Journal of Symbolic Computation 68 (2015) 75–86

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

The computation of generalized Ehrhart series in Normaliz

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ARTICLE INFO

Article history:
Received 15 October 2013
Accepted 29 March 2014
Available online 19 September 2014

MSC:
52B20
13F20
14M25
91B12

Keywords:
Rational polytope
Integral
Generalized Ehrhart series

ABSTRACT

We describe an algorithm for the computation of generalized (or weighted) Ehrhart series based on Stanley decompositions as implemented in the offspring NmzIntegrate of Normaliz. The algorithmic approach includes elementary proofs of the basic results. We illustrate the computations by examples from combinatorial voting theory.

Let $M \subset \mathbb{Z}^n$ be an affine monoid endowed with a positive $\mathbb{Z}$-grading deg. Then the *Ehrhart* or *Hilbert series* is the generating function

$$E_M(t) = \sum_{x \in M} t^{\deg x} = \sum_{k=0}^{\infty} \#\{x \in M\colon \deg x = k\} t^k,$$

and $E(M,k) = \#\{x \in M\colon \deg x = k\}$ is the Ehrhart or Hilbert function of M (see Bruns and Gubeladze, 2009 for terminology and basic theory). It is a classical theorem that $E_M(t)$ is the power series expansion of a rational function of negative degree at $t_0 = 0$ and that $E(M,k)$ is given by a quasipolynomial

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http://dx.doi.org/10.1016/j.jsc.2014.09.004
0747-7171/

of degree $\operatorname{rank} M - 1$ with constant leading coefficient equal to the (suitably normed) volume of the rational polytope

$$P = \operatorname{cone}(M) \cap A_1$$

where $\operatorname{cone}(M) \subset \mathbb{R}^n$ is the cone generated by M and A_1 is the hyperplane of degree 1 points. See Beck and Robins (2007) for a gentle introduction to the fascinating area of Ehrhart series. In the following we assume that

$$M = \operatorname{cone}(M) \cap L$$

for a sublattice L of $\mathbb{Z}^n$. Then $E(M,k)$ counts the L-points in the multiple kP, and is therefore the Ehrhart function of P (with respect to L).

Monoids of the type just introduced are important for applications, and in some of them, like those discussed in Section 3, one is naturally led to consider *generalized* (or weighted) *Ehrhart series*

$$E_{M,f}(t) = \sum_{x \in M} f(x) t^{\deg x}$$

where f is a polynomial in n indeterminates. It is well-known that also the generalized Ehrhart series is the power series expansion of a rational function; see Baldoni et al. (2011; 2012).

In applications that involve strict linear inequalities M is to be replaced by $M' = M \cap (\operatorname{cone}(M) \setminus \mathscr{F})$ where $\mathscr{F}$ is a union of faces (not necessarily facets) of $\operatorname{cone}(M)$. Our approach covers this "semi-open" situation as well.

In 2012 we have implemented an offspring of Normaliz (Bruns et al., no date) called NmzIntegrate[1] that computes generalized Ehrhart series. The input polynomials of NmzIntegrate must have rational coefficients, and we assume that f is of this type although it is mathematically irrelevant. This note describes the computation of generalized Ehrhart series based on Stanley decompositions (Stanley, 1982). Apart from taking the existence of Stanley decompositions as granted, we give complete and very elementary proofs of the basic facts. They follow exactly the implementation in NmzIntegrate (or vice versa). The semi-open case mentioned above has already been implemented in the current development versions of Normaliz and NmzIntegrate. It will be contained in the next public version.

The generalized Ehrhart function is given by a quasipolynomial $q(k)$ of degree $\leq \deg f + \operatorname{rank} M - 1$, and the coefficient of $k^{\deg f + \operatorname{rank} M - 1}$ in $q(k)$ can easily be described as the integral of the highest homogeneous component of f over the polytope P. Therefore we have also included (and implemented) an approach to the computation of integrals of polynomials over rational polytopes in the spirit of the Ehrhart series computation. See Baldoni et al. (2012) and De Loera et al. (2013) for more sophisticated approaches. Our algorithm and its implementation in NmzIntegrate have been developed independently from LattE integrale (DeLoera et al., no date). It is a consequent extension of the Normaliz algorithm for the computation of ordinary Ehrhart series.

1. The computation of generalized Ehrhart series

Via a Stanley decomposition and substitution the computation of generalized Ehrhart series can be reduced to the case in which M is a free monoid, and for free monoids one can split off the variables of f successively so that one ends at the classical case $M = \mathbb{Z}_+$. We take the opposite direction, starting from $\mathbb{Z}_+$.

1.1. The monoid $\mathbb{Z}_+$

Let $M = \mathbb{Z}_+$. By linearity it is enough to consider the polynomials $f(k) = k^m$, $k \in \mathbb{Z}_+$, for which the generalized Ehrhart series is given by

[1] NmzIntegrate version 1.2 is available as part of the Normaliz 2.11 distribution.

$$\sum_{k=0}^{\infty} k^m t^{um}, \quad u = \deg 1,$$

and if necessary we can assume $u = 1$, substituting $t \mapsto t^u$ in the final result.

The rising factorials

$$(k+1)_m = (k+1)\cdots(k+m)$$

form a $\mathbb{Z}$-basis of the polynomial ring $\mathbb{Z}[k]$. Therefore we can write

$$k^m = \sum_{j=0}^{m} s_{m,j}(k+1)_j \tag{1.1}$$

and use that

$$\sum_{k=0}^{\infty}(k+1)_r t^k = \frac{d^r}{dt^r}\sum_{j=r}^{\infty}(t^j) = \frac{d^r}{dt^r}\sum_{j=0}^{\infty}(t^j) = \frac{d^r}{dt^r}\left(\frac{1}{1-t}\right) = \frac{r!}{(1-t)^{r+1}}. \tag{1.2}$$

Eqs. (1.1) and (1.2) solve our problem for $M = \mathbb{Z}_+$ and $f(k) = k^m$:

$$\sum_{k=0}^{\infty} k^m t^k = \frac{A_{m,u}(t)}{(1-t^u)^{m+1}}, \quad A_{m,u}(t) \in \mathbb{Z}[t]. \tag{1.3}$$

It is enough to compute $A_{m,1}(t)$ because $A_{m,u}(t) = A_{m,1}(t^u)$. One should note that $A_{m,u}$ is a polynomial of degree m. Therefore the rational function in (1.3) has negative degree.

Since the coefficient $s_{m,m}$ of $(k+1)_m$ in the representation of k^m is evidently equal to 1, we have

$$\sum_{k=0}^{\infty} k^m t^{um} = \frac{m!}{(1-t)^{m+1}} + \text{terms of smaller pole order at } t = 1. \tag{1.4}$$

Remark 1. The coefficients $s_{m,j}$ in (1.1) and the coefficients of the polynomials $A_{m,1}$ are well-known combinatorial numbers.

(a) $s_{m,j} = (-1)^{m-j} S(m+1, j+1)$ where $S(p,q)$ is the Stirling number of the second kind that counts the number of partitions of a p-set into q blocks. This follows immediately from the classical identity $k^{m+1} = \sum_{j=1}^{m+1}(-1)^{m+1-j} S(m+1, j)(k)_j$ (for example, see Stanley, 1986, 4.3, c).

(b) For $m = 0$ we have $A_{0,1} = 1$ and $A_{m,1} = \sum_{j=1}^{m} A(m, j)t^j$ for $m > 0$ where $A(m, j)$ is the Eulerian number (Stanley, 1986, 4.3, d).

1.2. The monoid $\mathbb{Z}_+^d$

Next we consider $M = \mathbb{Z}_+^d$. The crucial observation is that the problem is multiplicative for products of polynomials in disjoint variables. Suppose that $f(x) = g(y)h(z)$, $y = (x_1, \dots, x_r)$, $z = (x_{r+1}, \dots, x_d)$. Then

$$E_{M,f}(t) = \sum_{x \in \mathbb{Z}_+^d} f(x)t^{\deg x} = \left(\sum_{y \in \mathbb{Z}_+^r} g(y)t^{\deg y}\right)\left(\sum_{z \in \mathbb{Z}_+^{d-r}} h(z)t^{\deg z}\right) \tag{1.5}$$

by multiplication of power series.

In order to exploit (1.5) we split the last variable off,

$$f(x) = \sum_i f_i(x_1, \dots, x_{d-1})x_d^i,$$

and obtain

$$E_{M,f}(t)=\sum_i\Big(\Big(\sum_{x'\in\mathbb{Z}_+^{d-1}} f_i(x')t^{\deg x'}\Big)\Big(\sum_{k=0}^{\infty}k^i t^{ui}\Big)\Big)$$
$$=\sum_i\Big(\frac{A_{i,u}(t)}{(1-t^u)^{i+1}}\sum_{x'\in\mathbb{Z}_+^{d-1}} f_i(x')t^{\deg x'}\Big) \qquad (1.6)$$

with $u=\deg e_d$.

Applying this formula inductively allows us to eliminate all variables x_i and to end with the desired representation of $E_{\mathbb{Z}_+^d,f}(t)$.

Generalizing (1.4), let us consider the case in which f is a monomial, $f(x_1,\dots,x_d)=x_1^{m_1}\cdots x_d^{m_d}$, and $\mathbb{Z}_+^d$ is endowed with its *standard degree*, $\deg(x)=x_1+\cdots+x_d$. Then Eqs. (1.5) and (1.4) imply that

$$E_{M,f}(t)=\frac{m_1!\cdots m_d!}{(1-t)^{m_1+\cdots+m_d+d}}+\text{terms of smaller pole order at } t=1. \qquad (1.7)$$

1.3. Using the Stanley decomposition

We now turn to general $M\subset\mathbb{Z}^n$. Normaliz computes a triangulation Σ of $C=\operatorname{cone}(M)$ into full dimensional simplicial subcones σ. Moreover, it computes a *disjoint* decomposition

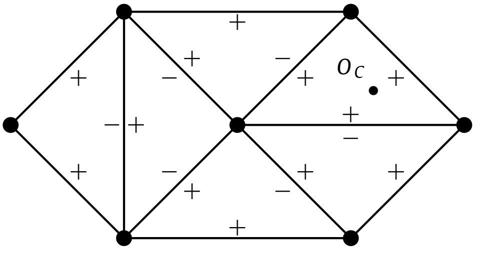
$$\operatorname{cone}(M)=\bigcup_{\sigma\in\Sigma}\sigma\setminus S_\sigma \qquad (1.8)$$

where S_σ is a union of facets of σ. The existence of such a decomposition is a nontrivial fact. Classically it is derived from the Brugesser–Mani theorem on the existence of line shellings (see Stanley, 1982). Instead of a line shelling, Normaliz (now) uses a method of Köppe and Verdoolaege that we describe in the following remark; also see Köppe and Verdoolaege (2008) and Bruns et al. (2014, Section 4). It is computationally much better than line shellings.

Remark 2. In order to compute the Stanley decomposition one starts with a vector O_C in the interior of one of the simplicial cones in the triangulation O_C that avoids all hyperplanes spanned by the facets of all $\sigma\in\Sigma$. In practice, one chooses O_C in the interior of the first simplicial cone σ in the triangulation and works with an infinitesimal perturbation; see Bruns et al. (2014, Section 4). For $\sigma\in\Sigma$ one then collects in S_σ all facets F of σ such that O_C and σ lie on different sides of the hyperplane through F.

It is important to note the following: S_σ is never the union of all facets of σ. This could only happen if $O_C\in-\sigma$. But since C is pointed we have $-\sigma\cap C=\{0\}$, and $O_C\neq 0$.

The following figure indicates the decomposition of a (cross-section of a) cone computed by the method just described.



Every simplicial subcone (of full dimension) is generated by linearly independent vectors $v_1, \dots, v_d \in M$, $d = \operatorname{rank} M$. They generate a free submonoid M_σ of M. For every σ Normaliz computes the set

$$E_\sigma = \{x \in \operatorname{gp}(M)\colon x = \alpha_1 v_1 + \cdots + \alpha_d v_d,\ \alpha_i \in [0,1)\}$$

where $\operatorname{gp}(M)$ denotes the group generated by M. For $x \in E_\sigma$ we let $\epsilon(x)$ be the sum of those v_i for which (i) $\alpha_i = 0$ and (ii) the facet of σ opposite to v_i lies in the excluded set S_σ: $\epsilon(x) \neq 0$ if and only if x lies in the excluded set, and the translation by $\epsilon(x)$ moves x out off S_σ. Then it is not hard to see that we have a disjoint decomposition

$$M = \bigcup_{\sigma \in \Sigma} \bigcup_{x \in E_\sigma} x + \epsilon(x) + M_\sigma. \tag{1.9}$$

It is called a *Stanley decomposition* since its existence is originally due to Stanley (1982).

In the following we set $\widetilde{x} = x + \epsilon(x)$ and

$$N_{\sigma,x} = \widetilde{x} + M_\sigma.$$

Then

$$E_{M,f}(t) = \sum_\sigma \sum_{x \in E_\sigma} E_{N_{\sigma,x},f}(t).$$

Set $d = \operatorname{rank} M$, and for given σ consider the linear map

$$\alpha_\sigma : \mathbb{Z}_+^d \to \mathbb{Z}^n, \quad \alpha_\sigma(y_1, \dots, y_d) = y_1 v_1 + \cdots + y_d v_d, \tag{1.10}$$

where $v_1, \dots, v_d$ is the generating set of M_σ as above. With

$$\begin{aligned} &\deg_\sigma y = \deg \alpha_\sigma(y), \\ &g_{\sigma,x}(y) = f(\alpha_\sigma(y) + \widetilde{x}), \end{aligned} \tag{1.11}$$

we have

$$E_{N_{\sigma,x},f}(t) = t^{\deg \widetilde{x}} \sum_{y \in \mathbb{Z}_+^d} g_{\sigma,x}(y) t^{\deg_\sigma y}. \tag{1.12}$$

For a precise estimate of the degrees arising in (1.12) one should note that

$$\deg \widetilde{x} < \deg v_1 + \cdots + \deg v_d. \tag{1.13}$$

Since $\widetilde{x} = \xi_1 v_1 + \cdots + \xi_d v_d$ with $0 \le \xi_i \le 1$ for $i = 1, \dots, d$, one must only exclude equality in (1.13). In fact, equality would only be possible with $\xi_i = 1$ for all i, and in its turn this would imply that S_σ consists of all facets of σ. However, this is impossible as observed in Remark 2.

Eq. (1.12) transforms the summation over $N_{\sigma,x}$ into a summation over $\mathbb{Z}_+^d$. Then we can apply (1.6) inductively to

$$\widetilde{E}_{\sigma,f}(t) = \sum_{x \in E_\sigma} E_{N_{\sigma,x},f}(t). \tag{1.14}$$

Finally, we sum the rational functions $\widetilde{E}_{\sigma,f}(t)$ over the triangulation Σ.

Remark 3. (a) Instead of applying (1.6) to every σ, we accumulate the polynomials $g_{\sigma,x}$ over all σ that induce the same degree $\deg_\sigma$ on $\mathbb{Z}^d$ (the classes formed in this way are called *denominator classes*).

(b) The time critical steps in the algorithm are

(1) the coordinate transformation (1.11), and
(2) the inductive application of (1.6).

In order to speed up (1), we factor the polynomial f, transform the factors separately, and multiply the transformed factors. If f happens to decompose into linear factors, then multiplication of linear polynomials becomes a time critical step. In order to speed up (2) we have introduced the denominator classes.

(c) Note that $\sum_{y\in\mathbb{Z}_+^d} g_{\sigma,x}(y)t^{\deg_\sigma y}$ is invariant under permutations of variables y_i that preserve the degrees $\deg_\sigma e_i$. Therefore one can go over $g_{\sigma,x}$ monomial by monomial and reorder the exponent vectors in such a way that the exponents of variables corresponding to the same degree become decreasing. The reordering significantly reduces the number of monomials in the polynomials to which (1.6) must be applied, saves memory and also speeds up (1.6).

(d) We want to point out that (1.6) is *not* applied recursively. Instead the right hand side is expanded after the elimination of x_d, and x_{d-1} is then eliminated from the resulting polynomial whose coefficients are rational functions in t. This procedure is repeated until all x_i have been eliminated.

1.4. *The semi-open case*

In applications like those sketched in Section 3 one is interested in counting lattice points in semi-open rational cones of type

$$C' = C \setminus \mathscr{F}$$

where $\mathscr{F}$ is a union of faces of C. The monoid $M = C \cap L$ is then to be replaced by its ideal

$$M' = C' \cap L.$$

One should note that counting lattice points in M' is intrinsically more difficult than counting those in M, even if $\mathscr{F}$ is a union of facets. For example, let C be the cone over the unit square, i.e.,

$$C = \mathbb{R}_+\{(0,0,1),(1,0,1),(1,1,1),(0,1,1)\}$$

and $\mathscr{F}$ be the union of the faces of C through two opposite edges of the square. Then the ordinary Ehrhart series of M' is $(2t^2 - t^3)/(1-t)^3$. This excludes a Stanley decomposition of type (1.9) which would give a numerator polynomial with nonnegative coefficients since the coefficient of t^k in the numerator just counts the elements $x + \epsilon(x)$ that have degree k. From the geometric viewpoint, the difficulty is demonstrated by the fact that C' has no decomposition of type (1.8): in at least one of the simplicial cones one must remove an edge without any of the two facets in which the edge is contained.

For this reason Normaliz and NmzIntegrate treat the semi-open case by inclusion–exclusion. This principle is applicable since taking Ehrhart series is additive in the sense of measure theory. But we do not go the most obvious way by computing the Ehrhart series for each of the involved faces and evaluating the sieve formula at the very end. Instead inclusion–exclusion is applied to all simplicial cones σ.

Let F be a face of C. By (1.9) we have

$$F \cap L = \bigcup_{\sigma\in\Sigma}\bigcup_{x\in E_\sigma} F \cap (\widetilde{x} + M_\sigma)$$

where, as above $\widetilde{x} = x + \epsilon(x)$. The face F is an extreme subset of C: $y + z \in F$ implies $y, z \in F$ for all $y, z \in C$. This fact makes the computation of $F \cap (\widetilde{x} + M_\sigma)$ very easy:

$$F \cap (\widetilde{x} + M_\sigma) = \begin{cases} \emptyset & \text{if } \widetilde{x} \notin F, \\ \widetilde{x} + (F \cap M_\sigma) & \text{otherwise,} \end{cases}$$

and $F \cap M_\sigma$ is simply the free submonoid of M_σ that is generated by those v_i that lie in F (notation as in (1.10)). This simple observation shows that the decomposition of C directly induces a decomposition of F into components whose ordinary Ehrhart series can easily be computed.

For generalized Ehrhart series this approach is even more advantageous: the expensive coordinate transformation in (1.12) needs to be done only once for $\widetilde{x}$ since it can simply be restricted to $F \cap \sigma$ by selecting those terms that do not contain any indeterminate representing a generator outside F.

However, one should note that the application of inclusion–exclusion to each simplicial cone usually increases the number of components that must be taken into account for the Ehrhart series of a face F since $\dim F \cap \sigma < \dim F$ in general, and the Stanley decomposition of $M \cap F$ is no longer full-dimensional.

2. The quasipolynomial, its virtual leading coefficient, and integration

2.1. The quasipolynomial

All rational functions in t that come up in (1.14) can be written over the denominator

$$\left(1 - t^\ell\right)^{\deg f + \operatorname{rank} M}$$

where ℓ is the least common multiple of the numbers $\deg x$ for the generators x of M that appear in the triangulation. This follows from (1.6) if one observes that $1 - t^u$ divides $1 - t^\ell$. Moreover, all summands have negative degree as rational functions in t, as follows from (1.13). Therefore Stanley (1986, 4.4.1) implies the following proposition.

Proposition 4.

$$E_{M,f}(t) = \sum_{k=0}^{\infty} q(k) t^k$$

where q is a rational quasipolynomial of period π dividing ℓ and of degree $\le \deg f + \operatorname{rank} M - 1$.

The statement about the quasipolynomial means that there exist polynomials $q^{(j)}$, $j = 0, \dots, \pi - 1$, of degree $\le \deg f + \operatorname{rank} M - 1$ such that

$$q(k) = q^{(j)}(k), \quad j \equiv k \ (\operatorname{mod} \pi),$$

and

$$q^{(j)}(k) = q_0^{(j)} + q_1^{(j)} k + \dots + q_{\deg f + \operatorname{rank} M - 1}^{(j)} k^{\deg f + \operatorname{rank} M - 1}$$

with coefficients $q_i^{(j)} \in \mathbb{Q}$. As we will see below, it is justified to call

$$\operatorname{ed}(M, f) = \deg f + \operatorname{rank} M - 1$$

the *expected degree* of q.

2.2. The virtual leading coefficient and Lebesgue integration

Let $m = \deg f$ and write $f = f_m + g$ where f_m is the degree m homogeneous component of f. Then $\deg g < m$, and it follows from Proposition 4 that g does not contribute to the coefficient $q_{\operatorname{ed}(M,f)}^{(j)}$. Moreover, this coefficient is independent of j and given by an integral, as we will see in Proposition 5 below.

For the representation as an integral we must norm the measure in such a way that it is compatible with the lattice structure. We will integrate over the polytope

$$P = \operatorname{cone}(M) \cap A_1, \quad A_1 = \left\{x \in \mathbb{R}^n : \deg x = 1\right\}.$$

Let $L_0 = L \cap \mathbb{R}M \cap A_0$ where $A_0 = \{x \in \mathbb{R}^n\colon \deg x = 0\}$ is the linear subspace of degree 0 elements. Then L_0 is a (saturated) sublattice of L of rank $d-1$ ($d = \operatorname{rank} M$), and we choose a basis $u_1, \dots, u_{d-1}$ of L_0. Note that $H = \mathbb{R}M \cap A_1$ has dimension $d-1$ and contains a point $z \in L$ since we have required that deg takes the value 1 on $\operatorname{gp}(M)$, and we can consider the *basic L_0-simplex* $\delta = \operatorname{conv}(z, z+u_1, \dots, z+u_{d-1})$ in H. Now we norm the Lebesgue measure λ on H by giving volume $1/(d-1)!$ to the basic L_0-simplex. (The measure is independent of the choice of δ since two basic L_0-simplices differ by an affine-integral automorphism of H.) We call λ the *L-Lebesgue measure* on H.

The following Propositions 5, 6 and 7 are quite elementary, as their proofs will show. They may have appeared elsewhere, and we do not claim originality for them.

Proposition 5. *For all $j = 0, \dots, \pi - 1$ one has*

$$q^{(j)}_{\mathrm{ed}(M,f)} = \int_P f_m \, d\lambda. \tag{2.1}$$

Proof. We may assume that f is homogeneous of degree m. Let

$$L_c = \frac{1}{c} L.$$

Then

$$\int_P f_m \, d\lambda = \lim_{c\to\infty} \sum_{x \in P \cap L_c} \frac{1}{c^{d-1}} f(x)$$

by elementary integration theory.

Note that

$$f(x) = \frac{1}{c^m} f(cx)$$

by homogeneity and that $x \in P \cap L_c$ if and only if $cx \in L \cap cP$. Thus

$$\int_P f_m \, d\lambda = \lim_{c\to\infty} \sum_{y \in cP \cap L} \frac{1}{c^{m+d-1}} f(y).$$

On the other hand, we obtain $q^{(j)}_{\mathrm{ed}(M,f)}$ as the limit over the subsequence $(b\pi + j)_{b \in \mathbb{Z}_+}$:

$$q^{(j)}_{\mathrm{ed}(M,f)} = \lim_{b\to\infty} \sum_{y \in (b\pi + j)P \cap L} \frac{1}{(b\pi + j)^{m+d-1}} f(y)$$

by Proposition 4. This concludes the proof. □

In view of Proposition 5 it is justified to call $q_{\mathrm{ed}(M,f)} = q^{(j)}_{\mathrm{ed}(M,f)}$ the *virtual leading coefficient*, and the proposition justifies the term "expected degree" for $\deg f + \operatorname{rank} M - 1$ the. In analogy with the definition of multiplicity in commutative algebra (for example, see Bruns and Herzog, 1998), we call

$$\operatorname{vmult}(M, f) = \operatorname{ed}(M, f)! q_{\mathrm{ed}(M,f)}$$

the *virtual multiplicity of* (M, f). It is an integer if P is a lattice polytope and f_m has integral coefficients, as we will see below.

2.3. Computing the integral

It is natural to compute the integral by summation over the triangulation: the triangulation of $\operatorname{cone}(M)$ into simplicial subcones σ induces a triangulation of the polytope P into simplices $\delta = \sigma \cap P$. As usual let $v_1, \dots, v_d \in M$ be the generators of σ. Then δ is spanned by the degree 1 vectors $v_i/\deg(v_i)$, $i = 1, \dots, n$. Let $e_1, \dots, e_d$ be the unit vectors in $\mathbb{R}^d$. Then the substitution $e_i \mapsto v_i/\deg(v_i)$ induces a linear map $\mathbb{R}^d \to \mathbb{R}M$ that in its turn restricts to an affine map α from the standard degree 1 hyperplane in $\mathbb{R}^d$ spanned by $e_1, \dots, e_d$ to the hyperplane $H = A_1 \cap \mathbb{R}M$, and the image of the unit simplex Δ is just δ.

Proposition 6. *One has*

$$\int_\delta f\, d\lambda = \frac{|\det_L(v_1, \dots, v_d)|}{\deg(v_1)\cdots\deg(v_d)} \int_\Delta (f \circ \alpha)\, d\mu \tag{2.2}$$

where μ is the $\mathbb{Z}^d$-Lebesgue measure on the hyperplane $\widetilde{H}$ of standard degree 1 in $\mathbb{R}^d$ and $\det_L(v_1, \dots, v_d)$ is the determinant of the coefficient matrix of $v_1, \dots, v_d$ with respect to a basis of $L \cap \mathbb{R}M$.

Proof. This is just the substitution rule if one observes that the absolute value of the functional determinant of $\alpha|\widetilde{H}$ is given by the factor in front of the integral. For an affine map the functional determinant is constant. So we can assume $f = 1$ and it remains to relate the volumes of δ and Δ. But Δ has volume $1/(d-1)!$ with respect to μ and δ has volume

$$\frac{1}{(d-1)!}\frac{|\det_L(v_1, \dots, v_d)|}{\deg(v_1)\cdots\deg(v_d)}$$

with respect to λ; see Bruns et al. (2014, Section 4). □

After the substitution it remains to evaluate the integral over Δ, and this can be done monomial by monomial:

Proposition 7.

$$\int_\Delta y_1^{m_1} \cdots y_d^{m_d}\, d\mu = \frac{m_1! \cdots m_d!}{(m_1 + \cdots + m_d + d - 1)!}. \tag{2.3}$$

Proof. Let $g = y_1^{m_1} \cdots y_d^{m_d}$ and $M = \mathbb{Z}_d^+$. Then

$$E_{M,g}(t) = \frac{m_1! \cdots m_d!}{(1-t)^{(m_1+\cdots+m_d+d)}} + \text{terms of smaller pole order at } t = 1,$$

as stated in (1.7).

The quasipolynomial is a true polynomial in this case, and the (virtual) multiplicity is given by the value of the numerator polynomial at $t = 1$, namely $m_1! \cdots m_d!$ (for example, see Bruns and Herzog, 1998, 4.1.9). Now Proposition 5 gives the integral. □

One can also derive the formula in Proposition 7 by iterated use of the classical Beta integral

$$\int_0^1 y_1^{m_1}(1-y_1)^{m_2}\, dy_1 = \frac{m_1! m_2!}{(m_1 + m_2 + 1)!} = \frac{\Gamma(m_1+1)\Gamma(m_2+1)}{\Gamma(m_1+m_2+2)};$$

see Andrews et al. (1999, Theorem 1.1.4).

Table 1
Inequalities expressing that A beats the other 3 candidates.

λ_1:	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	1	−1	1	1	−1	−1	1	−1
λ_2:	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	−1	−1	−1	1	1	1	−1	−1	−1
λ_3:	1	1	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1

3. Computational examples

We illustrate the use of NmzIntegrate by three related examples coming from combinatorial voting theory that are discussed in Schürmann (2013). We refer the reader to Lepelley et al. (2008), Schürmann (2013) or Wilson and Pritchard (2007) for a more extensive treatment.

Consider an election in which each of the k voters fixes a linear preference order of n candidates. In other words, voter i chooses a linear order of the candidates $1, \dots, n$. Each such order represents a permutation of $1, \dots, n$. Set $N = n!$. The result of the election is an N-tuple $(x_1, \dots, x_N)$ in which x_p is the number of voters that have chosen the preference order labeled p. Then $x_1 + \dots + x_N = k$, and $(x_1, \dots, x_N)$ can be considered as a lattice point in the positive orthant of $\mathbb{R}^N_+$, or, more precisely, as a lattice point in the simplex

$$U_k^{(n)} = \mathbb{R}^N_+ \cap A_k = k\left(\mathbb{R}^N_+ \cap A_1\right) = kU^{(n)}$$

where A_k is the hyperplane defined by $x_1 + \dots + x_N = k$, and $U^{(n)} = U_1^{(n)}$ is the unit simplex of dimension $N - 1$ naturally embedded in N-space. We assume that all lattice points in the simplex $U_k^{(n)}$ have equal probability of being the outcome of the election.

The following three problems have been considered in Schürmann (2013) for 4 candidates A, B, C, D:

(1) the Condorcet paradox,
(2) the Condorcet efficiency of plurality voting,
(3) plurality voting versus cutoff.

For $n = 4$ one has $N = 24$, and the dimension of the polytope $U^{(4)}$ is already quite large.

Let us say that candidate A *beats* candidate B if the number of voters that prefer candidate A to candidate B is larger than the number of voters with the opposite preference. Candidate A is the *Condorcet winner* if A beats all other candidates. As the Marquis de Condorcet noticed, the relation "beats" is nontransitive for some outcomes of the election, and there may be no Condorcet winner. This phenomenon is called the *Condorcet paradox*. Problem (1) asks for its asymptotic probability as the number k of voters goes to ∞, or even for the precise number of election results without a Condorcet winner, depending on the number k of voters.

It is not hard to see that the outcomes that have A as the Condorcet winner can be described by three homogeneous linear inequalities $\lambda_i(x) > 0$ whose coefficients are given in Table 1 (relative to the lexicographic order of the permutations of A, B, C, D). They cut out a rational polytope from $U^{(n)}$, and the probability of Condorcet's paradox can be computed from the volume of the polytope. Finding the precise number of election results without (or with) a Condorcet winner requires the computation of the Ehrhart function of the semi-open polytope.

Problems (2) and (3) can be described by similar systems of linear inequalities. Since version 2.8, Normaliz can indeed compute the volumes and the Ehrhart series in dimension 24 that arise from tasks (1), (2) and (3) although the triangulations to be evaluated for (2) and (3) are formidable (see Table 3 or Bruns et al., 2014).

As Schürmann (2013) observed, the computations can be considerably simplified by exploiting the symmetries in the inequalities: some variables share the same coefficients in each inequality, for example the first 6 variables in Table 1. Therefore they can be replaced by their sum, and the replacement constitutes a projection of the original polytopes, monoids or cones onto objects of smaller dimension. For the Condorcet paradox the system of inequalities reduces to Table 2. However, instead of simply counting lattice points, one must now count them with their numbers of preimages. These

Table 2
Inequalities exploiting the symmetries in Table 1.

1	−1	1	1	1	−1	−1	−1
1	1	−1	1	−1	1	−1	−1
1	1	1	−1	−1	−1	1	−1

Table 3
Computation times (real) for Ehrhart series in dimension 24.

Computation	Triangulation size	Real time
Condorcet paradox	1,473,107	00:00:30 h
Condorcet efficiency	347,225,775,338	218:13:55 h
Plurality vs. cutoff	257,744,341,008	175:11:26 h

Table 4
Computation times (real) for symmetrized data.

Computation	Rank	deg f	# Triangulation / # Stanley dec	Normaliz time	Gen. Ehrhart series time	Lead. coeff. time
Condorcet paradox	8	16	17 / 21	0.01 s	2.3 s	0.04 s
– semi-open					2.5 s	
Condorcet efficiency	13	11	17,953 / 23,453	0.34 s	1:53 h	22 min
– semi-open					2:02 h	
Plurality vs. cutoff	6	18	3 / 4	0.01 s	8.1 s	0.09 s
– semi-open					13.6 s	

are given by polynomials, namely products of binomial coefficients. In our example the polynomial is

$$\binom{y_1+5}{5}(y_2+1)(y_3+1)(y_4+1)(y_5+1)(y_6+1)(y_7+1)\binom{y_8+5}{5}$$

where $y_1 = x_1 + \cdots + x_6$ etc. In other words, the Ehrhart function (or the volume) of a high dimensional polytope is replaced by a generalized Ehrhart function of a polytope of much lower dimension (or the virtual leading coefficient of the quasipolynomial).

A priori it may not be clear that the replacement of combinatorial complexity in high dimension by multivariate polynomial arithmetic in low dimension pays dividends, but this is indeed the case. Tables 3 and 4 compare both approaches. The computations in Table 3 and the Condorcet efficiency in Table 4 were run on a SUN xFire 4450 with 20 parallel threads. The other computations in Table 4 were done on the same machine, but serially.

If the computations in Table 3 are restricted to volumes, they become faster by a factor of approximately 3. The times are given for the Ehrhart series of the closed polytopes. For the semi-open versions one must approximately add another 30%, but we hope that a refined implementation will reduce the extra time somewhat.

The last 3 columns of Table 4 list the times for the following computations: (i) the time Normaliz needs for the computation of the Stanley decomposition, (ii) the time in which NmzIntegrate 1.2 computes the generalized Ehrhart series, and (iii) the NmzIntegrate time for the leading coefficient. Whether the extra computation time for the semi-open case can be further improved is not yet clear.

A welcome side effect of the computations of the generalized Ehrhart functions is that they have confirmed the results obtained by Normaliz.

J. Jeffries, J. Montaño and M. Varbaro (2013) have applied NmzIntegrate for the evaluation of integrals that compute certain multiplicities. A typical example is

$$\int\limits_{\substack{[0,1]^m \\ \sum x=t}} (x_1 \cdots x_m)^{n-m} \prod_{1\le i<j\le m} (x_j - x_i)^2 \mathrm{d}\mu,$$

taken over the intersection of the unit cube in $\mathbb{R}^m$ and the hyperplane of constant coordinate sum t. It is supposed that $t \le m \le n$. For $t = 2$, $m = 4$, $n = 6$ the computation time is $\ll 1$ s.

Remark 8. While NmzIntegrate accepts polynomials with rational coefficients as input, in version 1.2 all internal computations are based on integers of the CoCoALib type `BigInt` that is essentially a wrapper for the GMP type `mpz_class`. The use of integral arithmetic is possible since a common denominator can be computed beforehand.

Version 1.0 had used rational arithmetic instead. The change from rational to integer arithmetic has saved about 50 % of the computation time.

Acknowledgements

We gratefully acknowledge the support we received from John Abbott and Anna Bigatti in using CoCoALib (Abbott and Bigatti, no date), on which the multivariate polynomial algebra in NmzIntegrate is based. Christian Krattenthaler kindly provided reference (Andrews et al., 1999) for the Beta integral.

The first author thanks the Mathematical Sciences Research Institute, Berkeley CA for support and hospitality during Fall 2012 when the first version of this paper was written.

Both authors thank the Deutsche Forschungsgemeinschaft for support of the Normaliz project through the SPP 1489 "Experimental methods in algebra, geometry and number theory".

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