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HILBERT REGULARITY OF $\mathbb{Z}$-GRADED MODULES OVER POLYNOMIAL RINGS

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ABSTRACT. Let M be a finitely generated $\mathbb{Z}$-graded module over the standard graded polynomial ring $R = K[X_1, \dots, X_d]$ with K a field, and let $H_M(t) = Q_M(t)/(1-t)^d$ be the Hilbert series of M. We introduce the Hilbert regularity of M as the lowest possible value of the Castelnuovo-Mumford regularity for an R-module with Hilbert series H_M. Our main result is an arithmetical description of this invariant which connects the Hilbert regularity of M to the smallest k such that the power series $Q_M(1-t)/(1-t)^k$ has no negative coefficients. Finally, we give an algorithm for the computation of the Hilbert regularity and the Hilbert depth of an R-module.

1. Introduction. This note may be considered as part of a program that aims at estimating numerical invariants of a graded module M over a polynomial ring $K[X_1, \dots, X_d]$ (K is a field) in terms of the Hilbert series $H_M(t)$. For the notions of commutative algebra we refer the reader to Bruns and Herzog **[2]**. Well-known examples of such estimates are the bounds of Bigatti **[1]** and Hulett **[6]** on the Betti numbers or the bounds of Elias, Robbiano and Valla **[4]** on the number of generators for ideals primary to $\mathfrak{m} = (X_1, \dots, X_d)$.

A more recent result is the upper bound on $\operatorname{depth} M$ (or, equivalently, a lower bound on $\operatorname{projdim} M$) given by the third author **[11]**, namely, the *Hilbert depth* $\operatorname{Hdepth} M$. It is defined as the maximum

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value of depth N for a module N with $H_M(t) = H_N(t)$. We must emphasize that we will always consider the standard grading on R under which all indeterminates have degree 1. As soon as this hypothesis is dropped, matters become extremely difficult as witnessed by the paper **[9]** of the second and third author.

The objective of this paper is to bound the Castelnuovo-Mumford regularity $\operatorname{reg} M$ in terms of $H_M(t)$. Of course, the bound is the lowest possible value of $\operatorname{reg} N$ for a module N with $H_M(t) = H_N(t)$, which we term *Hilbert regularity* $\operatorname{Hreg} M$.

Both Hilbert depth and Hilbert regularity can be computed in terms of Hilbert decompositions introduced by Bruns, Krattenthaler and Uliczka **[3]** for arbitrary gradings; for a method computing Hilbert depth for $\mathbb{Z}^n$-graded modules, see Ichim and the second author **[7]**. The approach by Hilbert decompositions is related to Stanley depth and Stanley regularity, see Herzog **[5]** for a survey. Stanley regularity for quotients by monomial ideals was considered by Jahan **[8]**. Also, Herzog introduced Hilbert regularity via decompositions.

Write $H_M(t) = Q(t)/(1-t)^d$ with $d = \dim M$ and $Q \in \mathbb{Z}[t]$ (we may certainly assume that M is generated in degrees ≥ 0). Then $\operatorname{Hdepth} M = d - m$, where m is the smallest value of all natural numbers j such that $Q(t)/(1-t)^j$ is a positive power series, i.e., a power series with nonnegative coefficients **[11]**. Note that the Hilbert series $Q(t)/(1-t)^d$ has nonnegative coefficients. Hilbert regularity cannot always be described in such a simple way, but it is closely related to the smallest k for which $Q(1-t)/(1-t)^k$ is positive, see Theorems 4.7 and 4.10.

Our main tool for the analysis of Hilbert series is

$$H(t) = \sum_{i=0}^{k-1} \frac{f_i t^i}{(1-t)^n} + \frac{ct^k}{(1-t)^n} + \sum_{j=0}^{d-n-1} \frac{g_j t^k}{(1-t)^{d-j}},$$

which we call (n,k)-*boundary presentations* since the pairs of exponents (u,v) occurring in the numerator and denominator of the terms $t^i/(1-t)^n$, $t^k/(1-t)^n$, and $t^k/(1-t)^{d-j}$ occupy the lower and the right boundary of a rectangle in the u-v-plane whose right lower corner is (k,n).

Using the description of Hilbert regularity in terms of Hilbert decompositions, one easily sees that $\operatorname{Hreg} M$ is the smallest k for which a $(0,k)$-boundary representation with *nonnegative* coefficients f_i, c, g_j exists. (Without the requirement of nonnegativity the smallest such k is $\deg H_M(t)$.) The bridge to power series expansions of $Q(1-t)/(1-t)^k$ are given by the fact that the coefficients g_j appear in such expansions.

The paper is structured as follows. We introduce Hilbert regularity in Section 2 and discuss boundary representations in Section 3. Hilbert regularity is then determined in Section 4, whereas Section 5 contains an algorithm that computes Hilbert depth and Hilbert regularity simultaneously.

2. Hilbert regularity. Let K be a field, and let M be a finitely generated graded module over a positively graded K-algebra R. The Castelnuovo-Mumford regularity of M is given by

$$\operatorname{reg} M = \max\left\{i + j : H^i_{\mathfrak{m}}(M)_j \neq 0\right\},$$

where $\mathfrak{m}$ is the maximal ideal of R generated by the elements of positive degree.

In the sequel, we only consider $R = K[X_1, \ldots, X_d]$. In this case, a theorem of Eisenbud and Goto [**2,** 4.3.1] yields

$$\operatorname{reg} M = \max\left\{j - i : \operatorname{Tor}^R_i(K, M)_j \neq 0\right\},$$

where K is naturally identified with $R/\mathfrak{m}$.

Definition 2.1. The (plain) *Hilbert regularity* of a finitely generated graded R-module is

$$\operatorname{Hreg} M := \min\left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there is an f.g. gr. } R\text{-module } N \\ \text{with } H_N = H_M \text{ and } \operatorname{reg} N = r. \end{array} \right\}.$$

Let F_i be a graded free module over $K[X_1, \ldots, X_i]$, $i = 1, \ldots, d$, considered as an R-module via the retraction $R \to K[X_1, \ldots, X_i]$ that sends $X_{i+1}, \ldots, X_d$ to 0. The module $F_0 \oplus \cdots \oplus F_d$ is called a *Hilbert decomposition* of M if the Hilbert functions of M and $F_0 \oplus \cdots \oplus F_d$ coincide. This leads us to the following:

Definition 2.2. The *decomposition Hilbert regularity* of M is

$$\operatorname{decHreg} M = \min \{\operatorname{reg} N : H_N(t) = H_M(t)\},$$

where now N ranges over direct sums $F_0 \oplus \cdots \oplus F_d$, i.e., over the Hilbert decompositions of M.

It is in particular clear that $\operatorname{decHreg} M \geq \operatorname{Hreg} M$. As we shall see below both numbers coincide in our setting of standard graded polynomial rings. However, both definitions make sense in much more generality if one replaces the $K[X_1, \ldots, X_i]$ by graded retracts of $K[X_1, \ldots, X_d]$, see **[3]**. In the more general setting the equality is a completely open problem, for regularity as well as for depth. In fact, proving equality for depth in the multigraded setting would come close to proving the Stanley conjecture for depth, see **[5]**.

Remark 2.3.

(a) The notion of Hilbert decomposition is the same as that in **[3]**, except that the F_i are further decomposed into cyclic modules there.

(b) Hilbert depth and Hilbert regularity are companions in the following sense: the Hilbert depth determines the smallest width of a Betti table admitting the given Hilbert series, Hilbert regularity determines the smallest such possible height. The Betti table is given in terms of the graded Betti numbers $\beta_{i,j} = \dim_K \operatorname{Tor}_i^R(K, M)_j$ by

$$\begin{array}{cccc} \beta_{0,0} & \beta_{1,1} & \cdots & \beta_{p,p} \\ \vdots & \vdots & & \vdots \\ \beta_{0,r} & \beta_{1,r+1} & \cdots & \beta_{p,r+p}, \end{array}$$

where $p = \operatorname{projdim} M$ and $r = \operatorname{reg} M$.

The decomposition Hilbert regularity can be described in terms of *positive representations* $\mathcal{P} = (Q_d, \ldots, Q_0)$ of the Hilbert series:

$$H_M(t) = \frac{Q_d(t)}{(1-t)^d} + \cdots + \frac{Q_1(t)}{(1-t)^1} + Q_0(t),$$

where each Q_i is a polynomial with nonnegative coefficients. Such polynomials will be called *nonnegative*. It is well known that there is always a Hilbert decomposition of M. This simple fact will be proved (again) in Proposition 2.5.

Let $F_0 \oplus \cdots \oplus F_d$ be a Hilbert decomposition of M. Then, we have

$$H_{F_i} = Q_i(t)/(1-t)^i$$

with a nonnegative polynomial Q_i, and we immediately get a positive representation of the Hilbert series. Conversely, given a positive representation of the Hilbert series, we find a direct sum $F_0 \oplus \cdots \oplus F_d$ by choosing F_i as the free module over $K[X_1, \cdots, X_i]$ that has a_{ij} basis elements of degree i where

$$Q_i = \sum_j a_{ij} t^j.$$

Moreover, $\operatorname{reg} F_i = \deg Q_i$, and therefore, we have

Proposition 2.4.

$$\operatorname{decHreg} M = \min_{\mathcal{P}} \max_i \deg Q_i, \quad \mathcal{P} = (Q_d, \ldots, Q_0),$$

where $\mathcal{P}$ ranges over the positive representations of $H_M(t)$.

For Hilbert depth, we can similarly give a "plain" or a "decomposition" definition: the *Hilbert depth* of M is defined as:

$$\operatorname{Hdepth} M := \max \left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there is an f.g. gr. } R\text{-module } N \\ \text{with } H_N = H_M \text{ and } \operatorname{depth} N = r \end{array} \right\}.$$

The Hilbert depth of M turns out to coincide with the arithmetical invariant

$$\operatorname{p}(M) := \max \{ r \in \mathbb{N} \mid (1-t)^r H_M(t) \text{ is nonnegative} \},$$

called the *positivity* of M, see **[11,** Theorem 3.2]. The inequality $\operatorname{Hdepth} M \le \operatorname{p}(M)$ follows from general results on Hilbert series and regular sequences. The converse can be deduced from the main result of **[11,** Theorem 2.1], which states the existence of a representation

$$H_M(t) = \sum_{j=0}^{\dim M} \frac{Q_j(t)}{(1-t)^j} \quad \text{with nonnegative } Q_j \in \mathbb{Z}[t, t^{-1}].$$

The decomposition version, or positivity, is close to Stanley decompositions and Stanley depth. The same holds true for Hilbert regularity, as we shall now show; our proof will also confirm the equivalence of the two notions of Hilbert depth.

Proposition 2.5. *There exists a Hilbert decomposition of regularity equal to* $\operatorname{reg} M$ *and depth equal to* $\operatorname{depth} M$.

Proof. If M is a free R-module, there is nothing to prove: M is already in Hilbert decomposition form.

Now, suppose that M is not free. Let m be the maximal degree of a generator of M. Then $m \leq \operatorname{reg} M$, and we can choose elements $v_1, \dots, v_n \in M$ of degree $\leq m$ such that $n = \operatorname{rank} M$ and $v_1, \dots, v_n$ are linearly independent. (This is a well-known general position argument; we may have to pass to an infinite field K, but that is no problem.) We set $F_n = Rv_1 + \cdots + Rv_n$. For the sake of Hilbert series computations we can replace M by $F_n \oplus M/F_n$.

Note that $\operatorname{depth} M/F_n = \operatorname{depth} M$ since $\operatorname{depth} M < \operatorname{depth} F_n$ by assumption on M and standard depth arguments. We obtain $\dim M/F_n < n$ since $\operatorname{rank} M/F_n = 0$ as an R-module.

For the regularity, we observe that M/F_n is generated in degrees $\leq m$ and $\dim M/F_n < n$. Since F_n is free, $\operatorname{Tor}_j^R(K, M/F_n) = \operatorname{Tor}_j^R(K, M)$ for $j \geq 2$, and therefore, 1 is the only critical homological degree for the regularity of M/F_n. There is a homogeneous exact sequence

$$\operatorname{Tor}_1^R(K, F_n) = 0 \longrightarrow \operatorname{Tor}_1^R(K, M) \longrightarrow \operatorname{Tor}_1^R(K, M/F_n) \\ \longrightarrow \operatorname{Tor}_0^R(K, F_n).$$

This is $\operatorname{Tor}_0^R(K, F_n)_i = 0$ and $\operatorname{Tor}_1^R(K, M)_i = \operatorname{Tor}_1^R(K, M/F_n)_i$, except for $i \leq m$. Thus, the only critical arithmetical degree is m. However, we subtract 1 from the highest shift in homological degree 1 in order to compute regularity, and it does not affect the inequality $\operatorname{reg} M/F_n \leq \operatorname{reg} M$ if $\operatorname{Tor}_1^R(K, M/F_n)_i \neq 0$ for some $i \leq m$.

On the other hand, $\operatorname{reg} M \leq \max(\operatorname{reg} F_n, \operatorname{reg} M/F_n)$, and altogether we conclude that $\operatorname{reg} M/F_n = \operatorname{reg} M$.

Let $S = R/\operatorname{Ann} M$, and choose a degree 1 Noether normalization R' in S. We first view M/F_n as a module over R'. Then,

$$\operatorname{reg}_R M/F_n = \operatorname{reg}_S M/F_n = \operatorname{reg}_{R'} M/F_n,$$

since regularity does not change under finite graded extensions. Now, we can identify R' with one of the algebras $K[X_1, \dots, X_i]$ for some $i < n$. Hence, we can proceed by induction considering M/F_n.

Eventually, the procedure ends when the dimension of the Noether normalization has reached the depth of M, since the quotient of M then attained is free over the Noether normalization, and we are in the case of a free module. $\square$

Remark 2.6. The proof shows that regularity may be considered as a measure for filtrations

$$0 = U_0 \subset U_1 \subset \cdots \subset U_q = M$$

in which U_{i+1}/U_i is always a free module over some polynomial subquotient of R: there exists such a filtration in which each free module is generated in degree $\leq \operatorname{reg} M$, but there is no such a filtration in which all base elements have smaller degree. A similar statement holds for depth.

Corollary 2.7. *Let M be a finitely generated graded R-module. Then,*

$$\operatorname{Hreg} M = \operatorname{decHreg} M.$$

In fact, if N is a module whose regularity attains the minimum, we can replace it by a Hilbert decomposition as in Proposition 2.5.

A specific example follows. Let M be the first syzygy module of the maximal ideal in the polynomial ring $K[X_1, \dots, X_5]$. It has been shown [**3,** Theorem 3.5] that it has multigraded Hilbert depth 4. It follows that the standard graded Hilbert depth is also 4, but this is much easier to see; the Hilbert series is

$$\frac{10t^2 - 10t^3 + 5t^4 - t^5}{(1-t)^5} = \frac{10t^2}{(1-t)^4} + \frac{t^4}{(1-t)^4} + \frac{4t^4}{(1-t)^5}. \tag{2.1}$$

Thus, we can get away with the worst denominator $(1-t)^4$ for the Hilbert depth.

Let us look at the Hilbert regularity: the decomposition

$$\frac{10t^2-10t^3+5t^4-t^5}{(1-t)^5} = \frac{4t^2}{(1-t)^5}+\frac{3t^2}{(1-t)^4}+\frac{2t^2}{(1-t)^3}+\frac{t^2}{(1-t)^2} \tag{2.2}$$

shows that $\operatorname{Hreg} M = 2$. It cannot be smaller since M has no generators in degree < 2. On the other hand, the decomposition (2.2) is the only one with regularity 2, since the powers of the series $1/(1-t)$ are linearly independent, and it comes from a filtration as in the proof of Proposition 2.5. (In this example, $\operatorname{Hreg} M$ could be determined more easily since $\operatorname{Hreg} M \geq 2$ and $\operatorname{reg} M = 2$.) This shows that, in general, optimization of depth and regularity cannot simultaneously occur.

More generally, if M is a module with all generators in degree r and of regularity r, then $\operatorname{Hreg} M = \operatorname{reg} M$.

However, in general, Hilbert regularity is smaller than regularity. Let N be the sum of the modules in the Hilbert decomposition (2.1). Then $\operatorname{Hreg} N < \operatorname{reg} N$ holds, as (2.2) shows.

A simple lower bound is as follows.

Proposition 2.8. *Let M be a finitely generated graded R-module. Then*

$$\operatorname{Hreg} M \geq \deg H_M(t).$$

In fact, for $j > \operatorname{Hreg} M$, the Hilbert polynomial and the Hilbert function of M coincide, and the smallest number k such that the Hilbert polynomial and the Hilbert function coincide in all degrees $j > k$ is $k = \deg H_M(t)$, the degree of H_M as a rational function; see [**2,** 4.1.12].

3. Boundary presentation. In this section, we introduce the fundamental tool for examination of the Hilbert regularity.

Definition 3.1. Let $H(t) = Q(t)/(1-t)^d$. For integers $0 \leq n \leq d$ and $k \geq 0$, an (n,k)*-boundary presentation* of H is a decomposition of H in the form

$$H(t) = \sum_{i=0}^{k-1} \frac{f_i t^i}{(1-t)^n} + \frac{ct^k}{(1-t)^n} + \sum_{j=0}^{d-n-1} \frac{g_j t^k}{(1-t)^{d-j}} \quad \text{with } f_i, c, g_j \in \mathbb{Z}. \tag{3.1}$$

If $c = 0$, the boundary presentation is called *corner-free.*

Note that $Q(t)/(1-t)^d$ may be viewed as a $(d, \deg Q)$-boundary presentation of H. If $\deg Q \le d$, there is also a $(d - \deg Q, 0)$-boundary presentation: let $Q(1-t) = \sum_i \widetilde{q}_i t^i$, then

$$H(t) = \frac{Q(t)}{(1-t)^d} = \frac{\sum_{i=0}^{\deg Q} \widetilde{q}_i (1-t)^i}{(1-t)^d} = \sum_{i=0}^{\deg Q} \frac{\widetilde{q}_i}{(1-t)^{d-i}}.$$

In the sequel, the polynomial $Q(1-t)$ will be needed several times; therefore, we introduce the notation

$$\widetilde{Q}(t) := Q(1-t)$$

for an arbitrary $Q \in \mathbb{Z}[t]$.

Example 3.2. Let

$$H(t) = \frac{1 - 2t + 3t^2 - t^3}{(1-t)^3}.$$

A $(1, 3)$-boundary presentation of H is given by

$$H(t) = \frac{1}{1-t} + \frac{2t^2}{1-t} + \frac{2t^3}{(1-t)^2} + \frac{t^3}{(1-t)^3}.$$

The term “boundary presentation” is motivated by visualization of a decomposition of a Hilbert series. A decomposition

$$\frac{Q(t)}{(1-t)^d} = \sum_{i=0}^{d} \sum_{j \ge 0} a_{ij} \frac{t^j}{(1-t)^i}$$

can be depicted as a square grid with the box at position (i, j) labeled by a_{ij}.

In the case of an (n, k)-boundary presentation the nonzero labels in this grid form the bottom and the right edges of a rectangle with $d - n + 1$ rows and $k + 1$ columns. The coefficient in the “corner”

3				1	
2				2	
1	1	0	2	0	
0					
	0	1	2	3	4

3	1				
2	-1				
1	0				
0	1				
	0	1	2	3	4

FIGURE 1. Two boundary presentations of $(1-2t+3t^2-t^3)/(1-t)^3$.

$(d-n,k)$ plays a dual role since it belongs to both edges; therefore, it is denoted by an extra letter.

Next, we deduce a description for the coefficients in a boundary presentation.

Lemma 3.3. *Let $H(t) = Q(t)/(1-t)^d$ be a series with (n,k)-boundary presentation* (3.1)*. Moreover, let*

$$\frac{Q(t)}{(1-t)^{d-n}} = \sum_{i=0}^{\infty} a_i t^i \quad \text{and} \quad \frac{\widetilde{Q}(t)}{(1-t)^k} = \sum_{i=0}^{\infty} b_i t^i.$$

Then

$$f_i = a_i \quad \text{for } i = 0,\dots,k-1,$$
$$c = a_k - \sum_{i=0}^{d-n-1} b_i = b_{d-n} - \sum_{i=0}^{k-1} a_i,$$
$$g_j = b_j \quad \text{for } j = 0,\dots,d-n-1.$$

Proof. Multiplication of (3.1) by $(1-t)^n$ yields

$$\frac{Q(t)}{(1-t)^{d-n}} = \sum_{i=0}^{k-1} f_i t^i + ct^k + \sum_{j=0}^{d-n-1} \frac{g_j t^k}{(1-t)^{d-n-j}}.$$

Hence, the f_i agree with the first k coefficients of the power series $\sum_{i=0}^{\infty} a_i t^i$, while $a_k = c + \sum_{j=0}^{d-n-1} g_j$.

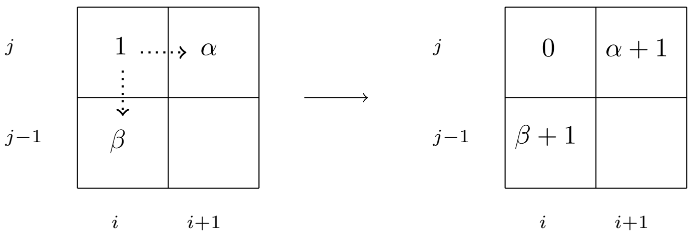

FIGURE 2.

Next, we look at (3.1) with t substituted by $1-t$:

$$\frac{Q(1-t)}{t^d} = \sum_{i=0}^{k-1} \frac{f_i(1-t)^i}{t^n} + \frac{c(1-t)^k}{t^n} + \sum_{j=0}^{d-n-1} \frac{g_j(1-t)^k}{t^{d-j}}.$$

This time, multiply by $t^d/(1-t)^k$ and obtain

$$\frac{\widetilde{Q}(t)}{(1-t)^k} = \frac{Q(1-t)}{(1-t)^k} = \sum_{i=0}^{k-1} \frac{f_i t^{d-n}}{(1-t)^{k-i}} + ct^{d-n} + \sum_{j=0}^{d-n-1} g_j t^j;$$

hence, $g_j = b_j$ for $j = 0, \ldots, d-n-1$ and $c = b_{d-n} - \sum_{i=0}^{k-1} f_i$. $\square$

Since the coefficients in the power series expansion of a rational function are unique, Lemma 3.3 has an immediate consequence:

Corollary 3.4. *The coefficients in an (n,k)-boundary presentation of $H(t) = Q(t)/(1-t)^d$ are uniquely determined.*

In the rest of this section we will make extensive use of the relation

$$\frac{t^i}{(1-t)^j} = \frac{t^{i+1}}{(1-t)^j} + \frac{t^i}{(1-t)^{j-1}}, \quad j > 1. \tag{3.2}$$

Repeated application of this relation allows us to transform an (n,k)-boundary presentation of a rational function H into an $(n-1,k)$, respectively, $(n,k+1)$-boundary presentation. We give a formula for the coefficients of the new boundary presentation in terms of the old coefficients next.

Lemma 3.5. *Let*

$$H(t)=\sum_{i=0}^{k-1}\frac{f_it^i}{(1-t)^n}+\frac{ct^k}{(1-t)^n}+\sum_{j=0}^{d-n-1}\frac{g_jt^k}{(1-t)^{d-j}}$$

be an (n,k)-boundary presentation. Then, there exists a corner-free $(n,k+1)$-boundary presentation; its coefficients $f^{(k+1)},g^{(k+1)}$ are given by

$$f_i^{(k+1)}=\begin{cases}f_i & \text{for } i=0,\dots,k-1\\ c+\sum_{r=0}^{d-n-1}g_r & \text{for } i=k\end{cases}$$

$$g_j^{(k+1)}=\sum_{r=0}^{j}g_r\quad \text{for } j=0,\dots,d-n-1.$$

If $n>0$, then there is also a corner-free $(n-1,k)$-boundary presentation with coefficients $f^{(n-1)},g^{(n-1)}$ given by

$$f_i^{(n-1)}=\sum_{r=0}^{i}f_r\quad \text{for } i=0,\dots,k-1$$

$$g_j^{(n-1)}=\begin{cases}g_j & \text{for } j=0,\dots,d-n-1\\ c+\sum_{r=0}^{k-1}f_r & \text{for } j=d-n.\end{cases}$$

In particular, an expansion of a corner-free boundary presentation leads to a boundary presentation with the entries next to the corner being equal.

Corollary 3.6. *Let*

$$H(t)=\sum_{i=0}^{k-1}\frac{f_it^i}{(1-t)^n}+\sum_{j=0}^{d-n-1}\frac{g_jt^k}{(1-t)^{d-j}}$$

be a corner-free (n,k)-boundary presentation. If $k>0$, then there exists an $(n,k-1)$-boundary presentation; its coefficients $f^{(k-1)},c^{(k-1)},g^{(k-1)}$ are given by

$$f_i^{(k-1)}=f_i\quad \text{for } i=0,\dots,k-2,$$
$$c^{(k-1)}=f_{k-1}-g_{d-n-1},$$

$$g_j^{(k-1)} = \begin{cases} g_0 & \textit{for } j = 0 \\ g_j - g_{j-1} & \textit{for } j = 1, \dots, d-n-1. \end{cases}$$

If $n < d$, then there is also an $(n+1, k)$-boundary presentation with coefficients $f^{(n+1)}$, $c^{(n+1)}$, $g^{(n+1)}$ given by

$$\begin{aligned} f_i^{(n+1)} &= \begin{cases} f_0 & \textit{for } i = 0 \\ f_i - f_{i-1} & \textit{for } i = 1, \dots, k-1, \end{cases} \\ c^{(n+1)} &= g_{d-n-1} - f_{k-1}, \\ g_j^{(n+1)} &= g_j \quad \textit{for } j = 0, \dots, d-n-2. \end{aligned}$$

Corollary 3.7. *If a rational function H admits an (n, k)-boundary presentation, then there is also an (n', k')-boundary presentation for every pair (n', k') with $n' \leq n$, $k' \geq k$; for $(n', k') \neq (n, k)$, this presentation is corner-free. Moreover, the coefficients of this (n', k')-boundary presentation are nonnegative, provided that the same holds for the (n, k)-boundary presentation.*

In particular, there exists an (n, k)-boundary presentation of the series $Q(t)/(1-t)^d$ for every $k \geq \deg Q$ and $n = 0, \dots, d-1$. Note that, in these cases, the formula of Lemma 3.5 provides an alternative proof for the equality of the coefficients f_i and the first coefficients of $Q(t)/(1-t)^{d-n}$. Analogously, if $d \geq \deg Q$, then the $(d - \deg Q, 0)$-boundary presentation can be expanded to an (n, k)-boundary presentation for $n = 0, \dots, d - \deg Q$ and $k \geq 1$, also confirming the description of the g_j.

Lemma 3.8. *An (n, k)-boundary presentation of a rational function $H(t) = Q(t)/(1-t)^d$ exists if and only if $k - n \geq \deg H = \deg Q - d$. It is corner-free if and only if $k - n > \deg H$; otherwise the entry in the corner is given by $(-1)^{d-n} a_q$, where a_q denotes the leading coefficient of Q.*

Proof. An (n, k)-boundary presentation with $n, k > 0$ can be transformed into an $(n-1, k-1)$-boundary presentation, as the relation

(3.2) implies:

$$\begin{aligned}
&\sum_{i=0}^{k-1}\frac{f_i t^i}{(1-t)^n}+\frac{ct^k}{(1-t)^n}+\sum_{j=0}^{d-n-1}\frac{g_j t^k}{(1-t)^{d-j}}\\
&\quad=\sum_{i=0}^{k-2}\frac{\left(\sum_{r=0}^{i} f_r\right)t^i}{(1-t)^{n-1}}-\frac{ct^{k-1}}{(1-t)^{n-1}}+\sum_{j=0}^{d-n-1}\frac{g_j t^{k-1}}{(1-t)^{d-j}}\\
&\qquad+\frac{\left(c-g_{d-n-1}+\sum_{r=0}^{k-1} f_r\right)t^{k-1}}{(1-t)^n}.
\end{aligned}$$

Hence, the existence of a non-corner-free (n,k)-boundary presentation of H for (n,k) with $k-n=\deg Q-n$ and the assertion on the entry in its corner follow by induction on $d-n$ beginning with the $(d,\deg Q)$-boundary presentation of H. The remainder is clear since an (n,k)-boundary presentation which is not corner-free cannot be obtained by expanding some (n',k')-boundary presentation with $n'\geq n$, $k'\leq k$. $\square$

Since any (n,k)-boundary presentation with $k>\deg Q$ can be obtained as an expansion of the $(d,\deg Q)$-boundary presentation of $Q(t)/(1-t)^d$, we obtain a second description of the coefficients g_j:

Proposition 3.9. *Let*

$$H(t)=\frac{Q(t)}{(1-t)^d}=\sum_{i=0}^{k-1}\frac{f_i t^i}{(1-t)^n}+\sum_{j=0}^{d-n-1}\frac{g_j^{(k)} t^k}{(1-t)^{d-j}}$$

with $k>d$. Then, the coefficient $g_j^{(k)}$ for $j=1,\dots,d-n-1$ agrees with the $(k-1)$th coefficient of the power series expansion of $Q(t)/(1-t)^{j+1}$. In particular, for $Q(t)/(1-t)^k=\sum_{n\geq 0} a_n^{(k)}t^n$ and $\widetilde{Q}(t)/(1-t)^k=\sum_{n\geq 0} b_n^{(k)}t^n$, we have

$$b_j^{(k)}=a_{k-1}^{(j+1)}\quad \textit{for } k\geq \deg Q \textit{ and } j=0,\dots,d-1.$$

Proof. Let $0\leq j\leq d-1$. We consider the $(d-1-j,k)$-boundary presentation of H with $k>\deg Q$. Since this can be viewed as an expansion of the corner-free $(d, \deg(Q)+1)$-boundary presentation,

$$\frac{Q(t)}{(1-t)^d} + \frac{0 \cdot t^{\deg(Q)+1}}{(1-t)^d},$$

we have $f_{k-1}^{(d-1-j)} = g_j^{(k)}$; thus, by Lemma 3.3, the coefficient $g_j^{(k)}$ agrees with the $(k-1)$th coefficient of

$$\frac{Q(t)}{(1-t)^{d-(d-1-j)}} = \frac{Q(t)}{(1-t)^{j+1}}.$$

Expanding the $(d-1-j, k)$-boundary presentation downwards does not affect $g_j^{(k)}$; therefore, this equality is also valid for any (n, k)-boundary presentation with $n \le d-1-j$. The second part follows immediately from Lemma 3.3. □

4. Arithmetical characterization of the Hilbert regularity. In this section, we continue our investigation of the Hilbert regularity so we restrict our attention to nonnegative series $H(t) = Q(t)/(1-t)^d$. As mentioned above, such a series admits a Hilbert decomposition. In particular, H is the Hilbert series of some finitely generated graded R-module M; we set $\operatorname{Hdepth} H := \operatorname{Hdepth} M$ and $\operatorname{Hreg} H := \operatorname{Hreg} M$. It is easy to see that H also admits a boundary presentation with nonnegative coefficients. In the sequel, such a boundary presentation will be called *nonnegative* for short.

Lemma 4.1. *Let*

$$H(t) = \sum_{i=n}^{d} \frac{Q_i(t)}{(1-t)^i}$$

be a Hilbert decomposition, and let $k = \max_i \deg Q_i$. *Then, there exists a nonnegative* (n, k)*-boundary presentation of* H.

Proof. Obviously, a Hilbert decomposition can be rewritten as

$$\sum_{i=n}^{d} \frac{Q_i(t)}{(1-t)^i} = \sum_{j=n}^{d} \sum_{i=0}^{k} \frac{a_{ij} t^i}{(1-t)^j} \quad \text{with } a_{ij} \in \mathbb{N}. \tag{4.1}$$

It is sufficient to show that this decomposition can be turned into one of the form

$$\sum_{j=n}^{p}\sum_{i=0}^{k}\frac{b_{ij}t^i}{(1-t)^j}+\sum_{j=p+1}^{d}\frac{b_{kj}t^k}{(1-t)^j}\quad\text{with } b_{ij}>0$$

for any p with $n \leq p \leq d$. Repeated application of relation (3.2) yields

$$\sum_{i=0}^{k}\frac{b_{ij}t^i}{(1-t)^j}=\sum_{i=0}^{k-1}\frac{(\sum_{r=0}^{i}b_{rj})t^i}{(1-t)^{j-1}}+\frac{(\sum_{r=0}^{k}b_{rj})t^k}{(1-t)^j}.$$

Since the coefficients on the right-hand side are still nonnegative, the claim follows by reverse induction on $p \leq d$, starting with the vacuous case $p = d$. □

Corollary 4.2.

(a) *Let* $H(t) = Q(t)/(1-t)^d$ *be a nonnegative series. Then,* H *admits a nonnegative* $(0, \operatorname{Hreg} H)$*-boundary presentation as well as a nonnegative* $(\operatorname{Hdepth} H, k)$*-boundary presentation with suitable* $k \geq 0$.

(b) *If* H *admits a non-corner-free* $(0, k)$*-boundary presentation, then it holds that* $\operatorname{Hreg} H \geq k$.

Proof. Statement (a) is clear from the definition of $\operatorname{Hreg} H$, respectively, $\operatorname{Hdepth} H$. Part (b) follows from (a) and the fact that a non corner-free boundary presentation cannot be obtained by expansion of another boundary presentation. □

Remark 4.3. By Lemma 3.8, a $(0, k)$-boundary presentation of H exists if and only if $k \geq \deg H$; together with Corollary 4.2 (a), this yields another proof of Proposition 2.8.

Corollary 4.2 implies that, for computations of Hilbert regularity (and also of Hilbert depth), we may exclusively consider boundary presentations. This observation leads to an estimate for $\operatorname{Hreg} M$ in the flavor of the equality $\mathrm{p}(M) = \operatorname{Hdepth} M$. In order to formulate this inequality we need the following notion:

Definition 4.4. For any $Q \in \mathbb{Z}[t]$ and $k \in \mathbb{N}$, let $Q(t)/(1-t)^k = \sum_{n\geq 0} a_n^{(k)} t^n$. For any $d \in \mathbb{N}$, we set

$$\delta_d(Q) := \min\left\{k \in \mathbb{N} \mid a_0^{(k)}, \dots, a_{d-1}^{(k)} \text{ nonnegative}\right\}$$

and

$$\delta(Q) := \min\left\{k \in \mathbb{N} \mid \frac{Q(t)}{(1-t)^k} \text{ nonnegative}\right\}.$$

Note that $\delta_d(Q)$ is finite if and only if the lowest nonvanishing coefficient of Q is nonnegative, as one easily sees by induction on d. By [**11,** Theorem 4.7], $\delta(Q)$ is finite if and only if Q, viewed as a real-valued function of one variable, takes positive values in the open interval $(0, 1)$.

For a finitely generated graded R-module M with Hilbert series

$$H_M(t) = \frac{Q_M(t)}{(1-t)^{\dim M}},$$

the equality $\operatorname{Hdepth} M = \mathrm{p}(M)$ implies

$$\delta(Q_M) = \dim M - \operatorname{Hdepth} M;$$

thus, according to [**2,** Proposition 1.5.15] and the Auslander-Buchsbaum theorem, $\delta(Q_M)$ could be called $\operatorname{Hprojdim} M$, the *Hilbert projective dimension.* Note that $\operatorname{Hprojdim} M$ depends only upon Q_M but not upon $\dim M$.

The above-mentioned estimate for the Hilbert regularity reads as follows:

Proposition 4.5. *Let $H(t) = Q(t)/(1-t)^d$ be a nonnegative series. Then*

$$\operatorname{Hreg} H \geq \delta_d(\widetilde{Q}).$$

Proof. Since $\widetilde{Q}(0) = Q(1) > 0$, $\delta_d(\widetilde{Q})$ is finite. Let $\operatorname{Hreg} H = k$. Then there exists a $(0, k)$-boundary presentation

$$H(t) = \sum_{i=0}^{k-1} f_i t^i + ct^k + \sum_{j=0}^{d-1} \frac{g_j t^k}{(1-t)^{d-j}}$$

with nonnegative coefficients. By Lemma 3.3, the first d coefficients of $\widetilde{Q}(t)/(1-t)^k$ agree with the coefficients g_j, and therefore, they are nonnegative. Hence, $\delta_d(\widetilde{Q}) \leq k = \operatorname{Hreg} H$. $\square$

Proposition 4.6. *Under the hypothesis of Proposition* 4.5, *in addition, we have* $\operatorname{Hreg} H \geq \delta(\widetilde{Q})$.

Proof. An (n,k)-boundary presentation of the series $Q(t)/(1-t)^d$ induces an $(n+m,k)$-boundary presentation of $Q(t)/(1-t)^{d+m}$, $m \in \mathbb{N}$, with the same coefficients. The $(0, \operatorname{Hreg} H)$-boundary presentation of $Q(t)/(1-t)^d$ has nonnegative coefficients; hence, the same holds for the $(m, \operatorname{Hreg} H)$-boundary presentation of $Q(t)/(1-t)^{d+m}$, and, by Corollary 3.7, the $(0, \operatorname{Hreg} H)$-boundary presentation of $Q(t)/(1-t)^{d+m}$ is also nonnegative. This implies $\delta_{d+m}(\widetilde{Q}) \leq \operatorname{Hreg} H$ for all $m \in \mathbb{N}$, and thus, $\delta(\widetilde{Q}) \leq \operatorname{Hreg} H$, as desired. $\square$

Theorem 4.7. *Under the hypothesis of Proposition* 4.5 *and the additional assumption of either* (i) $\delta_d(\widetilde{Q}) \geq \deg Q$ *or* (ii) $\deg Q \leq d$, *we have*

$$\operatorname{Hreg} H = \delta_d(\widetilde{Q}) = \delta(\widetilde{Q}).$$

Proof. In both cases, expansion of the $(d, \deg Q)$, respectively, the $(d - \deg Q, 0)$-boundary presentation, yields a $(0, \delta_d(\widetilde{Q}))$-boundary presentation of H, which is nonnegative by the nonnegativity of H and the definition of $\delta_d(\widetilde{Q})$. Hence,

$$\delta_d(\widetilde{Q}) \geq \operatorname{Hreg} H \geq \delta(\widetilde{Q}) \geq \delta_d(\widetilde{Q}). \qquad \square$$

The next example shows that, contrary to $\operatorname{Hdepth} M \leq \mathrm{p}(M)$ in case of the Hilbert depth, the inequality $\operatorname{Hreg} H \geq \delta_d(\widetilde{Q})$ may be strict.

Example 4.8. For

$$H(t) = \frac{1 - t + 2t^2 - 2t^3 + t^4}{(1-t)^2},$$

we obtain $\widetilde{Q}(t) = Q(t)$. Therefore,

$$\frac{\widetilde{Q}(t)}{1-t} = \frac{Q(t)}{1-t} = 1 + 0t + 2t^2 + 0t^3 + \sum_{n\geq 4} t^n$$

implies $\delta_2(\widetilde{Q}) = 1 = \operatorname{Hprojdim} H$. The $(0,2)$-boundary presentation of H is given by

$$H(t) = 1 + t + t^2 + \frac{t^2}{1-t} + \frac{t^2}{(1-t)^2}.$$

Since this is not corner-free, Lemma 3.8 implies $\operatorname{Hreg} H = 2 > 1 = \delta_2(\widetilde{Q})$. In particular, the Hilbert regularity of $Q(t)/(1-t)^d$ depends on d: for

$$H'(t) = \frac{1 - t + 2t^2 - 2t^3 + t^4}{(1-t)^d}$$

with $d \geq 4$, we have $\operatorname{Hreg} H' = 1$ by Theorem 4.7.

Example 4.8 also explains that non-negativity of $\widetilde{Q}(t)/(1-t)^k$ for some $k \in \mathbb{N}$ does not ensure $\operatorname{Hreg} H \leq k$. The decomposition

$$\frac{\widetilde{Q}(t)}{(1-t)^k} = \sum_{i=0}^{k} \frac{\widetilde{Q}_i(t)}{(1-t)^i}$$

with nonnegative $\widetilde{Q}_i \in \mathbb{Z}[t]$ according to [**11,** Theorem 2.1] can be turned into one of

$$\frac{Q(t)}{(1-t)^{\max\{\deg \widetilde{Q}_i\}}}$$

by exchanging t and $1-t$, but, if $d < \max\{\deg \widetilde{Q}_i\}$, this does not yield a decomposition of $Q(t)/(1-t)^d$.

Due to the difficulty illustrated by the previous example the general description of Hilbert regularity is less straightforward than that of Hilbert depth. In the remaining case of $\deg Q > d, \delta(\widetilde{Q})$, the $(0, \deg Q)$-boundary presentation is nonnegative, and hence, $\operatorname{Hreg} H \leq \deg Q$. If $\operatorname{Hreg} H < \deg Q$, then the $(0, \deg Q)$-boundary presentation can be reduced to a nonnegative $(0, k)$-boundary presentation with smaller k. Such a reduction may be performed in steps; therefore, we investigate whether a reduction from k to $k-1$ is possible in what follows.

Proposition 4.9. *Let*

$$H(t) = \sum_{i=0}^{k-1} \frac{f_i t^i}{(1-t)^n} + \frac{ct^k}{(1-t)^n} + \sum_{j=0}^{d-n-1} \frac{g_j t^k}{(1-t)^{d-j}}$$

with nonnegative coefficients. Then,

$$\operatorname{Hreg} H \le k-1 \iff \begin{cases} c = 0 \\ f_{k-1} \ge g_{d-n-1} \\ g_{j+1} \ge g_j & \textit{for } j = 0, \dots, d-n-2. \end{cases}$$

Proof.

$\Rightarrow$. Let $\operatorname{Hreg} H \le k-1$. Then there exists a boundary presentation

$$H(t) = \sum_{i=0}^{k-2} \frac{f'_i t^i}{(1-t)^n} + \frac{c' t^{k-1}}{(1-t)^n} + \sum_{j=0}^{d-n-1} \frac{g'_j t^{k-1}}{(1-t)^{d-j}} \tag{4.2}$$

with nonnegative coefficients. By Lemma 3.5, this presentation may be transformed into

$$H(t) = \sum_{i=0}^{k-2} \frac{f_i t^i}{(1-t)^n} + \frac{(c' + \sum_{j=0}^{d-n-1} g'_j) t^{k-1}}{(1-t)^n} + \sum_{j=0}^{d-n-1} \frac{(\sum_{i=0}^{j} g'_i) t^k}{(1-t)^{d-j}},$$

and, by uniqueness of the (n, k)-boundary presentation, we have

$$f_{k-1} = c' + \sum_{j=0}^{d-n-1} g'_j \ge \sum_{j=0}^{d-n-1} g'_j = g_{d-n-1}.$$

Necessity of the other conditions was already noted in Corollary 4.2 (b) and Proposition 4.5.

$\Leftarrow$. If the conditions on the right are satisfied then Corollary 3.6 yields a nonnegative $(0, k-1)$-boundary presentation (4.2). $\square$

The $(0, \operatorname{Hreg} H)$-boundary presentation can be achieved by iterated reduction steps starting from the $(0, \deg Q)$-boundary presentation. The reduction continues as long as the conditions of Proposition 4.9 remain valid. Hence, it ends in one of the three cases illustrated by the diagrams in Figure 3.

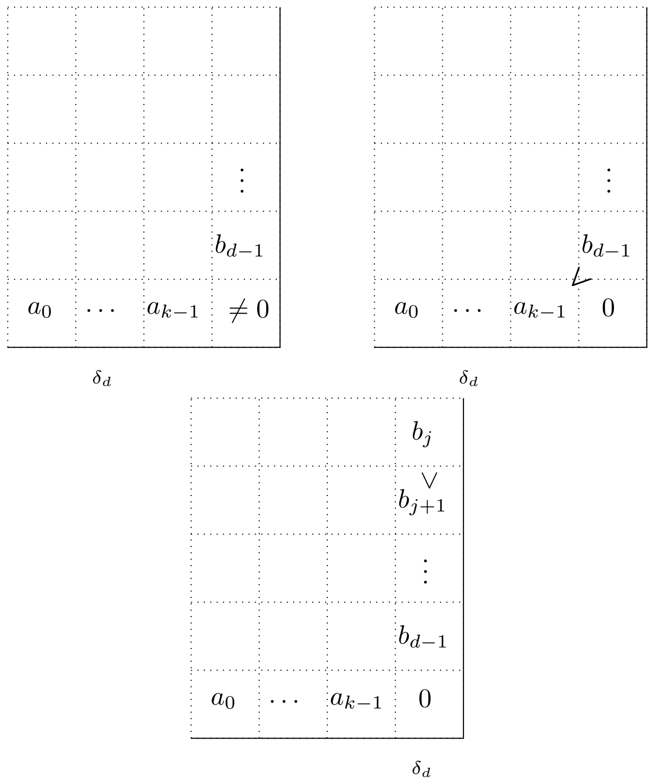

FIGURE 3.

Construction of the $(0, \operatorname{Hreg} H)$-boundary presentation may be described as follows. Beginning with $k = \deg Q$, we consider the $(0, k)$-boundary presentation. As long as $k > \delta_d(\widetilde{Q})$ and $f_{k-1} = g^{(k)}_{d-n-1}$, there is also a nonnegative and corner free $(0, k-1)$-boundary presentation; thus, we continue with $k-1$ instead of k. As soon as $k = \delta_d(\widetilde{Q})$ or $f_{k-1} \neq g^{(k)}_{d-n-1}$, we have reached the minimal k for which a nonnegative and corner-free $(0, k)$-boundary presentation exists. If $k = \delta_d(\widetilde{Q})$ or $f_{k-1} < g^{(k)}_{d-n-1}$, no further reduction is possible; hence, $\operatorname{Hreg} H = k$. However, if $k > \delta_d(\widetilde{Q})$ and $f_{k-1} \geq g^{(k)}_{d-n-1}$, one last reduction step, leading to a non corner-free boundary presentation, may be performed;

thus, $\operatorname{Hreg} H = k-1$ in this case. Note that, here, Lemma 3.8 implies $\operatorname{Hreg} H = \deg H$.

Theorem 4.10. *Let $H(t) = Q(t)/(1-t)^d = \sum_{n\geq 0} a_n t^n$ be a nonnegative series with $d > 0$, and let $\widetilde{Q}(t)/(1-t)^j = \sum_{n\geq 0} b_n^{(j)} t^n$ for $j \in \mathbb{N}$.*

(i) *If* $\deg Q \leq d$ *or* $\delta_d(\widetilde{Q}) \geq \deg Q$, *then* $\operatorname{Hreg} H = \delta_d(\widetilde{Q})$.
(ii) *Otherwise, with*

$$k := \min\{i \mid \delta_d(\widetilde{Q}) \leq i \leq \deg Q \quad \textit{and} \quad a_j = b_{d-1}^{(j+1)} \textit{ for all } j = i, \ldots, \deg Q\},$$

we have

$$\operatorname{Hreg} H = \begin{cases} k & \textit{if } k = \delta_d(\widetilde{Q}) \vee a_{k-1} < b_{d-1}^{(k)} \\ k-1 & \textit{if } k > \delta_d(\widetilde{Q}) \wedge a_{k-1} > b_{d-1}^{(k)}. \end{cases}$$

Proof. The cases in (i) were already treated in Theorem 4.7.

Part (ii) follows from the discussion preceding this theorem; the number k, which is well defined by Proposition 3.9, is merely the width of the minimal nonnegative and corner-free boundary presentation. □

The closing result of this section is the analogue of Proposition 4.6 for $\delta(Q)$.

Lemma 4.11. *Let $H(t) = Q(t)/(1-t)^d$ be nonnegative and*

$$e := \max\{\delta_d(\widetilde{Q}), \deg(Q) + 1\}.$$

Then, $\delta(Q) = \delta_e(Q)$.

Proof. The $(d - \delta_e(Q), \delta_e(Q))$-boundary presentation of the series H is nonnegative by Lemma 3.3, and the definition of $\delta_d(\widetilde{Q})$ and $\delta_e(Q)$. Hence the $(d - \delta_e(Q), \delta_{e+m}(Q))$-boundary presentation with $m \geq 0$ is nonnegative as well, but this implies that $\delta_{e+m}(Q) \leq \delta_e(Q)$ for all $m \in \mathbb{N}$. Therefore, $\delta(Q) = \delta_e(Q)$. □

5. Computation of Hilbert depth and Hilbert regularity. The aim of this section is an algorithm for computing the Hilbert depth and Hilbert regularity of a module with given Hilbert series $H(t) = Q(t)/(1-t)^d$, see Algorithm 5.1. An algorithm solely for the Hilbert depth was given by Popescu [**10**].

The correctness of Algorithm 5.1 follows immediately from the previous results. The output could easily be extended by boundary presentations realizing Hdepth or Hreg since the required coefficients are computed in the course; for example, a nonnegative boundary presentation of the minimal height Hdepth H is given by

$$H(t) = \sum_{i=0}^{e-1} \frac{a_i^{(h)} t^i}{(1-t)^h} + \begin{cases} \displaystyle\sum_{j=0}^{d-h-1} \frac{a_{e-1}^{(j+1)} t^e}{(1-t)^{d-j}} & \text{for } e = \deg Q > \delta_d(\widetilde{Q}) \\ \displaystyle\sum_{j=0}^{d-h-1} \frac{b_j^{(\delta_d(\widetilde{Q}))} t^e}{(1-t)^{d-j}} & \text{for } e = \delta_d(\widetilde{Q}) \geq \deg Q, \end{cases}$$

with a and b used as in the description of the algorithm, and $h :=$ Hprojdim H.

For completeness, we give an upper bound for the number of repetitions of the loop in the second step of Algorithm 5.1. The idea is to replace $\widetilde{Q}(t) = \sum_i \widetilde{q}_i t^i$ with a polynomial $\widetilde{q}_0 + rt$ such that, for all $n, i \in \mathbb{N}$, the coefficient $c_n^{(k)}$ of $(\widetilde{q}_0 + rt)/(1-t)^k$ is not greater than the coefficient $b_n^{(k)}$ of $\widetilde{Q}(t)/(1-t)^k$. Such a polynomial may be obtained by repeated application of the map

$$f = \sum_{i=0}^{m} h_i t^i \longmapsto \sum_{i=0}^{m-2} h_i t^i + \min\{h_{m-1}, h_{m-1} + h_m\} t^{m-1}$$

to the polynomial $\widetilde{Q}$. Since

$$\begin{aligned} \frac{\widetilde{q}_0 + rt}{(1-t)^k} &= \sum_{n\geq 0} \left[\widetilde{q}_0 \binom{n+k-1}{n-1} + r \binom{n+k-2}{n-2} \right] t^n \\ &= \sum_{n\geq 0} \left[\frac{\prod_{j=0}^{k-2}(n+j)}{k!} \left(\widetilde{q}_0(n+k-1) + r(n-1) \right) \right] t^n, \end{aligned}$$

Algorithm 5.1: Computing Hilbert depth and Hilbert regularity.

Input: $Q \in \mathbb{Z}[t], d \in \mathbb{Z}$ with $H(t) = Q(t)/(1-t)^d$ nonnegative

1 $\widetilde{Q}(t) := Q(1-t)$;

2 - - Determine $\delta_d(\widetilde{Q})$:

$k := -1$;

repeat

  $k := k + 1$;

  Compute the first d coefficients $b_0^{(k)}, \dots, b_{d-1}^{(k)}$ of $\widetilde{Q}(t)/(1-t)^k$;

until $b_0^{(k)}, \dots, b_{d-1}^{(k)}$ *nonnegative*;

$\delta_d(\widetilde{Q}) = k$;

3 - - Determine Hprojdim H:

$e := \max\{\delta_d(\widetilde{Q}), \deg(Q) + 1\}$;

$k := -1$;

repeat

  $k := k + 1$;

  Compute the first e coefficients $a_0^{(k)}, \dots, a_{e-1}^{(k)}$ of $Q(t)/(1-t)^k$;

until $a_0^{(k)}, \dots, a_{e-1}^{(k)}$ *nonnegative*;

Hprojdim $H = k$;

4 Hdepth $H = d -$ Hprojdim H;

5 - - Determine Hreg H:

if $\deg Q \leq d$ **or** $\delta_d(\widetilde{Q}) \geq \deg Q$ **then**

  Hreg $H = \delta_d(\widetilde{Q})$;

else

  Compute the ith coefficient a_i of H for $i = 0, \dots, \deg Q$;

  Compute the $(d-1)$th coefficient $b_{d-1}^{(j)}$ of $\dfrac{\widetilde{Q}(t)}{(1-t)^j}$ for $j = \delta_d(\widetilde{Q}), \dots, \deg Q$;

  $k := \min\{i \mid \delta_d(\widetilde{Q}) \leq i \leq \deg Q$ and $a_j = b_{d-1}^{(j+1)}$ for all $j = i, \dots, \deg Q\}$;

  if $a_{k-1} \geq b_{d-1}^{(k)}$ **and** $k > \delta_d(\widetilde{Q})$ **then**

    Hreg $H = k - 1$;

  else

    Hreg $H = k$;

  end

end

Output: Hdepth H, Hreg H

we want to determine the least k such that

$$\widetilde{q}_0(n+k-1)+r(n-1)=(\widetilde{q}_0+r)(n-1)+k\widetilde{q}_0 \geq 0 \tag{5.1}$$

holds for $0 \leq n \leq d-1$. Without loss of generality, we may assume that $q+r<0$. Then, (5.1) is equivalent to

$$n \leq 1-\frac{\widetilde{q}_0 k}{\widetilde{q}_0+r}.$$

This inequality must be valid, in particular, for $n=d-1$, and thus, for

$$k \geq \frac{(2-d)(\widetilde{q}_0+r)}{\widetilde{q}_0},$$

the first d coefficients of $(\widetilde{q}_0+rt)/(1-t)^k$ and a fortiori those of $\widetilde{Q}(t)/(1-t)^k$ are nonnegative.

Example 5.1. Let $H(t)=\dfrac{2-5t+t^2+4t^3}{(1-t)^7}$. Then $\widetilde{Q}(t)=Q(1-t)=2-9t+13t^2-4t^3$, and we find $\delta_7(\widetilde{Q})=7$ since

$$\begin{aligned}
\frac{\widetilde{Q}(t)}{(1-t)^5} &= 2+t-2t^2-4t^3+0t^4+17t^5+56t^6+\cdots\\
\frac{\widetilde{Q}(t)}{(1-t)^6} &= 2+3t+t^2-3t^3-3t^4+14t^5+70t^6+\cdots\\
\frac{\widetilde{Q}(t)}{(1-t)^7} &= 2+5t+6t^2+3t^3+0t^4+14t^5+84t^6+\cdots.
\end{aligned}$$

In order to determine the Hilbert depth we first compute the $\delta_7(\widetilde{Q})=7$ coefficients of $Q(t)/(1-t)^k$ for $k \geq 0$. Since

$$\begin{aligned}
\frac{Q(t)}{(1-t)^5} &= 2+5t+6t^2+4t^3+0t^4-3t^5+0t^6+\cdots\\
\frac{Q(t)}{(1-t)^6} &= 2+7t+13t^2+17t^3+17t^4+14t^5+14t^6+\cdots,
\end{aligned}$$

we have $\operatorname{Hdepth} H=7-6=1$. The Hilbert regularity requires no further computations since $\deg Q=3<7=d$, and thus, $\operatorname{Hreg} H=$

$\delta_7(\widetilde{Q}) = 7$. Moreover, in this case, the boundary presentation

$$H(t) = \frac{2 + 7t + 13t^2 + 17t^3 + 17t^4 + 14t^5 + 14t^6}{1-t} + \frac{14t^7}{(1-t)^2} + \frac{3t^7}{(1-t)^4} + \frac{6t^7}{(1-t)^5} + \frac{5t^7}{(1-t)^6} + \frac{2t^7}{(1-t)^7}$$

simultaneously has the minimal height $\operatorname{Hdepth} H$ and the minimal width $\operatorname{Hreg} H$.

Finally, we give two examples illustrating the remaining cases occurring in Algorithm 5.1.

Example 5.2. Let $H(t) = \dfrac{1 - t + t^3}{(1-t)^2}$. Then, $\widetilde{Q}(t) = 1 - 2t + 3t^2 - t^3$ and $\delta_2(\widetilde{Q}) = 2$. Since $\deg Q$ exceeds $\delta_d(\widetilde{Q})$ as well as d, the final loop of our algorithm applies. By

$$\frac{Q(t)}{(1-t)^2} = 1 + t + t^2 + 2t^3 + \cdots$$
$$\frac{\widetilde{Q}(t)}{(1-t)^2} = 1 + 0t + \cdots$$
$$\frac{\widetilde{Q}(t)}{(1-t)^3} = 1 + t + \cdots,$$

we find $k = 2 = \delta_2(\widetilde{Q})$. Hence, $\operatorname{Hreg} H = 2$.

This example confirms that $\operatorname{Hreg} H = \delta_d(\widetilde{Q})$ may also occur if $\deg Q > d, \delta_d(\widetilde{Q})$.

Example 5.3. For $H(t) = \dfrac{1 - t + 2t^2 - t^3}{(1-t)^2}$, we have $\delta_2(\widetilde{Q}) = 1$, and the calculations may be summarized by

The third subcase of $\operatorname{Hreg} H > \delta_d(\widetilde{Q})$, which leads to a non corner-free $(0, \operatorname{Hreg} H)$-boundary presentation, was already shown in Example 4.8.

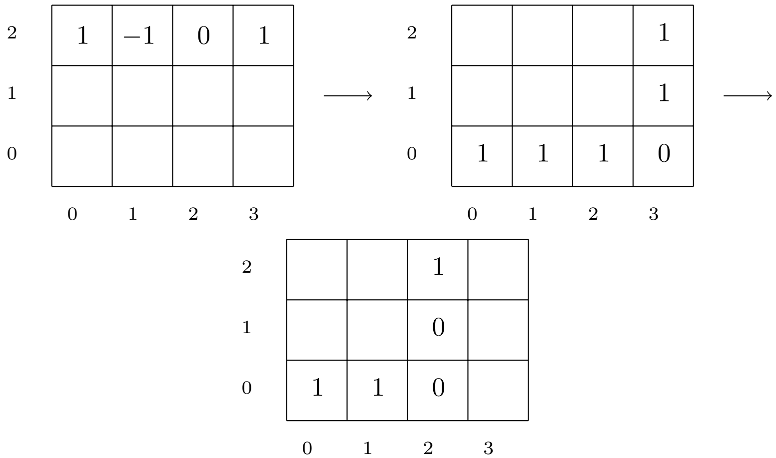

FIGURE 4. $H(t) = (1 - t + t^3)/(1 - t)^2$.

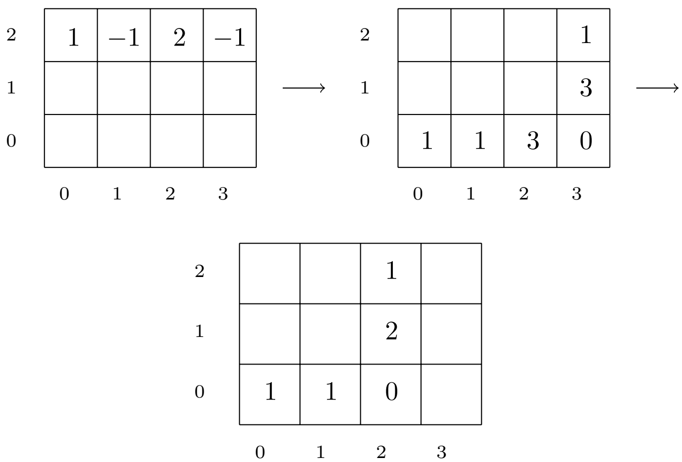

FIGURE 5. $H(t) = (1 - t + 2t^2 - t^3)/(1 - t)^2$.

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