)} \operatorname{Rees}(I, M) = 0$.

When R is the polynomial ring over a field and $M = R$ Theorem 5.2 part (2) has been proved in [5] extending earlier results of Römer [20].

Theorems 5.2 and 4.1 have been generalized to the case where the single ideal I is replaced with a set of ideals $I_1, \ldots, I_p$ and one looks at the regularity $\operatorname{reg}(I_1^{v_1} \cdots I_p^{v_p} M)$ as a function of $(v_1, \ldots, v_p) \in \mathbb{N}^p$. The main difference is that $\operatorname{reg}(I_1^{v_1} \cdots I_p^{v_p} M)$ is (only) a piecewise linear function unless each ideal I_i is generated in a single degree, see [3, 4, 16] for details.

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