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STANLEY DECOMPOSITIONS AND HILBERT DEPTH IN THE KOSZUL COMPLEX

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ABSTRACT. Stanley decompositions of multigraded modules M over polynomial rings have been discussed intensively in recent years. There is a natural notion of depth that goes with a Stanley decomposition, called the *Stanley depth*. Stanley conjectured that the Stanley depth of a module M is always at least the (classical) depth of M. In this paper we introduce a weaker type of decomposition, which we call *Hilbert decomposition*, since it only depends on the Hilbert function of M, and an analogous notion of depth, called *Hilbert depth*. Since Stanley decompositions are Hilbert decompositions, the latter set upper bounds to the existence of Stanley decompositions. The advantage of Hilbert decompositions is that they are easier to find. We test our new notion on the syzygy modules of the residue class field of $K[X_1, \dots, X_n]$ (as usual identified with K). Writing $M(n,k)$ for the k-th syzygy module, we show that the Hilbert depth of $M(n,1)$ is $\lfloor (n+1)/2 \rfloor$. Furthermore, we show that, for $n > k \geq \lfloor n/2 \rfloor$, the Hilbert depth of $M(n,k)$ is equal to $n-1$. We conjecture that the same holds for the Stanley depth. For the range $n/2 > k > 1$, it seems impossible to come up with a compact formula for the Hilbert depth. Instead, we provide very precise asymptotic results as n becomes large.

1. Introduction. In recent years *Stanley decompositions* of multigraded modules over polynomial rings $R = K[X_1, \dots, X_n]$ have been discussed intensively. Such decompositions, introduced by Stanley in [**14**], break the module M into a direct sum of graded vector subspaces, each of which is of type Sx where x is a homogeneous element and $S = K[X_{i_1}, \dots, X_{i_d}]$ is a polynomial subalgebra. Stanley conjectured that one can always find such a decomposition in which $d \geq \operatorname{depth} M$ for each summand. (For unexplained terminology of commutative algebra we refer the reader to [**3**].)

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One says that M has *Stanley depth* m, $\operatorname{Stdepth} M = m$, if one can find a Stanley decomposition in which $d \geq m$ for each polynomial subalgebra involved, but none with m replaced by $m+1$. With this notation, Stanley's conjecture says $\operatorname{Stdepth} M \geq \operatorname{depth} M$.

In this paper we introduce a weaker type of decomposition in which we no longer require the summands to be submodules of M, but only vector spaces isomorphic to polynomial subrings. Evidently, such decompositions depend only on the Hilbert series of M, and therefore they are called *Hilbert decompositions.* The *Hilbert depth* $\operatorname{Hdepth} M$ is defined accordingly.

Since Stanley decompositions are Hilbert decompositions, the latter set upper bounds to the existence of Stanley decompositions, and since they are easier to find, one may try to construct a Stanley decomposition by appropriately modifying a "good" Hilbert decomposition.

Moreover, our discussion shows that it is worthwhile to also consider the standard grading along with the multigrading, as already suggested implicitly by Stanley, who allows arbitrary gradings in his conjecture. In order to distinguish multigraded and standard graded invariants, we use the indices n and 1, respectively. All this is made precise in Section 2. In addition, in the same section, we collect several useful results from the literature that are proved in a concise way, some in an extended form.

While most papers are devoted to the case in which multigraded components have K-dimension ≤ 1 (and in which Hilbert decompositions and Stanley decompositions coincide under a mild hypothesis), we test our notions on the syzygy modules of the residue class field of $K[X_1, \dots, X_n]$ (as usual identified with K). The Stanley depth of the first syzygy module, the maximal ideal $\mathfrak{m} = (X_1, \dots, X_n)$, was found by Biró et al.: $\operatorname{Stdepth} \mathfrak{m} = \lfloor (n+1)/2 \rfloor$. By the standard inductive approach to the Koszul complex, it is then easily shown that the k-th syzygy module $M(n,k)$ has multigraded Stanley depth $\geq \lfloor (n+k)/2 \rfloor$.

Further investigations reveal a significant difference between the "lower" syzygy modules $M(n,k)$, $1 < k < \lfloor n/2 \rfloor$, and the "upper" ones. For the upper ones, one can easily determine the multigraded Hilbert depth: if $\lfloor n/2 \rfloor \leq k < n$, then $\operatorname{Hdepth}_n M(n,k) = n-1$, which is the best possible value for a nonfree module. We believe that the multigraded Stanley depth has the same value and show that this holds for $k = n - 3$ (in addition to $k = n - 2, n - 1$). In the lower range, it seems impossible to find a simple expression even for Hdepth_1, since the binomial sum that must be evaluated for a precise bound (see Proposition 3.7 and Remark 3.8) cannot be summed in closed form. The best we can offer in Section 4 (apart from experimental values for $n \le 22$) is asymptotic estimates for $\mathrm{Hdepth}_1 M(n,k)$ as n becomes large. We consider two "regimes": if k is fixed and n tends to ∞, then Theorem 4.1 provides a rather precise asymptotic approximation, showing in particular that the lower bound $\lfloor (n+k)/2 \rfloor \sim n/2$ has the correct leading asymptotic order, although it is still rather far away from the true value. This changes, if both k and n tend to ∞ at a fixed rate: as we show in Theorem 4.5, in that case $\mathrm{Hdepth}_1 M(n,k) \sim \varepsilon n$ with $\varepsilon > 1/2$, where ε depends on the ratio between k and n. In particular, again, this turns out to be much larger than the corresponding value provided by the lower bound $\lfloor (n+k)/2 \rfloor$ (see Remarks 4.6 (2)).

2. Stanley decompositions and Hilbert depth. We consider the polynomial ring $R = K[X_1, \dots, X_n]$ over a field K and two graded structures on R:

(1) the *multigrading*, more precisely, the $\mathbf{Z}^n$-grading in which the degree of X_i is the i-th vector e_i of the canonical basis;

(2) the *standard grading* over $\mathbf{Z}$ in which each X_i has degree 1.

All R-modules are assumed to be finitely generated.

In order to treat both cases in a uniform way, we use *graded retracts* of R, namely subalgebras $S \subset R$ such that there exists a graded epimorphism $\pi : R \to S$ with $\pi | S = \mathrm{id}$. In the multigraded case, these retracts are the subalgebras generated by a subset of the indeterminates, and, in the standard graded case, they are the subalgebras generated by a set of 1-forms.

Definition 2.1. Let M be a finitely generated graded R-module. A *Stanley decomposition* of M is a finite family

$$\mathcal{D} = (S_i, x_i)_{i \in I},$$

in which x_i is a homogeneous element of M and S_i is a graded K-

algebra retract of R for each $i \in I$ such that $S_i \cap \operatorname{Ann} x_i = 0$, and

$$M = \bigoplus_{i \in I} S_i x_i$$

as a graded K-vector space.

While M, is not decomposed as an R-module in the definition, the direct sum itself carries the structure of an R-module and has a well-defined depth. Following Herzog et al. [**8**] we make the following definition.

Definition 2.2. The *Stanley depth* $\operatorname{Stdepth} M$ of M is the maximal depth of a Stanley decomposition of M. (For convenience, we set $\operatorname{Stdepth} 0 = \infty$.)

In the following we will use the index n in order to denote invariants associated with the multigrading, and the index 1 for those associated with the standard grading. If no index appears in a statement, then it applies to both cases.

Remark 2.3. Stanley [**14**] introduced decompositions as in Definition 2.1 and conjectured that

$$\operatorname{Stdepth} M \geq \operatorname{depth} M \tag{2.1}$$

for all modules M. However, one should note that the decompositions considered by us are more special than Stanley's since he allows arbitrary gradings on the polynomial ring.

The reason for our more restrictive definition is that we want the denominators of the Hilbert series of the rings S_i to divide the denominator of the Hilbert series of R.

It is not hard to see that Stanley's conjecture holds in the standard graded case, at least for infinite fields. It was actually proved by Baclawski and Garsia [**1**] before the conjecture was made; see also Theorem 2.7. For the multigraded case, Stanley decompositions have recently been investigated in several papers: Biró et al. [**2**], Cimpoeaş [**5**], Herzog et al. [**8**], Popescu [**10**] and Rauf [**11**].

From the combinatorial viewpoint, a module is often only an algebraic substrate of its Hilbert function, and we may ask what decompositions a given Hilbert function can afford.

Definition 2.4. Under the same assumptions on R and M as above, a *Hilbert decomposition* is a finite family

$$\mathcal{H} = (S_i, s_i)_{i \in I},$$

such that $s_i \in \mathbf{Z}^m$ (where $m = 1$ or $m = n$, respectively, depending on whether we are in the standard graded or in the multigraded case), S_i is a graded K-algebra retract of R for each $i \in I$, and

$$M \cong \bigoplus_{i \in I} S_i(-s_i)$$

as a graded K-vector space.

A Stanley decomposition breaks M into a direct sum of submodules over suitable subalgebras, whereas for a Hilbert decomposition we only require an isomorphism to the direct sum of modules over such subalgebras. Clearly, Hilbert decompositions of M depend only on the Hilbert function of M. As for Stanley decompositions, we can define $\operatorname{depth} \mathcal{H}$.

Definition 2.5. The *Hilbert depth* $\operatorname{Hdepth} M$ of M is the maximal depth of a Hilbert decomposition of M.

Weakening Stanley's conjecture, one may ask whether

$$\operatorname{Hdepth} M \geq \operatorname{depth} M, \tag{2.2}$$

or, equivalently,

$$\operatorname{Hdepth} M = \max\{\operatorname{depth} N : H(N, \_) = H(M, \_)\}. \tag{2.3}$$

(Here $H(N, \_)$ denotes the Hilbert function of N, $H(N, g) = \dim_K N_g$ for all $g \in \mathbf{Z}^m$.) It is clear that (2.3) implies (2.2), and the converse holds since M and N share all Hilbert decompositions. Moreover, a positive answer to Stanley's conjecture would evidently imply (2.2).

Hilbert series, in the standard as well as in the multigraded case, are rational functions of type

$$H_M(T) = \frac{Q_M(T)}{(1-T)^n}$$

and

$$H_M(T_1, \dots, T_n) = \frac{Q_M(T_1, \dots, T_n)}{(1-T_1)\cdots(1-T_n)},$$

respectively, where $Q_M(T) \in \mathbf{Z}[T^{\pm 1}]$ and $Q_M(T_1, \dots, T_n) \in \mathbf{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ are Laurent polynomials. A Hilbert decomposition in the standard graded case amounts to a representation of the numerator in the form

$$Q_M(T) = \sum_j q_j(T)\,(1-T)^{t_j},$$

where q_j is a Laurent polynomial with *positive* coefficients. Then the depth of the decomposition is $n - \max_j t_j$. In the multigraded case, it amounts to a representation

$$Q_M(T_1, \dots, T_n) = \sum_j q_j(T_1, \dots, T_n) \prod_{i \in I_j} (1 - T_i),$$

where the I_j's are subsets of $\{1, \dots, n\}$, and the polynomials q_j are nonzero and have nonnegative coefficients. Here, the depth of the decomposition is $n - \max_j |I_j|$.

Consider the following example: $R = K[X, Y]$, $M = K \oplus YR/(X) \oplus YR$. Then

$$H_M(T) = H_R(T) = \frac{1}{(1-T)^2}$$

and

$$H_M(T_1, T_2) = \frac{1 - T_1 + T_2}{(1-T_1)(1-T_2)} = \frac{1}{1-T_2} + \frac{T_2}{(1-T_1)(1-T_2)}.$$

It follows immediately that $\mathrm{Hdepth}_1 M = 2$ and $\mathrm{Hdepth}_2 M = 1$, whereas $\mathrm{Stdepth}_1 M = \mathrm{Stdepth}_2 M = 0$.

The following example shows that $\mathrm{Stdepth}_n M < \mathrm{Stdepth}_1 M$ in general. Let $R = K[X,Y,Z]$ and $M = R/(XZ, YZ, Z^2)$. Then $\mathrm{Stdepth}_3 M = 0$ by Remark 2.14 below, since $\operatorname{depth} M = 0$ and $\dim_K M_a \leq 1$ for all $a \in \mathbf{Z}^3$. On the other hand,

$$M \cong k[X] \cdot 1 + k[X] \cdot Y + k[Y] \cdot (Y+Z) + k[X,Y] \cdot XY^2,$$

is a $\mathbf{Z}$-Stanley decomposition. Hence $\mathrm{Stdepth}_1 M \geq 1 = \mathrm{Hdepth}_1 M$. To sum up, $\mathrm{Stdepth}_1 M = 1 > 0 = \mathrm{Stdepth}_3 M$.

A priori, it is not clear that Stanley or Hilbert decompositions exist at all. In the multigraded case one can use a standard filtration argument. Under much more general assumptions, M has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_q = M$$

in which each quotient M_{i+1}/M_i is isomorphic to a shifted copy $R/\mathfrak{p}_i(-m_i)$ of a residue class ring modulo a graded prime ideal $\mathfrak{p}_i$. In the multigraded case, this fact establishes the existence of Stanley decompositions, since each of the prime ideals $\mathfrak{p}_i$ is generated by a subset of $X_1, \ldots, X_n$.

Proposition 2.6. *Let*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of graded R-modules. If U and N have Stanley decompositions, then so does M, and

$$\operatorname{Stdepth} M \geq \min(\operatorname{Stdepth} U, \operatorname{Stdepth} N).$$

The same statements apply to Hilbert decompositions and depth.

Proof. For Hilbert decompositions, the statement is completely trivial since M and $U \oplus N$ have the same Hilbert function. For Stanley decompositions, it is only necessary to lift the generators in a Stanley decomposition of N to homogeneous preimages in M. $\square$

In the standard graded case, a filtration as above does not yield a Stanley decomposition since the residue class rings $R/\mathfrak{p}_i$ fail to be retracts in general. This failure is however compensated by the existence of Noether normalizations in degree 1, provided K is infinite. By the following theorem of Baclawski and Garsia [**1**], Stanley decompositions exist in the standard graded case, at least under a mild restriction, and inequality (2.1) holds. For the convenience of the reader, we include the short proof.

Theorem 2.7. *Let K be an infinite field. Then, in the standard graded case, every R-module $M \neq 0$ has a Stanley decomposition, and*

$$\operatorname{Stdepth}_1 M \geq \operatorname{depth} M.$$

Proof. If $\dim M = 0$, the assertion is trivial, since M is a finite-dimensional K-vector space and K is a retract of R.

Now suppose that $\dim M > 0$. Note that for every graded R-module there exists a homogeneous system of parameters $y_1, \dots, y_d$, $d = \dim M$, in degree 1. The essential point is that $y_1, \dots, y_d$ generate a retract S of R. Since all graded retracts of S are graded retracts of R, and since $\operatorname{depth}_S M = \operatorname{depth}_R M$, we can replace R by S. In other words, we may assume that $\dim M = n$.

If $\operatorname{depth} M = n$, then M is a free R-module, and the claim is again obvious. Suppose that $\operatorname{depth} M < n$. Since $\dim M = \dim R$, M contains a free graded R-submodule F of rank equal to $\operatorname{rank} M$. Since $\operatorname{depth} M/F = \operatorname{depth} M$, but $\dim M/F < \dim M$, we can apply induction. $\square$

In the standard graded case, Hilbert decompositions were considered by Uliczka [**16**]. Among other things, he proved that

$$\operatorname{Hdepth} M = n - \min\{u : Q_M(T)/(1-T)^u \text{ is positive}\}. \tag{2.4}$$

Here $Q_M(T)$ is the numerator polynomial of the Hilbert series, and a rational function is called *positive* if its Laurent expansion at 0 has only nonnegative coefficients.

Our next result shows that, in the case that is certainly the most interesting one from the combinatorial viewpoint, a Hilbert decomposition is automatically a Stanley decomposition.

Proposition 2.8. *Suppose that* $\dim_K M_t \leq 1$*and* $R_s M_t \neq 0$ *whenever* $R_s, M_t, M_{s+t} \neq 0$*. Let* $\mathcal{H} = (S_i, s_i)_{i\in I}$ *be a Hilbert decomposition of* M*, and choose a homogeneous nonzero element* $x_i \in M$ *of degree* s_i *for each* i*. Then* $\mathcal{D} = (S_i, x_i)_{i\in I}$ *is a Stanley decomposition of* M*.*

The proof is straightforward: the supporting degrees of the vector spaces $S_i x_i$ do not overlap since $\dim_K M_t \leq 1$ for all t, and all degrees are reached.

In the general case, the choice of the elements x_i is of course critical. The next proposition gives a necessary and sufficient condition.

Proposition 2.9. *Let* $\mathcal{H} = (S_i, s_i)_{i\in I}$ *be a Hilbert decomposition of* M*, and choose a homogeneous nonzero element* $x_i \in M$ *of degree* s_i *for each* i*.*

(1) *The following properties are equivalent*:

(a) $\mathcal{D} = (S_i, x_i)_{i\in I}$ *is a Stanley decomposition.*

(b) *If* $\sum_{i\in I} a_i x_i = 0$ *with* $a_i \in S_i$*, then* $a_i = 0$ *for all* i*.*

(2) *In particular,* $\mathcal{D}$ *is a Stanley decomposition if for every degree* g *and the family* $\mathcal{G} = \{i : (S_i x_i)_g \neq 0\}$ *the elements* x_i*,* $i \in \mathcal{G}$*, are linearly independent.*

In fact, the type of restricted linear independence in (1) (b) is equivalent to the fact that the subspaces $S_i x_i$ form a direct sum. Then they must “fill” M since the direct sum has the same Hilbert function as M. That (1) (b) follows from (2) results immediately from the fact that every linear dependence relation of homogeneous elements decomposes into its homogeneous components.

For a special case, the following proposition can be found in [**11**].

Proposition 2.10. *Let* R *and* S *be polynomial rings over* K*, and let* M *and* N *be graded modules over* R *and* S*, respectively. Then*

$$\operatorname{Stdepth} M \otimes_K N \geq \operatorname{Stdepth} M + \operatorname{Stdepth} N,$$

and the analogous inequality holds for Hdepth*.* (*Here,* $M \otimes_K N$ *is considered as a module over* $R \otimes_K S$).

The proposition is obvious since the tensor product is distributive with respect to direct sums. The following proposition was proved in [**11**, 1.8] for the multigraded case.

Proposition 2.11. *With the standard assumptions on R and M, suppose that $a_1, \dots, a_r$ is a homogeneous M-sequence such that $K[a_1, \dots, a_r]$ is a graded retract of R. Then*

$$\operatorname{Stdepth} M \geq \operatorname{Stdepth} M/(a_1, \dots, a_r) + r,$$

and the analogous inequality holds for Hdepth.

Proof. Suppose $\mathcal{D}' = (S_i', x_i')$ is a Stanley decomposition of $M' = M/(a_1, \dots, a_r)$. Then we lift the x_i' to homogeneous elements of the same degree in M and claim that $\mathcal{D} = (S_i[a_1, \dots, a_r], x_i)$ is a Stanley decomposition of M.

By induction it is enough to treat the case $r = 1$. Let $R' = R/(a_1)$. First one should convince oneself that, in the multigraded case, a_1 is an indeterminate that does not occur in any of the S_i'. Since S_i' is a retract of R, the same holds for $S_i'[a_1]$; we may assume $a_1 = X_n$ in this case. In the standard graded case, we choose subspaces V_i of R_1 such that $\dim_K V_i = \dim_K(S_i')$ and V_i is mapped onto $(S_i')_1$ by the epimorphism $R \to R' \to S_i'$. Clearly, $a_i \notin V_i$, and so R_i is again a retract.

Since $H_M(T) = H_{M'}(T)/(1-T)$ in the standard graded case and $H_M(T_1, \dots, T_n) = H_{M'}(T_1, \dots, T_{n-1}, T_n)/(1-T_n)$ in the multigraded case, our desired Stanley decomposition is at least a Hilbert decomposition. (This argument proves the assertion about Hilbert depth.)

We use Proposition 2.9 to prove that it is indeed a Stanley decomposition. Consider a critical relation $b_1 x_{i_1} + \cdots + b_r x_{i_r} = 0$, and expand each b_i as a polynomial in a_1 with coefficients in S_i'. Reduction modulo a_1 yields that the constant terms of the b_i must be zero, and we can factor a_1 from the remaining terms. But a_1 is not a zero divisor, and it can be canceled. This reduces the a_1-degree of our coefficients by 1, and we are done. $\square$

Note that Proposition 2.11 implies the inequality in Theorem 2.7; more precisely, it reduces the proof of the theorem to the case where $\operatorname{depth} M = 0$, since one can find a suitable M-sequence of 1-forms.

Corollary 2.12. *Let M be the j-th graded syzygy of a graded R-module N. Then* $\operatorname{Stdepth} M \geq j$.

For the proof it is enough to note that every R-sequence of length j is an M-sequence (see, for example, Bruns and Vetter [**4**, (16.33)]).

We use Proposition 2.11 to prove that Stanley's conjecture holds in the multigraded case if $\operatorname{depth} M = 1$. This was already stated by Cimpoeaş [**5**]; however, the proof in [**5**] is not correct.

Proposition 2.13. *Suppose that* $\operatorname{depth} M \geq 1$. *Then* $\operatorname{Stdepth}_n M \geq 1$.

Proof. Set $U_{n+1} = M$, $U_0 = 0$, and define

$$U_i = \{x \in U_{i+1} : X_i^j x = 0 \text{ for some } j > 0\}, \quad i = 1, \dots, n.$$

Then we have a filtration of multigraded modules

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n+1} = M,$$

so that $\operatorname{Stdepth}_n M \geq \min_i \operatorname{Stdepth}_n U_{i+1}/U_i$. Moreover, $U_1/U_0 \cong U_1 = H^0_{\mathfrak{m}}(M)$ where $\mathfrak{m} = (X_1, \dots, X_n)$ and $H_{\mathfrak{m}}$ denotes local cohomology.

By hypothesis, $H^0_{\mathfrak{m}}(M) = 0$, and, by construction, X_i is not a zero divisor of U_{i+1}/U_i for $i = 1, \dots, n$. Therefore $\operatorname{Stdepth}_n M \geq 1$. □

Remark 2.14. The converse of Proposition 2.13 does not hold in general, as documented by the example given in [**5**, 1.6].

However, it is easy to see that $\operatorname{Stdepth} M = 0$ if $H^0_{\mathfrak{m}}(M)$ contains a full graded component M_g of M. For, then we must have $H^0_{\mathfrak{m}}(M) \cap S_i x_i \neq 0$ for some component of the Stanley decomposition, so that $H^0_{\mathfrak{n}}(S_i) \cong H^0_{\mathfrak{n}}(S_i x_i) \neq 0$ for the ideal $\mathfrak{n}$ generated by the indeterminates of S_i. This forces $S_i = K$.

Evidently, the assumption on $H^0_{\mathfrak{m}}(M)$ is satisfied if all homogeneous components have dimension ≤ 1 over K, and for this case this remark appeared already in [**5**].

3. The Koszul complex. In the following we want to investigate the syzygy modules of K, viewed as an R-module by identification with

$R/\mathfrak{m}$, $\mathfrak{m} = (X_1, \dots, X_n)$. With this R-module structure, K is resolved by the Koszul complex

$$\mathcal{K}(X_1, \dots, X_n; R) : 0 \longrightarrow \bigwedge^n R^n \xrightarrow{\partial} \bigwedge^{n-1} R^n \xrightarrow{\partial} \cdots \xrightarrow{\partial} R^n \xrightarrow{\partial} R \longrightarrow 0$$

where the basis vector $e_{i_1} \wedge \cdots \wedge e_{i_k}$ of $\bigwedge^k R^n$, $i_1 < \cdots < i_k$, has degree $X_{i_1} \cdots X_{i_k}$ (we identify monomials with their exponent vectors when we speak of degrees). In the standard grading, the degree of $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is simply k.

Let $M(n,k)$ be the k-th syzygy module of K. The Hilbert series of this module can be immediately read off the free resolution; its numerator polynomial is

$$\binom{n}{k} T^k - \binom{n}{k+1} T^{k+1} + \cdots + (-1)^{n-k} T^n \tag{3.1}$$

in the standard graded case, and

$$Q(n,k) = \sigma_{n,k} - \sigma_{n,k+1} + \cdots + (-1)^{n-k} \sigma_{n,n} \tag{3.2}$$

in the multigraded case, where $\sigma_{n,j}$ denotes the j-th elementary symmetric polynomial in the indeterminates $T_1, \dots, T_n$. Just for the record, the multigraded Hilbert series of $M(n,k)$ is given by

$$H_{M(n,k)}(T_1, \dots, T_n) = \sum_{a \in \mathbf{Z}_+^n} \binom{|\operatorname{supp}(a)| - 1}{k-1} T_1^{a_1} \cdots T_n^{a_n}; \tag{3.3}$$

here $\operatorname{supp}(a)$ denotes the set of indices i with $a_i \neq 0$. The standard graded Hilbert series is contained in Proposition 3.7 (with $s = n$).

For $k = 1$, one has the following result.

Theorem 3.1. *We have*

$$\operatorname{Hdepth}_1 \mathfrak{m} = \operatorname{Hdepth}_n \mathfrak{m} = \operatorname{Stdepth}_n \mathfrak{m} = \lfloor (n+1)/2 \rfloor.$$

Proof. For the difficult result on Stanley depth, see Biró et al. [**2**]. In order to estimate Hdepth_1, one considers the numerator polynomial of the Hilbert series,

$$nT - \binom{n}{2} T^2 \pm \cdots .$$

It is clear that we have to multiply by at least the power $1/(1-T)^v$, with

$$v = \left\lceil \frac{\binom{n}{2}}{n} \right\rceil = \lceil (n-1)/2 \rceil,$$

in order to get a positive rational function. Hence, by (2.4), $\lfloor (n+1)/2 \rfloor$ is an upper bound for $\operatorname{Hdepth}_1 \mathfrak{m}$, and the theorem follows. (See also [**16**] for a direct computation of $\operatorname{Hdepth}_1 \mathfrak{m}$.) $\square$

The Koszul complex allows (at least) two well-known inductive approaches.

Lemma 3.2. *For all n and k one has*

$$\operatorname{Stdepth} M(n,k) \geq \operatorname{Stdepth} M(n-1,k)$$

and

$$\operatorname{Stdepth} M(n,k) \geq 1+\min\bigl\{\operatorname{Stdepth} M(n-1,k), \operatorname{Stdepth} M(n-1,k-1)\bigr\},$$

and the analogous inequalities hold for Hdepth.

Proof. Here and in the following we will write $[i_1,\dots,i_k]$ for $\partial(e_{i_1} \wedge \cdots \wedge e_{i_k})$. Consider the submodule L of $M(n,k)$ generated by the elements $[i_1,\dots,i_{k-1},n]$. An inspection of $M(n,k+1)$ yields that $M(n,k)/L$ is annihilated by X_n. Thus $\operatorname{rank} M(n,k)/L = 0$, and $\operatorname{rank} L = \operatorname{rank} M(n,k) = \binom{n-1}{k-1}$. Since L is generated by exactly this number of elements, it is a free submodule. Therefore $\operatorname{Stdepth} M(n,k) \geq \operatorname{Stdepth} M(n,k)/L$.

Let $R' = K[X_1,\dots,X_{n-1}]$. The natural epimorphism $R \to R'$ that sends X_i to itself for $i \neq n$ and X_n to 0, can be lifted to a chain map of the Koszul complexes that sends e_i to "itself" and e_n to 0. This map induces an epimorphism $M(n,k)/L \to M(n-1,k)$, which is an isomorphism since the modules have the same Hilbert function. This proves the first inequality.

For the second inequality we use the inductive construction of the Koszul complex by iterated tensor products over R (see [**3**, 1.6.12]):

$$\mathcal{K}(X_1,\dots,X_n;R) = \mathcal{K}(X_1,\dots,X_{n-1};R) \otimes_R \mathcal{K}(X_n;R).$$

It yields an exact sequence

$$0 \longrightarrow N(n-1,k) \longrightarrow M(n,k) \longrightarrow N(n-1,k-1) \longrightarrow 0,$$

where $N(n-1,j)$ is the j-th syzygy module of $R/(X_1,\dots,X_{n-1})$. On the other hand, $N(n-1,j) = M(n-1,j) \otimes_K K[X_n]$, and the inequality follows from Propositions 2.6 and 2.10. □

If we combine Theorem 3.1 inductively with the second inequality, then we obtain a significant improvement of the bound $\operatorname{Stdepth} M(n,k) \geq k$ that one gets for free from Corollary 2.12.

Corollary 3.3. *Let $M(n,k)$ be the k-th syzygy module of K. Then*

$$\operatorname{Stdepth}_n M(n,k) \geq \lfloor (n+k)/2 \rfloor.$$

Remark 3.4. Theorem 3.1 has been generalized to ideals generated by monomial regular sequences $y_1,\dots,y_m$ as follows: $\operatorname{Stdepth}_n(y_1,\dots,y_m) = n - \lfloor m/2 \rfloor$; see Shen [**12**, 2.4]. Since R/I is resolved by the Koszul complex $\mathcal{K}(y_1,\dots,y_m;R)$, a similar induction as in the proof of Corollary 3.3 shows that the k-th syzygy module of $R/(y_1,\dots,y_m)$ has multigraded Stanley depth $\geq n-m+\lfloor (m+k)/2 \rfloor$. In the induction, one must observe that the indeterminate factors of the y_i form pairwise disjoint sets.

The upper half of the resolution poses no problems for Hilbert depth.

Theorem 3.5. *Suppose $n > k \geq \lfloor n/2 \rfloor$. Then*

$$\operatorname{Hdepth}_1 M(n,k) = \operatorname{Hdepth}_n M(n,k) = n-1.$$

Proof. Note that the maximal value n is excluded. It can only be attained by a module of Krull dimension n with a positive numerator polynomial in its Hilbert series, standard graded or multigraded. It is therefore enough to consider the multigraded case.

Now we look at the multigraded numerator polynomial, given by equation (3.2). Consider the set Y_u of squarefree monomials in $T_1, \dots, T_n$ of degree u, summing up to σ_{nu}. For $u \geq \lfloor n/2 \rfloor$ one has an injective map $Y_u \to Y_{u-1}$ that assigns each monomial a divisor (cf. [**15**, page 35]). It follows that we can write $Q(n,k)$ as a sum of monomials and polynomials of type $\mu(1-T_p)$ where μ is a monomial. Exactly those terms $\mu(1-T_p)$ appear for which μ is the image of μT_p under the injection.

This leads to a Hilbert decomposition in which the summands are of type R and $R/(X_p)$ (with appropriate shifts). More precisely, the decomposition is given by

$$(K[F_i'], X^{F_i}),$$

where

- F_i runs through the subsets of $\{1, \dots, n\}$ with $k+j$ elements, j even,
- X^{F_i} is the product of the indeterminates dividing F_i,
- $F_i' = \{1, \dots, n\}$ if T^{F_i} is not in the image of the injection, and $F_i' = \{1, \dots, n\} \setminus \{p\}$ if T^{F_i} is the image of $T^{F_i \cup \{p\}}$,
- $K[F_i']$ is the polynomial ring in the indeterminates X_q, $q \in F_i'$. □

One can try to convert the Hilbert decomposition indicated in the proof of Theorem 3.5 into a Stanley decomposition by the following method. To simplify notation, we denote the element $[i_1, \dots, i_k]$ by w_G where $G = \{i_1, \dots, i_k\}$. We call these elements *generators* and the products μw_G, μ a monomial in R, *monomials*. In the multigraded structure of the Koszul complex, the degree of μw_G is μX^G (where we again identify a monomial with its exponent vector).

For each pair $(K[F_i'], F_i)$ in the decomposition, we now choose a monomial $h_i = \mu w_G$ such that $\mu X^G = X^{F_i}$. Let us call h_i the *hook* of $(K[F_i'], X^{F_i})$. In the total set of monomials that we obtain by multiplying h_i by the monomials in $K[F_i']$ and collecting over all i, each multidegree appears with the right multiplicity (because we are starting from a Hilbert decomposition). The crucial point is to make these monomials (of the same degree) linearly independent over K.

Note that each hook produces a given multidegree at most once. Fix a multidegree, and consider all hooks that contribute to it. Each of them has the form μw_G, and it is enough to make the family of generators w_G associated with the given multidegree linearly independent over R (Proposition 2.9).

For a given monomial μ in R, let the squarefree part $\operatorname{sqf}(\mu)$ be the product of the indeterminates dividing μ. Clearly, a generator is associated with a given multidegree ν if and only if it is associated with $\operatorname{sqf}(\nu)$ (since all hooks are squarefree). This observation reduces the test for linear independence to the squarefree degrees.

To prove the desired linear independence, we use the following simple criterion: if we can order a family $(w_G)_{G\in\mathcal{G}}$ in such a way that $G_1 \cup \cdots \cup G_m \supsetneq G_1 \cup \cdots \cup G_{m-1}$ for all m, then the family $\mathcal{G}$ is linearly independent.

Let us now consider the special case $n = 5$, $k = 2$. One has $\operatorname{Stdepth}_n M(5,2) \geq 3$ by Corollary 3.3, but in fact $\operatorname{Stdepth}_n M(5,2) = 4$, as we will see now. Following [**15**, page 35], we obtain an injection $Y_3 \to Y_2$ if we go through the monomials μ in Y_3 lexicographically and choose for each μ the lexicographically smallest divisor that is still available: $123 \mapsto 12$, $124 \mapsto 14$, ... , $345 \mapsto 34$. Furthermore $12345 \mapsto 1234$.

For the squarefree monomials of degree 2, there is only a single choice of hooks, namely the corresponding generator, and this leads to no problem in degree 3: if the total degree of a squarefree monomial is 3, then there are exactly two generators associated with it, and they are automatically linearly independent.

Now we come to total degree 4, and the choice of hooks becomes critical. Consider 1234. There are exactly 6 monomials of this multidegree. Of these two are already in use, namely $13[24] = X_1X_3[24]$ (24 is the image of 245 in our injection) and $12[34]$ ($345 \mapsto 34$). Since $14[23]$ is linearly dependent on the first two over K, it is also excluded, and we choose $34[12]$ as the hook of 1234. It is "good," since $[24], [34], [12]$ are linearly independent over R.

Further choices: $1235 \mapsto 23[15]$, $1245 \mapsto 25[14]$, $1345 \mapsto 45[13]$, $2345 \mapsto 45[23]$. Again we get linearly independent families of generators for each squarefree multidegree of total degree 4.

The generators associated with multidegree 12345 are [15], [14], [13], [23]. They are linearly independent, and we are done.

Using Lemma 3.2, one obtains that

$$\operatorname{Stdepth} M(n, n-3) = n - 1 \tag{3.4}$$

for $n \geq 5$. We believe that $\operatorname{Stdepth} M(n,k) = n-1$ for all $k \geq \lfloor n/2 \rfloor$. It suffices to show this for n odd, $k = (n-1)/2$. The general statement would follow by induction.

In the lower half of the Koszul complex the situation is much more complicated, and it seems impossible to give a precise, simple expression even for Hdepth_1. The proposition below provides a trivial upper bound.

Proposition 3.6. *Let $k < \lfloor n/2 \rfloor$. Then*

$$\operatorname{Hdepth}_1 M(n,k) \leq n - \left\lceil \frac{n-k}{k+1} \right\rceil .$$

Proof. Simply consider the quotient of the second, negative term in the numerator polynomial by the first term. $\square$

Naively one might think that the proposition gives the correct value as it does in the case $k = 1$ (and for $k \geq \lfloor n/2 \rfloor$). A computer experiment confirms this value for $n \leq 22$. However for $n = 23$ it fails for $k = 3, 4, 5$. As we shall see in the next section, the upper bound in Proposition 3.6 is very far from the truth, see Theorems 4.1 and 4.5. As a preparatory step, we prove the following result, which, in combination with (2.4), forms the key for proving these theorems.

Proposition 3.7. *Let $Q_{n,k}$ be the numerator polynomial of the* **Z**-*graded Hilbert series of $M(n,k)$. Then*

(3.5)

$$\begin{aligned}\frac{Q_{n,k}}{(1-T)^s} &= \sum_{j=0}^{\infty} \left((-1)^j \binom{n-s}{k+j} + \sum_{t=1}^{s} \binom{n-t}{k-1} \binom{s-t+j}{s-t} \right) T^{j+k} \\ &= \sum_{j=0}^{\infty} \left((-1)^j \binom{n-s}{k+j} \right.\end{aligned}$$

$$+\sum_{\ell=0}^{k-1}\binom{j+\ell}{\ell}\binom{n-s-j-\ell-1}{k-\ell-1}\binom{s+j+\ell}{s-1}\Bigg)T^{j+k}. \tag{3.6}$$

Proof. By (3.1), equation (3.5) is true for $s=0$. For the induction, one observes that the term in the inner sum is the degree j value of the Hilbert function of the free module of rank $\binom{n-t}{k-1}$ over the polynomial ring in $s-t+1$ variables. In other words, its sum over j is the Hilbert series of this module. Multiplication by $1/(1-T)$ increases the number of variables by 1. Thus the multiplication by $1/(1-T)$ replaces s by $s+1$ in these terms, as desired.

In order to complete the proof of (3.5), it remains to show that

$$\begin{aligned}\frac{1}{1-T}&\sum_{j=0}^{\infty}(-1)^j\binom{n-s}{k+j}T^{j+k}\\ &=\sum_{j=0}^{\infty}\left((-1)^j\binom{n-(s+1)}{k+j}+\binom{n-(s+1)}{k-1}\right)T^{j+k}.\end{aligned}$$

After the replacement of $n-s$ by n this is the case $s=1$, and the easy verification is left to the reader.

In order to establish the second form (3.6), we rewrite the inner sum in (3.5) as follows:

$$\begin{aligned}&\sum_{t=1}^{s}\binom{n-t}{k-1}\binom{s-t+j}{s-t}\\ &=\sum_{t=1}^{s}(-1)^{k-1}\binom{-n+t+k-2}{k-1}\binom{s-t+j}{s-t}\\ &=\sum_{t=1}^{s}(-1)^{k-1}\sum_{\ell=0}^{k-1}\binom{-s-j+t-1}{\ell}\binom{s+j+k-n-1}{k-\ell-1}\binom{s-t+j}{s-t}\\ &=\sum_{\ell=0}^{k-1}\binom{n-s-j-\ell-1}{k-\ell-1}\sum_{t=1}^{s}\binom{s+j+\ell-t}{\ell}\binom{s-t+j}{j}\\ &=\sum_{\ell=0}^{k-1}\binom{n-s-j-\ell-1}{k-\ell-1}\binom{j+\ell}{\ell}\sum_{t=1}^{s}\binom{s+j+\ell-t}{j+\ell}\end{aligned}$$

$$= \sum_{\ell=0}^{k-1} \binom{n-s-j-\ell-1}{k-\ell-1} \binom{j+\ell}{\ell} \binom{s+j+\ell}{j+\ell+1}.$$

Here, to arrive at the second line and at the last line, we used special instances of the Chu-Vandermonde summation (cf., e.g., [**7**, Section 5.1, (5.27)]). ◻

Remark 3.8. In hypergeometric terms (cf., [**13**] for definitions), the inner sums (over t and ℓ, respectively) in (3.5) and (3.6) are ${}_3F_2$-series, namely

$$\binom{n-1}{k-1}\binom{s+j-1}{s-1} {}_3F_2\left[\begin{matrix} 1-s, k-n, 1 \\ 1-j-s, 1-n \end{matrix}; 1\right]$$
$$= \binom{n-s-j-1}{k-1}\binom{s+j}{s-1}$$
$$\times {}_3F_2\left[\begin{matrix} 1-k, 1+j, 1+j+s \\ 2+j, 1+j-n+s \end{matrix}; 1\right].$$

There are no summation formulas available for these ${}_3F_2$-series, and therefore one cannot expect that they can be summed in closed form. Indeed, by looking at special values of k and s, respectively, by applying the Gosper-Zeilberger algorithm in order to find a recurrence for these series and subsequently applying the Petkovšek algorithm to the recurrence (cf. [**9**]), one can *prove* that these series cannot be further simplified. It is for this reason, that, given k and n, it is difficult to find the smallest s such that *all* the coefficients in the polynomial (3.5) (respectively in (3.6)) are non-negative, that is, to find the Hilbert depth of $M(n,k)$ for the standard grading (cf. (2.4)).

If we combine Proposition 3.7 with (2.4), then we obtain a monotonicity property for the Hilbert depth of the syzygy modules $M(n,k)$.

Corollary 3.9. *For all k one has*

$$\mathrm{Hdepth}_1 M(n,k) \le \mathrm{Hdepth}_1 M(n,k+1).$$

Proof. For s fixed, the quotient of the negative terms on the right-hand side of (3.5) is smaller than the quotients of the corresponding positive terms. ◻

4. An asymptotic discussion. In view of the apparent impossibility (addressed in Remark 3.8) of finding a compact expression for $\mathrm{Hdepth}_1 M(n,k)$, the next best result that one can hope for is asymptotic approximations of $\mathrm{Hdepth}_1 M(n,k)$ as n becomes large. This will be the subject of this final section. Our results, given in Theorems 4.1 and 4.5 below, show that the general bounds in Corollary 3.3 and Proposition 3.6 are far from the truth for large n, that is, they can be substantially improved. We shall discuss two "regimes" for large n. In the first part of this section, we let k be fixed, while n tends to ∞. On the other hand, in the second part, we let both k and n tend to ∞ at a fixed rate.

4.1. The case of fixed k and large n. The theorem below provides rather precise asymptotics for $\mathrm{Hdepth}_1 M(n,k)$ for the case where k is fixed and n tends to ∞.

Theorem 4.1. *For a fixed positive integer k, we have*

(4.1)
$$\mathrm{Hdepth}_1 M(n,k) = \frac{1}{2}n + \frac{1}{2}\sqrt{(k-1)n\log n} + \frac{1}{4}\sqrt{\frac{(k-1)n}{\log n}}\log\log n + o\left(\sqrt{\frac{n}{\log n}}\log\log n\right),$$

as $n \to \infty$.

Proof. By Theorem 3.1, we know that (4.1) is correct if $k = 1$. We may therefore assume that $k \geq 2$ in the sequel.

In all of this proof, we let k be fixed and

$$s = \frac{1}{2}n - \frac{1}{2}\sqrt{(k-1)n\log n} - \delta\sqrt{\frac{(k-1)n}{\log n}}\log\log n, \quad \text{as } n \to \infty, \tag{4.2}$$

where δ is a fixed positive real number. We shall prove that the quotient of

$$\sum_{\ell=0}^{k-1} \binom{j+\ell}{\ell}\binom{n-s-j-\ell-1}{k-\ell-1}\binom{s+j+\ell}{s-1} \tag{4.3}$$

and

$$\binom{n-s}{k+j} \tag{4.4}$$

is (asymptotically) less than 1 *for some* j if $\delta > 1/4$, and larger than 1 *for all* j with $0 \le j \le n-s-k$ if $\delta < 1/4$. Clearly, in view of Proposition 3.7 and (2.4), this would establish the assertion of the theorem.

In order to establish this claim, we proceed in several steps. In the first step, we show that, for large n, the summands in the sum (4.3) can be bounded by a constant times the term for $\ell = k-1$, so that it suffices to prove the above claim for the quotient

$$\begin{aligned}\binom{j+k-1}{k-1}\binom{s+j+k-1}{s-1}\bigg/\binom{n-s}{k+j}&\\ &= \frac{(j+1)_{k-1}}{(k-1)!}\,\frac{\Gamma(s+j+k)}{\Gamma(s)}\,\frac{\Gamma(n-s-j-k+1)}{\Gamma(n-s+1)}.\end{aligned} \tag{4.5}$$

Here, $(j+1)_{k-1}$ is the standard notation for shifted factorials (Pochhammer symbols),

$$(j+1)_{k-1} = (j+1)(j+2)\cdots(j+k-1),$$

and $\Gamma(x)$ denotes the classical gamma function (cf. [**13**]).

In the second step, we consider the right-hand side of (4.5) as a continuous function in the *real* variable j, $0 \le j \le n-s-k$, and we determine the (asymptotic) value of j for which the expression (4.5) is minimal. Finally, in the third step, we estimate (4.5) as $n \to \infty$ for this value of j. The conclusion will be that it will be less than 1 if $\delta > 1/4$, while it will be larger than 1 if $\delta < 1/4$.

Step 1. The quotient of the $(\ell+1)$-st and the ℓ-th summand in (4.3) equals

$$\frac{(j+\ell+1)}{(\ell+1)}\,\frac{(k-\ell-1)}{(n-s-j-\ell-1)}\,\frac{(s+j+\ell+1)}{(j+\ell+2)},\qquad \ell = 0,1,\dots,k-2, \tag{4.6}$$

for which we have

$$\begin{aligned}\frac{(j+\ell+1)}{(\ell+1)}\,\frac{(k-\ell-1)}{(n-s-j-\ell-1)}\,\frac{(s+j+\ell+1)}{(j+\ell+2)}&\\ \ge \frac{j+1}{(k-1)}\,\frac{1}{(n-s-1)}\,\frac{(s+1)}{(j+k)} &\ge \frac{1}{2k(k-1)}\end{aligned}$$

for n large enough, where we have taken into account our choice (4.2) of s. Hence, all the summands in (4.3) are bounded by a constant times the term for $\ell = k - 1$.

Step 2. The reader should recall that k is fixed, s is given by (4.2), and that we consider large n. It is a simple fact that the product

$$\Gamma(s + j + k)\,\Gamma(n - s - j - k + 1)$$

occurring in (4.5) attains its minimum when the arguments of the gamma functions are equal to each other, that is, for $j = ((n + 1)/2) - s - k$. It is then not difficult to see that this implies that, as a function in j, the expression (4.5) cannot attain its minimum at the boundary of the defining interval for j, that is, at $j = 0$ or at $j = n - s - k$. (The term $(j + 1)_{k-1}$ cannot compensate the difference in orders of magnitude of (4.5) at $j = ((n + 1)/2) - s - k$ and at $j = 0$, respectively at $j = n - s - k$.) Therefore, in order to determine places of minima of the function (4.5) (in j), we compute its logarithmic derivative with respect to j, which we shall subsequently equate to 0. Let $\psi(x)$ denote the classical *digamma function*, which, by definition, is the logarithmic derivative of the gamma function. Using this notation, the logarithmic derivative of (4.5) is given by

$$\sum_{i=1}^{k-1} \frac{1}{j + i} + \psi(s + j + k) - \psi(n - s - j - k + 1).$$

Let us for the moment write $s = (n/2) - s_1$, where $s_1 = o(n)$, for short. Then, equating the above logarithmic derivative to 0 means to solve the equation

$$\sum_{i=1}^{k-1} \frac{1}{j + i} + \psi\left(\frac{n}{2} - s_1 + j + k\right) - \psi\left(\frac{n}{2} + s_1 - j - k + 1\right) = 0 \tag{4.7}$$

for j. For our purposes, it will not be necessary to determine solutions j exactly (which is impossible anyway), but it suffices to get appropriate asymptotic estimates.

For the following considerations we need the first few terms in the asymptotic series for the digamma function (cf. [**6**, 1.18 (7)]):

$$\psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right), \quad \text{as } x \to \infty. \tag{4.8}$$

If we suppose that $j \sim \alpha n$ as $n \to \infty$, where $\alpha > 0$, then, using (4.8), the limit of the left-hand side of (4.7) as $n \to \infty$ can be computed: it equals

$$\log \frac{(1/2)+\alpha}{(1/2)-\alpha} \neq 0,$$

a contradiction to the equation (4.7). Hence, we must have $j = o(n)$ as $n \to \infty$.

Let us, for convenience, write $j = s_1 + j_1$. Then (4.7) becomes

$$\sum_{i=1}^{k-1} \frac{1}{s_1+j_1+i} + \psi\left(\frac{n}{2}+j_1+k\right) - \psi\left(\frac{n}{2}-j_1-k+1\right) = 0. \tag{4.9}$$

Using (4.8), the estimate

$$\begin{aligned}\log\left(\frac{n}{2}+j_1+k\right) &= \log\frac{n}{2} + \log\left(1+\frac{2(j_1+k)}{n}\right)\\ &= \log\frac{n}{2} + \frac{2(j_1+k)}{n} + O\left(\frac{j_1^2}{n^2}\right), \quad \text{as } n \to 0,\end{aligned}$$

and an analogous estimate for $\log((n/2)-j_1-k+1)$, the left-hand side of (4.9) is asymptotically

$$\frac{k-1}{s_1+j_1} + O\left(\frac{1}{(s_1+j_1)^2}\right) + \frac{4j_1}{n} + O\left(\frac{j_1^2}{n^2}\right). \tag{4.10}$$

If the equation (4.9) wants to be true, then the asymptotically largest terms in (4.10) must cancel each other. If we suppose that $j_1 \ll \sqrt{n/\log n}$, then, taking into account that $s_1 \sim (1/2)\sqrt{(k-1)n\log n}$, the term $(k-1)/(s_1+j_1)$ would be asymptotically strictly larger than all other terms in (4.10), a contradiction. On the other hand, if we suppose that $j_1 \gg \sqrt{n/\log n}$, then the term $4j_1/n$ would be asymptotically strictly larger than all other terms in (4.10), again a contradiction. Hence, we must have $j_1 \sim \alpha\sqrt{n/\log n}$ for some $\alpha > 0$. If we substitute this in (4.10) and equate (asymptotically) the first and the third term in this expression, then we obtain $\alpha = -(1/2)\sqrt{k-1}$.

In summary, under our assumptions, the value(s)[1] for j which minimize the expression (4.5) is (are) asymptotically equal to

$$j_0 = \frac{1}{2}\sqrt{(k-1)n\log n}\left(1 - \frac{1}{\log n} + o\left(\frac{1}{\log n}\right)\right), \quad \text{as } n \to \infty. \tag{4.11}$$

Step 3. Now we substitute (4.2) and (4.11) in (4.5) and determine the asymptotic behavior of the resulting expression. For the term $(j_0+1)_{k-1}$, we use the estimation

$$\begin{aligned}\log\big((j_0+1)_{k+1}\big) &= \log\left(j_0^{k-1}\left(1+O\left(\frac{1}{j_0}\right)\right)\right)\\ &= (k-1)\log j_0 + O\left(\frac{1}{j_0}\right)\\ &= (k-1)\left(\frac{1}{2}\log n+\frac{1}{2}\log\log n+\frac{1}{2}\log(k-1)-\log 2\right)\\ &\quad + o(1).\end{aligned}$$

In order to approximate the gamma functions in (4.5), we need Stirling's formula in the form

$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi)+o(1), \quad \text{as } x\to\infty. \tag{4.12}$$

Writing, as earlier, $s=(n/2)-s_1$, application of (4.12) to the term $\Gamma(s)$ gives

$$\begin{aligned}\Gamma(s) &= \left(\frac{n}{2}-s_1-\frac{1}{2}\right)\log\left(\frac{n}{2}-s_1\right)-\left(\frac{n}{2}-s_1\right)\\ &\quad+\frac{1}{2}\log(2\pi)+o(1)\\ &= \left(\frac{n}{2}-s_1-\frac{1}{2}\right)\left(\log\frac{n}{2}-\frac{2s_1}{n}-\frac{1}{2}\frac{(2s_1)^2}{n^2}+O\left(\frac{s_1^3}{n^3}\right)\right)\\ &\quad-\left(\frac{n}{2}-s_1\right)+\frac{1}{2}\log(2\pi)+o(1).\end{aligned}$$

The terms $\Gamma(s+j+k)$, $\Gamma(n-s-j-k+1)$ and $\Gamma(n-s+1)$ are treated analogously. If everything is put together, after a considerable amount of simplification, we obtain

$$\begin{aligned}&\frac{(j+1)_{k-1}}{(k-1)!}\,\frac{(s+j+k-1)!}{(s-1)!}\,\frac{(n-s-j-k)!}{(n-s)!}\\ &\quad= \exp\left((k-1)\left(\frac{1}{2}-2\delta\right)\log\log n+\frac{1}{2}(k-1)\log(k-1)\right.\\ &\qquad\left.-(k-1)\log 2-\log\big((k-1)!\big)+o(1)\right), \quad \text{as } n\to\infty.\end{aligned} \tag{4.13}$$

We can now clearly see that the right-hand side is $\ll 1$ if $\delta > 1/4$, and $\gg 1$ if $\delta < 1/4$. This completes the proof of the theorem. $\square$

Remark 4.2. The alert reader may wonder whether the estimation given by the right-hand side of (4.1) provides a lower or upper bound for $\mathrm{Hdepth}_1 M(n,k)$. We shall now show that (at least for $k \ge 4$) it is indeed an upper bound, that is, for fixed $k \ge 4$ and large enough n, we have

$$\begin{aligned}\mathrm{Hdepth}_1 M(n,k) \le \frac{1}{2}n + \frac{1}{2}\sqrt{(k-1)n\log n}\\ + \frac{1}{4}\sqrt{\frac{(k-1)n}{\log n}}\log\log n.\end{aligned} \tag{4.14}$$

To see this, we return to (4.6), which expresses the quotient of the $(\ell+1)$-st and the ℓ-th summand in (4.3). Using this expression, we see that, for $j = j_0$ (cf. (4.11)), this quotient is asymptotically equal to

$$\frac{k-\ell-1}{\ell+1} + o(1), \quad \ell = 0, 1, \dots, k-2,$$

as $n \to \infty$. If we denote the ℓ-th summand in the sum (4.3) by t_ℓ, then this implies that

$$t_\ell = t_{k-1}\left(\binom{k-1}{\ell} + o(1)\right).$$

Thus, we infer that the sum in (4.3) is asymptotically equal to

$$\begin{aligned}\sum_{\ell=0}^{k-1} t_\ell &= t_{k-1}\sum_{\ell=0}^{k-1}\left(\binom{k-1}{\ell} + o(1)\right)\\ &= \binom{j+k-1}{k-1}\binom{s+j+\ell}{s-1}\big(2^{k-1} + o(1)\big).\end{aligned}$$

If we combine this with (4.13), in which we computed the asymptotics of the quotient of t_{k-1} and (4.4) with s given by (4.2) and $j = j_0$, then we obtain that the quotient of the sum (4.3) and the binomial coefficient (4.4), where s is given by (4.2) with δ specialized to $1/4$ and $j = j_0$, is equal to

$$\exp\left(\frac{1}{2}(k-1)\log(k-1) - \log\big((k-1)!\big) + o(1)\right), \quad \text{as } n \to \infty.$$

It is not difficult to show that, for $k \geq 4$, we have

$$\frac{1}{2}(k-1)\log(k-1) - \log\big((k-1)!\big) < 0.$$

In view of (2.4), this implies (4.14). We expect the same to be true as well for $k = 2$ and $k = 3$, but we did not perform the necessary asymptotic calculations using longer asymptotic series.

4.2. The case of large n and k. In this part, we consider the case where both k and n tend to ∞ at a fixed rate, say $k = \beta n + o(n)$ with $\beta > 0$. We shall see that then $\mathrm{Hdepth}_1 M(n,k) \sim (1-\gamma)n$, where $\gamma \leq (1/2) - \beta$. (See Theorem 4.5 for the exact definition of γ, and Remarks 4.6 (2) for a graph of γ as a function in β.) Note that this estimate is an (asymptotic) improvement of Corollary 3.3, which only yields $\mathrm{Hdepth}_1 M(n,k) \geq ((1/2) + (\beta/2) + o(1))n$.

Again, our starting point is (2.4) in combination with Proposition 3.7. We begin by providing an asymptotic estimate for the isolated binomial coefficient in (3.5).

Lemma 4.3. *Let $k = \beta n + o(n)$, $j = \alpha n + o(n)$ and $s = \gamma n + o(n)$, where α, β and γ are positive real numbers not exceeding 1. Then, as $n \to \infty$, we have*

$$\binom{n-s}{k+j} = \left(\frac{(1-\gamma)^{1-\gamma}}{(\alpha+\beta)^{\alpha+\beta}(1-\alpha-\beta-\gamma)^{1-\alpha-\beta-\gamma}}\right)^n \times \text{asymptotically smaller terms.} \tag{4.15}$$

Proof. This is a simple consequence of Stirling's formula (4.12). ◻

Next, we provide an asymptotic estimate for the inner sum in (3.5).

Lemma 4.4. *Let $k = \beta n + o(n)$, $j = \alpha n + o(n)$ and $s = \gamma n + o(n)$, where α, β and γ are positive real numbers not exceeding 1. Then, as $n \to \infty$, we have*

$$\sum_{t=1}^{s} \binom{n-t}{k-1}\binom{s+j-t}{s-t} = \left(\frac{(\alpha+\gamma)^{\alpha+\gamma}}{\alpha^\alpha \beta^\beta \gamma^\gamma (1-\beta)^{1-\beta}}\right)^n \times \text{asymptotically smaller terms.} \tag{4.16}$$

Proof. The summands of the sum on the left-hand side of (4.16) are monotone decreasing in t. In particular, they are bounded above by the summand with $t = 1$. Stirling's formula (4.12) applied to this summand yields the approximation given on the right-hand side of (4.16), and since the number of summands is $s \sim \gamma n$, the approximation given there is also valid for the whole sum. $\square$

Theorem 4.5. *Let β be a positive real number with $\beta \le 1/2$. For $k = \beta n + o(n)$, we have*

$$\mathrm{Hdepth}_1 M(n,k) = (1-\gamma)n + o(n), \quad \text{as } n \to \infty, \tag{4.17}$$

where γ is the smallest nonnegative solution of the equation

$$\frac{(\alpha+\gamma)^{\alpha+\gamma}(\alpha+\beta)^{\alpha+\beta}(1-\alpha-\beta-\gamma)^{1-\alpha-\beta-\gamma}}{\alpha^\alpha \beta^\beta \gamma^\gamma (1-\beta)^{1-\beta}(1-\gamma)^{1-\gamma}} = 1, \tag{4.18}$$

with

$$\alpha = \frac{1}{4}\left(1 - 2\beta - 2\gamma + \sqrt{(1-2\beta-2\gamma)^2 - 8\beta\gamma}\right).$$

Remarks 4.6. (1) Our computer calculations show that there is always *exactly one* solution to (4.18). More precisely, as a function in γ, the left-hand side of (4.18) seems always to be a monotone increasing function. In view of the daunting expression that one obtains by substituting the indicated value of α in (4.18), we did not try to prove this observation since this is also not essential for the assertion of Theorem 4.5.

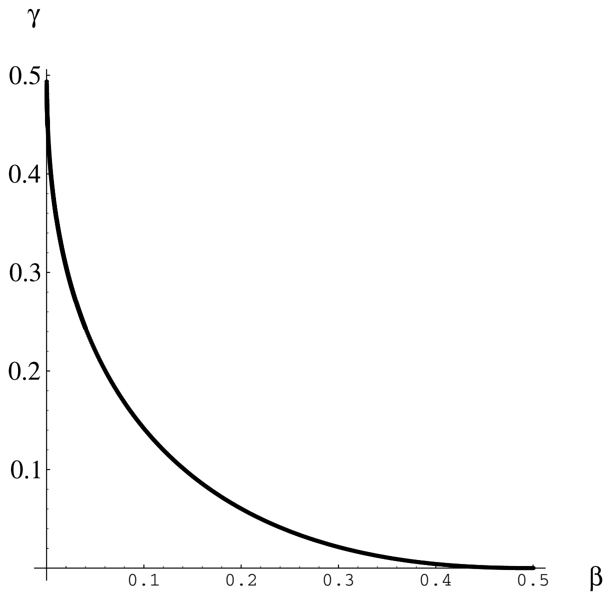

FIGURE 1. The value γ as a function in β.

(2) It is not difficult to see that the value γ in Theorem 4.5 satisfies $\gamma \le 1/2-\beta$. Indeed, except for β close to 0 or close to $1/2$, this is so by a large margin, as the graph in Figure 1 shows. As we already remarked at the beginning of this part, this yields a considerable improvement over the bound implied by Corollary 3.3.

Proof of Theorem 4.5. We proceed in a manner similar to the proof of Theorem 4.1. First, we form the quotient of (4.16) and (4.15), which is asymptotically

$$\left(\frac{(\alpha+\gamma)^{\alpha+\gamma}(\alpha+\beta)^{\alpha+\beta}(1-\alpha-\beta-\gamma)^{1-\alpha-\beta-\gamma}}{\alpha^\alpha\beta^\beta\gamma^\gamma(1-\beta)^{1-\beta}(1-\gamma)^{1-\gamma}}\right)^n \times \text{asymptotically smaller terms,} \tag{4.19}$$

as $n\to\infty$. In view of (2.4), we need to find the smallest γ such that the base of the exponential in (4.19) is larger than 1 for all α with

$0 \le \alpha \le 1-\beta-\gamma$. Hence, we should next consider this base,

$$f(\alpha) := \frac{(\alpha+\gamma)^{\alpha+\gamma}(\alpha+\beta)^{\alpha+\beta}(1-\alpha-\beta-\gamma)^{1-\alpha-\beta-\gamma}}{\alpha^\alpha\beta^\beta\gamma^\gamma(1-\beta)^{1-\beta}(1-\gamma)^{1-\gamma}},$$

and, given β and γ, discuss it as a function in α. More precisely, our goal is to determine the value(s) of α for which $f(\alpha)$ attains its minimum. In a subsequent step, we shall have to find the smallest possible γ such that this minimum is at least 1.

Our first observation is that both

$$f(0) = \frac{(1-\beta-\gamma)^{1-\beta-\gamma}}{(1-\beta)^{1-\beta}(1-\gamma)^{1-\gamma}}$$

and

$$f(1-\beta-\gamma) = \frac{1}{(1-\beta-\gamma)^{1-\beta-\gamma}\beta^\beta\gamma^\gamma}$$

are at least 1. For $f(1-\beta-\gamma)$ this is totally obvious since β, γ and $1-\beta-\gamma$ are numbers between 0 and 1, while for $f(0)$ this follows from the facts that $f(0)|_{\gamma=0} = 1$ and that $f(0)$ is monotone increasing in γ. Consequently, for given β and γ, the minimum of $f(\alpha)$ is either at least 1, or it is attained in the interior of the interval $[0, 1-\beta-\gamma]$. In order to find the places of minima in the interior of this interval, we compute the logarithmic derivative of $f(\alpha)$,

$$\frac{d}{d\alpha}\log f(\alpha) = \log\frac{(\alpha+\gamma)(\alpha+\beta)}{\alpha(1-\alpha-\beta-\gamma)}, \tag{4.20}$$

and equate it to 0. This equation leads to a quadratic equation in α with solutions

$$\alpha = \frac{1}{4}\left(1-2\beta-2\gamma \pm \sqrt{(1-2\beta-2\gamma)^2 - 8\beta\gamma}\right). \tag{4.21}$$

Since, from (4.20), we see that the derivative of $f(\alpha)$ is $+\infty$ at $\alpha = 0$ and at $\alpha = 1-\beta-\gamma$, the smaller of the two solutions in (4.21) must be the place of a local maximum of $f(\alpha)$, while the larger solution must be the place of a local minimum (if they are at all real numbers).

Finally, we must find the smallest γ such that the minimum of $f(\alpha)$, for α ranging in the interval $[0, 1-\beta-\gamma]$, is at least 1. In particular, the above described local minimum must be at least 1. Hence, we must substitute the larger value given by (4.21), α_0 say, in (4.19), and restrict our search to those values of γ, where the result is at least 1.

Now, if we do this substitution in (4.19) (the reader should observe that the result is exactly the left-hand side of (4.18)) and set $\gamma = 0$, then we obtain

$$f(\alpha_0)\big|_{\gamma=0} = \frac{1}{2\,(1-\beta)^{1-\beta}} \le \frac{1}{2\,(1/e)^{1/e}} = 0.72\ldots < 1.$$

Hence, the smallest γ such that the minimum of $f(\alpha)$, for α ranging in the interval $[0, 1-\beta-\gamma]$, is at least 1 is indeed the solution to the equation (4.18). $\square$

ENDNOTES

1. It could be proved, using estimates for the derivative of (4.7) with respect to j, that, for n large enough, there is a unique zero of (4.7), and hence, a unique j which minimizes (4.5). Since we do not really need this fact, we omit its proof.

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