F-Rationality of Determinantal Rings and Their Rees Rings

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Let X be an $m \times n$ matrix ($m \leq n$) of indeterminates over a field K of positive characteristic, and denote the ideal generated by its t-minors by I_t. We show that the Rees ring $\mathcal{R}(I_t)$ of $K[X]$, as well as the algebra A_t generated by the t-minors, are F-rational if char $K > \min(t, m-t)$. Without a restriction on characteristic this holds for $K[X]/I_{r+1}$ and the symbolic Rees ring $\mathcal{R}^s(I_t)$. The determinantal ring $K[X]/I_{r+1}$ is actually F-regular, as was previously proved by Hochster and Huneke [13] and Conca and Herzog [5] through different approaches.

Our main tool is the filtration induced by the straightening law. The associated graded ring with respect to this filtration is typically given by a Segre product $K[H] \#_{\mathbb{N}^m} F(X)$, where H is a normal semigroup representing the weights of the standard bitableaux present in the object under consideration, $\mathbb{N}^m$ represents all the possible weights, and $F(X)$ parameterizes the set of standard bitableaux of $K[X]$. The ring $F(X)$ itself is the Segre product $F_1(X) \#_{\mathbb{N}^m} F_2(X)$, where $F_1(X)$ (resp. $F_2(X)$) are the coordinate rings of the flag varieties associated with X (resp. the transpose of X).

We prove that $F(X)$ is F-regular. Normal semigroup rings are also F-regular since they are direct summands of polynomial rings. Furthermore, F-regularity is inherited by Segre products, and F-rationality is preserved under deformations. Hence a ring with an associated graded ring of type $K[H] \#_{\mathbb{N}^m} F(X)$ is (at least) F-rational. This applies especially to $K[X]/I_{r+1}$, $\mathcal{R}(I_t)$, and A_t.

The results and the method of this paper are a variant of the method applied by Bruns [1] in characteristic 0, where F-rationality is to be replaced by the property of having rational singularities. By a theorem of Smith [15], our results in positive characteristic actually imply those previously obtained in characteristic 0.

1. The Filtration Induced by the Straightening Law

In this section we discuss the filtration on $K[X]$ induced by the straightening law and identify its associated graded ring. The filtration was first described by De Concini, Eisenbud, and Procesi [7]. We use the language of Young tableaux; for unexplained terminology the reader is referred to Bruns and Vetter [4, Sec. 11].

Received January 17, 1997.

The visit of the first author to the University of Genova that made this paper possible was supported by the Vigoni program of the DAAD and the CRUI.

Michigan Math. J. 45 (1998).

In order to simplify notation we assume in the following that the $m \times n$ matrix X of indeterminates has at least as many columns as rows. Let $(\Sigma \mid \mathrm{T})$ be a bitableau with u rows. Its *shape* is the sequence $r_1, \dots, r_u$ of its row lengths. We define its *weight* by $w(\Sigma \mid \mathrm{T}) = (w_1, \dots, w_m)$ where $w_i = |\{ j : r_j = i \}|$. The weights of the bitableaux therefore correspond bijectively to the elements of the semigroup $\mathbb{N}^m$. It is clear that the shape of $(\Sigma \mid \mathrm{T})$ depends only on its weight. Therefore, given $w \in \mathbb{N}^m$, we may set

$$\gamma_t = \gamma_t(w) = \gamma_t(r_1, \dots, r_u) = \sum_{i=1}^{u} (r_i - t + 1)_+,$$

where $(a)_+ = \max\{a, 0\}$. The collection of the functions γ_t defines a partial order $\preceq$ on the semigroup $\mathbb{N}^m$ by

$$w \preceq w' \iff \gamma_t(w) \le \gamma_t(w'), \quad t = 1, \dots, m.$$

Bitableaux with ascending rows correspond bijectively to products of minors if one interprets the entries of the left tableau as row indices and those of the right tableau as column indices. The functions γ_t can therefore be applied to products of minors. Standard bitableaux represent standard monomials of minors.

Let J_w be the K-subspace of $K[X]$ generated by all standard bitableaux of weight $w' \succeq w$, and let $J_w^{\succ}$ be generated by those of weight $w' \succ w$. We quote the straightening law on $K[X]$ as given in [7]; see also [4, 11.3].

THEOREM 1.1. *Let $(\Sigma \mid \mathrm{T})$ be a bitableau of weight w, and let Ξ_w be the initial tableau of weight w. Then $(\Sigma \mid \Xi_w)$ and $(\Xi_w \mid \mathrm{T})$ have standard representations*

$$(\Sigma \mid \Xi_w) = \sum_i a_i (\Sigma_i \mid \Xi_w) \quad \textit{and} \quad (\Xi_w \mid \mathrm{T}) = \sum_j b_j (\Xi_w \mid \mathrm{T}_j)$$

for $a_i, b_j \in K$. Furthermore, $(\Sigma \mid \mathrm{T}) \equiv \sum_{i,j} a_i b_j (\Sigma_i \mid \mathrm{T}_j) \bmod J_w^{\succ}$.

The K-vector subspaces J_w are in fact ideals of $K[X]$. This follows easily from Theorem 1.1. (Furthermore, $J_w = \bigcap I_t^{(\gamma_t(w))}$; we will discuss the symbolic powers $I_t^{(u)}$ in what follows.)

It will be useful to consider filtrations that are more general than those parameterized by the semigroup $\mathbb{N}$ of natural numbers with its natural partial order. Let H be an additive semigroup with partial order $\le$ (which is, of course, supposed to be monotone with respect to addition). Then an $(H, \le)$-filtration on a ring R is a family $\mathcal{F} = (J_h)_{h \in H}$ of ideals J_h satisfying the conditions $J_g J_h \subset J_{g+h}$ for all $g, h \in H$ and $J_g \subset J_h$ for all $g, h \in H$ with $g \le h$. We define the associated graded ring by

$$\operatorname{gr}_{\mathcal{F}} R = \bigoplus_{g \in H} \frac{J_g}{J_g^{>}}, \quad J_g^{>} = \sum_{h > g} J_h.$$

With its natural multiplication, $\operatorname{gr}_{\mathcal{F}} R$ is an H-graded ring. However, note that we can speak of the leading form of an element $x \in R$ only if there exists a unique $g \in H$ with $x \in J_g \setminus J_g^{>}$.

The semigroup $\mathbb{N}^m$ of weights is partially ordered by $\preceq$; as we shall see, Theorem 1.1 gives us the associated $\mathbb{N}^m$-graded ring. Let $F_1(X)$ denote the subalgebra of $K[X]$ generated by all tableaux $(\Xi_w \mid \mathrm{T})$ and $F_2(X)$ the subalgebra generated by the tableaux $(\Sigma \mid \Xi_w)$. Then $F_1(X)$ and $F_2(X)$ are multihomogeneous coordinate rings of flag varieties, and it follows immediately from Theorem 1.1 that they are sub-ASLs of $K[X]$ in a natural way. (See [4] for the notion of ASL.) Moreover, they are $\mathbb{N}^m$-graded K-algebras whose homogeneous elements of multidegree w are the linear combinations of standard bitableaux of weight w. By

$$F(X) = F_1(X) \#_{\mathbb{N}^m} F_2(X) = \bigoplus_{w \in \mathbb{N}^m} F_1(X)_w \otimes_K F_2(X)_w$$

we denote their Segre product (as $\mathbb{N}^m$-graded K-algebras).

Let $\mathcal{S}$ denote the $(\mathbb{N}^m, \preceq)$-filtration $(J_w)_{w \in \mathbb{N}}$. Since we want to use some theorems that are available only for $\mathbb{N}$-filtrations (or could at best be formulated for $\mathbb{N}^m$ with its product partial order), we coarsen the filtration $\mathcal{S}$. Let

$$\lambda(w) = \sum_{i=1}^{m} w_i \quad \text{and} \quad J_j = \sum_{\lambda(w) \geq j} J_w;$$

then we may also consider the associated graded ring of $K[X]$ with respect to the $\mathbb{N}$-filtration $\mathcal{T} = (J_j)$.

THEOREM 1.2. $\operatorname{gr}_{\mathcal{T}} K[X] \cong \operatorname{gr}_{\mathcal{S}} K[X] \cong F(X)$.

Proof. The polynomial ring $K[X]$ has a K-basis given by the set of standard monomials. The ideals J_w and J_j are spanned over K by subsets of the standard basis, and the filtrations are separated. Therefore $\operatorname{gr}_{\mathcal{T}} K[X]$ and $\operatorname{gr}_{\mathcal{S}} K[X]$ are isomorphic to $K[X]$ as graded K-vector spaces, where in both cases the isomorphism maps a linear combination of standard monomials in $K[X]$ to the same linear combination of the initial forms of the standard monomials. (Note that a product of minors has an initial form in $\operatorname{gr}_{\mathcal{S}} K[X]$.)

Let * denote leading forms with respect to $\mathcal{S}$. It follows easily from the definition of weight that $(\delta_1 \cdots \delta_s)^* = \delta_1^* \cdots \delta_s^*$ for all minors $\delta_1, \ldots, \delta_s$ of X. In conjunction with the fact that the leading forms of the standard monomials form a K-basis, it is clear that the products $\delta_1^* \cdots \delta_s^*$ with $\delta_1 \leq \ldots \leq \delta_s$ constitute a K-basis of $\operatorname{gr}_{\mathcal{S}} K[X]$. Passing to leading forms in Theorem 1.1, we obviously obtain a straightening law on $\operatorname{gr}_{\mathcal{S}} K[X]$ that makes it an ASL over K. Furthermore Theorem 1.1 shows that the straightening law coincides with that of $F(X)$.

The very same arguments apply to $\operatorname{gr}_{\mathcal{T}} K[X]$. Additionally one need only notice that $w \prec w'$ implies $\lambda(w) < \lambda(w')$. (It follows that $\operatorname{gr}_{\mathcal{T}} K[X]$ has an $\mathbb{N}^m$-gradation that refines its $\mathbb{N}$-gradation.) $\square$

2. *F*-Regularity of Flag Varieties

In the sequel we will apply the following theorems of tight closure theory; R is always supposed to be a Noetherian ring of characteristic p.

(TC1) R regular $\Rightarrow$ R F-regular $\Rightarrow$ R weakly F-regular $\Rightarrow$ R F-rational $\Rightarrow$ R normal and Cohen–Macaulay (provided R is a homomorphic image of a Cohen–Macaulay ring) (Hochster and Huneke [12, (3.4), (4.2)]).

(TC2) A direct summand S of a F-regular ring R is F-regular (Hochster and Huneke [11, (4.12)]). (We say that a ring S is a *direct summand* of a ring T if there exists an injective ring homomorphism $\varphi\colon S \to T$ such that $\varphi(S)$ is a direct S-module summand of T.)

(TC3) If the associated graded ring of R with respect to a separated $\mathbb{N}$-filtration is F-rational, then R is F-rational (Conca, Herzog, and Valla [6, 2.3]).

(TC4) A Gorenstein F-rational ring is F-regular [12, (4.7)].

(TC5) If an excellent ring R is F-regular, then so is $R[X]$ [12, (7.31)].

(TC6) Let R_0 and S_0 be finitely generated $\mathbb{Z}$-algebras such that $R_0 \otimes_{\mathbb{Z}} K$ and $S_0 \otimes_{\mathbb{Z}} K$ are weakly F-regular for all fields K. Then $(R_0 \otimes_{\mathbb{Z}} K) \otimes_K (S_0 \otimes_{\mathbb{Z}} K)$ is weakly F-regular for all fields K. (This follows immediately from [12, (7.45)].)

It is important for us that the K-algebra $F(X)$ is F-regular. We first prove an auxiliary result about subalgebras of the homogeneous coordinate ring $G(X)$ of the Grassmannian variety of m-dimensional vector subspaces of K^n. As a K-algebra, $G(X)$ is generated by the m-minors of X. These form a distributive lattice $\Gamma(X)$ on which $G(X)$ is an ASL.

PROPOSITION 2.1. *Suppose that K is a field of characteristic $p > 0$. Let Λ be a subset of $\Gamma(X)$ for which the subalgebra $K[\Lambda]$ of $G(X)$ is a sub-ASL of $G(X)$. Then Λ is a sublattice and $K[\Lambda]$ is F-rational.*

Proof. According to Sturmfels [16, 3.2.9], the initial algebra $\mathrm{in}_\tau(G(X))$ with respect to a diagonal term order is generated by the diagonal monomials of the maximal minors. (We refer the reader to Eisenbud [8] for term orders and related notions.) This is not hard (to prove and) to generalize. In fact, if $\delta = \delta_1 \cdots \delta_s$ is a standard monomial of maximal minors, then $\mathrm{in}_\tau(\delta)$ is just the product of the diagonal monomials of the δ_i and δ can obviously be reconstructed from $\mathrm{in}_\tau(\delta)$. It follows that there is no cancellation of initial terms in linear combinations of standard monomials, and therefore $\mathrm{in}_\tau(K[\Lambda])$ is generated by the diagonal monomials of the elements of Λ.

Let μ and λ be incomparable elements of Λ. The only possibility for the product of their diagonals to occur on the right-hand side of the straightening relation is through the product $(\lambda \sqcap \mu)(\lambda \sqcup \mu)$. (By $\sqcap$ and $\sqcup$ we denote the lattice operations in $\Gamma(X)$.) Hence this product must appear on the right-hand side with coefficient 1. In particular it follows that Λ is a sublattice of $\Gamma(X)$.

For each distributive lattice $(L, \sqcup, \sqcap)$, the K-algebra

$$A_L = K[X_l : l \in L]\big/(X_k X_l - X_{k\sqcap l} X_{k\sqcup l},\ k, l \in L)$$

is an affine normal semigroup ring (Hibi [10]). The equations defining A_Λ hold for the initial monomials of the maximal minors $\lambda \in \Lambda$ as well. We thus have a surjective homomorphism $A_L \to \mathrm{in}_\tau(K[\Lambda])$. It is in fact an isomorphism, since the Hilbert functions of A_Λ, $K[\Lambda]$, and $\mathrm{in}_\tau(K[\Lambda])$ coincide. This holds for A_Λ and $K[\Lambda]$ since they are ASLs on isomorphic graded posets, and it holds for $K[\Lambda]$ and $\mathrm{in}_\tau(K[\Lambda])$ by the general properties of the initial algebra.

Thus $\mathrm{in}_\tau(K[\Lambda])$ is a normal affine semigroup ring. That makes it a direct summand of a polynomial ring over K (see e.g. Bruns and Herzog [3, 6.1.10]) and so it is F-regular by (TC2) and especially F-rational. By (TC3), $K[\Lambda]$ is also F-rational. $\square$

THEOREM 2.2. *Suppose that K is a field of characteristic $p > 0$. Then the K-algebras $F_1(X)$, $F_2(X)$, $F_1(X) \otimes_K F_2(X)$, and $F(X)$, as well as their polynomial extensions, are F-regular.*

Proof. Let $\tilde{X}$ be a $m \times (n + m)$ matrix. Then one has a "natural" surjective homomorphism $G(\tilde{X}) \to K[X]$ whose kernel is generated by $\varepsilon \pm 1$, where ε is the maximal element of $\Gamma(\tilde{X})$ (see [4, Sec. 4]). Under this homomorphism, the partially ordered set $\Delta(X)$ of minors of X corresponds bijectively to the partially ordered set $\Gamma(\tilde{X}) \setminus \{\varepsilon\}$. Let Λ be the subset of $\Gamma(\tilde{X})$ corresponding to the poset of minors generating $F_1(X)$. Then it is not hard to see that Λ indeed satisfies the hypothesis of Proposition 2.1. In fact, the straightening law in $K[X]$ is derived from that in $G(\tilde{X})$ by substituting ± 1 for ε. Because all terms in the straightening relations of $F_1(X)$ have two factors, such a substitution cannot occur when one goes from $K[\Lambda]$ to $F_1(X)$; in other words, $K[\Lambda] \cap (\varepsilon \pm 1) = 0$. It follows that $F_1(X)$ is isomorphic to $K[\Lambda]$ and is therefore F-rational.

Let U be the group of unipotent lower triangular $m \times m$ matrices. A matrix $M \in U$ acts on $K[X]$ by the substitution $X \mapsto MX$. The ring of invariants $K[X]^U$ is just $F_1(X)$, as follows immediately from [4, 11.6]. Since U is a connected algebraic group without nontrivial characters, $F_1(X)$ is a factorial K-algebra (see e.g. [4, 6.5.5]); furthermore, it is Cohen–Macaulay (as follows for example from the F-rationality) and is therefore Gorenstein. Such F-rational algebras are F-regular (TC4). The same argument applies to $F_2(X)$.

Note that the ASLs $F_1(X)$ and $F_2(X)$ can be defined over $\mathbb{Z}$. Therefore (TC6) implies that their tensor product is weakly F-regular. Since it is also factorial, F-regularity is guaranteed by (TC4).

The Segre product $F(X)$ is a direct summand of the tensor product. Thus it is F-regular, too. The assertion on polynomial extensions follows at once from (TC5). $\square$

3. *F*-Rationality of Determinantal Rings and Their Rees Rings

Before turning to the investigation of Rees rings, we draw a consequence of Theorem 2.2 for the determinantal rings $K[X]/I_{r+1}$ (previously proved by Hochster and Huneke [13, (7.14)] and, as part of more general results, by Glassbrenner and Smith [9] and Conca and Herzog [5, 5.2] through different approaches).

THEOREM 3.1. *Let K be a field of characteristic $p > 0$. Then $K[X]/I_{r+1}$ is F-regular.*

Proof. The standard basis of $R = K[X]/I_{r+1}$ is given by the standard bitableaux (which now represent residue classes of products of minors) with at most r columns. The filtration induced by $\mathcal{S}$ on R is effectively an $(\mathbb{N}^r, \preceq)$-filtration, and the associated graded ring is $F_1(X') \# F_2(X'')$, where X' consists of the first r rows of X and X'' consists of the first r columns. The same holds for the filtration induced by $\mathcal{T}$. Statement (TC3) thus implies that R is F-rational.

If X is a square matrix, then R is Gorenstein by a theorem of Svanes (see e.g. [3, 7.3.5]) and hence is F-regular. In the general case it is enough to note that R is a direct summand of $\tilde{R} = K[X]/I_{r+1}(\tilde{X})$, where $\tilde{X}$ is a $n \times n$ matrix of indeterminates. In fact, one obviously has a sequence $R \to \tilde{R} \to R$ whose composition is the identity. □

In the following we want to apply a filtration argument to a Rees ring, an object that itself is defined in terms of a filtration. Given an $\mathbb{N}$-filtration $\mathcal{G} = (I_i)$ on a ring R, we set $\mathcal{R}(\mathcal{G}, R) = \bigoplus_{i \in \mathbb{N}} I_i T^i$. Let $(H, \le)$ be a partially ordered semigroup, and let $\mathcal{F} = (J_h)$ be an $(H, \le)$-filtration. Then we can define filtrations $(\tilde{J}_h)$ and $(\tilde{I}_i)$ on $\mathcal{R}(\mathcal{G}, R)$ and $\operatorname{gr}_{\mathcal{F}} R$ (respectively) by setting

$$\tilde{J}_h = \bigoplus_{i \in \mathbb{N}} I_i \cap J_h \quad \text{and} \quad \tilde{I}_i = \bigoplus_{h \in H} \frac{J_h \cap (I_i + J_h^>)}{J_h^>}.$$

We again denote these filtrations by $\mathcal{F}$ and $\mathcal{G}$, respectively.

LEMMA 3.2. *Let R be a ring* (*a K-algebra*), *and let $\mathcal{F} = (J_h)_{h \in H}$ and $\mathcal{G} = (I_i)_{i \in \mathbb{N}}$ be filtrations on R. Then $\operatorname{gr}_{\mathcal{F}}(\mathcal{R}(\mathcal{G}, R))$ and $\mathcal{R}(\mathcal{G}, \operatorname{gr}_{\mathcal{F}} R)$ are isomorphic as bigraded rings* (*K-algebras*).

Proof. The bidegree (h, i) component of $\operatorname{gr}_{\mathcal{F}}(\mathcal{R}(\mathcal{G}, R))$ is $(I_i \cap J_h)/(I_i \cap J_h^>)$, and the corresponding component of $\mathcal{R}(\mathcal{G}, \operatorname{gr}_{\mathcal{F}} R)$ is

$$\frac{J_h \cap (I_i + J_h^>)}{J_h^>} \cong \frac{(J_h \cap I_i) + J_h^>}{J_h^>} \cong \frac{J_h \cap I_i}{J_h^> \cap I_i}.$$

Thus the two objects are naturally isomorphic as abelian groups (K-vector spaces). It is easily verified that the isomorphism respects multiplication. □

In the cases of interest, $\operatorname{gr}_{\mathcal{F}} R$ is a $\mathbb{N}^m$-graded ring, and the ideals forming the filtration $\mathcal{G}$ can be defined in terms of the filtration $\mathcal{F}$. Then the algebra $\mathcal{R}(\mathcal{G}, \operatorname{gr}_{\mathcal{F}} R)$ can be further analyzed: it is the Segre product of a Rees algebra of a semigroup ring with $\operatorname{gr}_{\mathcal{F}} R$.

Let H be a semigroup, and let $\mathcal{H} = (H_i)_{i \in \mathbb{N}}$ be a filtration of H by ideals H_i (i.e., $H_i + H \subset H_i$ and $H_i + H_j \subset H_{i+j}$ for all i, j). Then we also denote the filtration $(H_i K[H])_{i \in \mathbb{N}}$ of $K[H]$ by $\mathcal{H}$. Obviously $\mathcal{R}(\mathcal{H}, K[H])$ is itself H-graded in a natural way:

$$\mathcal{R}(\mathcal{H}, K[H])_h = \bigoplus_{(h,i)\,:\, h \in H_i} K X^h T^i.$$

Furthermore it is a semigroup ring: $\mathcal{R}(\mathcal{H}, K[H]) \cong K[\mathcal{R}(\mathcal{H}, H)]$, where

$$\mathcal{R}(\mathcal{H}, H) = \bigcup_i \{ (h, i) : h \in H_i \}$$

is the *Rees semigroup* of H with respect to the filtration $\mathcal{H}$.

As before, let $(H, \leq)$ be a partially ordered semigroup with filtration $(H_i)_{i\in\mathbb{N}}$ and let R be a ring with an $(H, \leq)$-filtration $\mathcal{F}$. We define a filtration $\mathcal{G} = (I_i)_{i\in\mathbb{N}}$ on R by putting $I_i = \bigoplus_{h\in H_i} R_h$. With these hypotheses, the following lemma holds.

LEMMA 3.3. $\operatorname{gr}_{\mathcal{F}}(\mathcal{R}(\mathcal{G}, R)) \cong \mathcal{R}(\mathcal{G}, \operatorname{gr}_{\mathcal{F}} R)) \cong \mathcal{R}(\mathcal{H}, K[H]) \#_H (\operatorname{gr}_{\mathcal{F}} R)$.

Proof. The first isomorphism is given by Lemma 3.2, and the second follows easily if one writes out the decomposition of $\mathcal{R}(\mathcal{G}, \operatorname{gr}_{\mathcal{F}} R)$ as the direct sum of its $(\mathbb{N} \times H)$-graded components:

$$\mathcal{R}(\mathcal{G}, \operatorname{gr}_{\mathcal{F}} R) = \bigoplus_{i\in\mathbb{N}} \bigoplus_{h\in H_i} (\operatorname{gr}_{\mathcal{F}} R)_h T^i;$$

$\mathcal{R}(\mathcal{H}, K[H]) \#_H \operatorname{gr}_{\mathcal{F}} R$ has the same decomposition. □

We now return to the polynomial ring, its determinantal ideals, the $(\mathbb{N}^m, \preceq)$-filtration $\mathcal{S}$, and its $\mathbb{N}$-coarsening $\mathcal{T}$.

THEOREM 3.4. *Suppose K is a field of characteristic $p > 0$, let $\mathcal{H} = (H_i)_{i\in\mathbb{N}}$ be a filtration of $\mathbb{N}^m$, and define the filtration $\mathcal{G} = (I_i)$ of $K[X]$ by $I_i = \sum_{w\in H_i} J_w$. Suppose that $\mathcal{R}(\mathcal{H}, K[\mathbb{N}^m])$ is a finitely generated normal K-algebra. Then the Rees algebra $\mathcal{R}(\mathcal{G}, K[X])$ is F-rational and therefore a normal Cohen–Macaulay K-algebra.*

Proof. Extending the filtrations $\mathcal{S}$ and $\mathcal{T}$ to $\mathcal{R}(\mathcal{G}, K[X])$, we obtain isomorphic K-algebras $\operatorname{gr}_{\mathcal{S}} \mathcal{R}(\mathcal{G}, K[X])$ and $\operatorname{gr}_{\mathcal{T}} \mathcal{R}(\mathcal{G}, K[X])$. This follows from the same argument as Theorem 1.2. It is therefore enough to show the F-rationality of $\operatorname{gr}_{\mathcal{S}} \mathcal{R}(\mathcal{G}, K[X])$. By virtue of Lemma 3.3 it is isomorphic to the Segre product of $S = \mathcal{R}(\mathcal{H}, K[\mathbb{N}^m])$ with $F(X)$. The Segre product is a direct summand of $S \otimes_K F(X)$. By hypothesis, S is a normal affine semigroup ring and therefore a direct summand of a polynomial ring over K. Altogether this shows that $S \#_{\mathbb{N}^m} F(X)$ is a direct summand of a polynomial extension of $F(X)$; it is therefore F-regular in view of Theorem 2.2. □

COROLLARY 3.5. *The symbolic Rees algebra $\mathcal{R}^s(I_t) = \bigoplus_{i=0}^{\infty} I_t^{(i)} T^i$ of $K[X]$ with respect to the ideal I_t is F-rational. The same holds for the “ordinary” Rees algebra $\mathcal{R}(I_t) = \bigoplus_{i=0}^{\infty} I_t^i T^i$ if* $\operatorname{char} K > \min(t, m - t)$.

Proof. Let $\lambda_1, \dots, \lambda_u$ be linear forms with nonnegative coefficients on $\mathbb{Q}^m$. Then the filtration $\mathcal{H} = (H_i)_{i\in\mathbb{N}}$ with $H_i = \{ z \in \mathbb{N}^m : \lambda_j(z) \geq i,\ i = 1, \dots, u \}$ satisfies the hypothesis of Theorem 3.4. In fact, $\mathcal{R}(\mathcal{H}, \mathbb{N}^m)$ is the subsemigroup $\tilde{H}$ of $\mathbb{N}^{m+1}$ consisting of all $z' \in \mathbb{N}^{m+1}$ such that

$$\lambda_i(z'_1,\ldots,z'_m) \geq z'_{m+1}, \quad i = 1,\ldots,u.$$

It is normal and finitely generated since it is the intersection of $\mathbb{Z}^{m+1}$ with finitely many rational halfspaces (see [3, 6.1]).

For the symbolic Rees algebra we set $u = 1$ and $\lambda_1 = \gamma_t$. Then the result follows since $I_t^{(i)}$ is spanned as a K-vector space by all standard bitableaux $(\Sigma \mid \mathrm{T})$ with $\gamma_t(\Sigma \mid \mathrm{T}) \geq i$.

For the ordinary Rees algebra we set $u = t$ and $\lambda_i = \gamma_i/(t-i+1)$ for $i = 1,\ldots,t$. The ideals I_t^i have the primary decomposition

$$I_t^i = \bigcap_{j=1}^{t} I_j^{(i(t-j+1))}$$

if $\operatorname{char} K = 0$ or $\operatorname{char} K > \min(t, m-t)$ (see [7] and [4]). In other words, I_t^i is spanned as a K-vector space by all standard bitableaux $(\Sigma \mid \mathrm{T})$ with $\gamma_j(\Sigma \mid \mathrm{T}) \geq i(t-j+1)$ for $j = 1,\ldots,t$. $\square$

REMARKS 3.6. (a) One can replace the polynomial ring $K[X]$ by a residue class ring $K[X]/I_{r+1}$ and modify all the data accordingly. The proof of Theorem 3.1 shows that no essential changes happen on the level of the associated graded rings, and Theorem 3.4 and Corollary 3.5 remain true after this adaptation to the more general situation. In particular, the (symbolic) Rees algebra of $K[X]/I_{r+1}$ with respect to I_t/I_{r+1} is F-rational. (Such a relative version is also available for Proposition 3.7.)

(b) It follows easily from Corollary 3.5 that the associated graded ring $\operatorname{gr}_{I_t} K[X]$ is Cohen–Macaulay if $\operatorname{char} K > \min(t, m-t)$.

(c) The ideals J_w are stable under the $\mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$ action on $K[X]$ (see [4, Sec. 11]). Hence this action can be naturally extended to the Rees algebra $\mathcal{R}(\mathcal{G}, K[X])$ of Theorem 3.4. One sees easily that $\mathcal{R}(\mathcal{H}, K[\mathbb{N}^m])$ is isomorphic to its ring of U-invariants, so we have deduced the F-rationality of the Rees algebra from the F-regularity of its ring of U-invariants in a manner similar to the case of characteristic 0 treated in [1]. (Here U must be chosen as the direct product of the group of unipotent lower triangular $m \times m$ matrices and that of the unipotent upper triangular $n \times n$ matrices.)

(d) The assertions about the F-rationality of the Rees rings can also be derived from the structure of the initial algebras that we have investigated in [2].

Our final result concerns the algebra generated by the t-minors.

PROPOSITION 3.7. *Let K be a field of characteristic $p > \min(m, m-t)$. Then the subalgebra A_t of $K[X]$ generated by the t-minors is F-rational and therefore a normal Cohen–Macaulay ring.*

Proof. Let $\mathcal{G}$ and $\mathcal{U}$ denote the restrictions of $\mathcal{S}$ and $\mathcal{T}$ (respectively) to A_t. Then one sees easily that $\operatorname{gr}_{\mathcal{U}} A_t \cong \operatorname{gr}_{\mathcal{G}} A_t \cong K[H] \# F(X)$, where H is the subsemigroup of $\mathbb{N}^m$ consisting of the weights w such that a bitableau of weight w belongs to A_t. It is not hard to show that H is a normal semigroup. $\square$

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