Introduction to Normaliz 2.5

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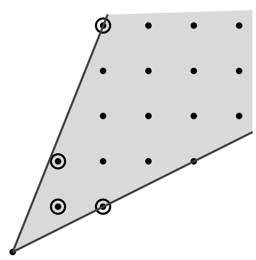
Abstract. In this paper we introduce the version 2.5 of Normaliz , a program for the computation of Hilbert bases of rational cones and the normalizations of affine monoids. It may also be used for solving diophantine linear systems of inequalities, equations and congruences. We present some of the new features of the program, as well as some recent achievements.

1 Introduction

Let C be a finitely generated pointed rational cone in $\mathbb{R}^d$, i.e. the set of linear combinations $\sum_{i=1}^{n} a_i v_i$ of finitely many integral vectors v_i with nonnegative real coefficients a_i such that $x, -x \in C$ is only possible for $x = 0$. The set of lattice points $C \cap \mathbb{Z}^d$ is an affine monoid with unique finite minimal system of generators, called its *Hilbert basis*. For the theory of affine monoids and the notions of commutative algebra used in this paper we refer the reader to [3]. In the figure the Hilbert basis is marked by open circles.



The program Normaliz [5], version 2.5 (in the following simply called Normaliz), is mainly a tool for computing Hilbert bases. Note that this task is NP-hard [11].

Several related computations are also integrated. Using Normaliz, one may compute the following:

(1) the Hilbert basis and the support hyperplanes of a rational cone. The cone may be given by:
  (i) a system of generators;
  (ii) a linear system of inequalities, equations and congruences (congruences since version 2.5);
  (iii) the binomial equations of the (monoid) generators (since version 2.5).

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(2) the lattice points and the support hyperplanes of an integral polytope;
(3) the generators of the integral closure of the Rees algebra of a monomial ideal $I \subseteq K[X_1, \ldots, X_n]$ and the generators of the integral closure of I.

If the associated monoid is homogeneous in a certain sense, then one may also compute the h-vector and Hilbert polynomial of the monoid.

For the algorithms implemented see [9] (starting with version 1.0), [6] (introduced in version 2.0) and [4] (some of the recent additions in version 2.5). A description of the user interface (version 2.2) is contained in [7].

2 Interactions with other Software Systems

We provide the library `normaliz.lib` that make Normaliz accessible from Singular and also the package `Normaliz.m2` [8] that make Normaliz accessible from Macaulay2. Thus Singular or Macaulay2 can be used as a comfortable environment for the work with Normaliz, and, moreover, Normaliz can be applied directly to objects belonging to the classes of toric rings and monomial ideals.

Thanks to Andreas Paffenholz, Normaliz has been made accessible from polymake (see [12]).

3 New Features of the Program

In the following, we present the changes in the new version of Normaliz.

(1) First, there are some deep changes in the *implementation* of the program.
   (i) The new version of Normaliz has full support for parallel computing. Much better computation times can now be obtained on a multi-core processor system.
   (ii) Memory usage has been optimized.
(2) We have also done some *algorithmic improvements*, which have allowed us to solve some computationally difficult questions (presented in [4]).
   (i) We have introduced a new method for computing the Hilbert basis using a partial triangulation (see [4] for details).
   (ii) A new algorithm for computing the support hyperplanes has been implemented. It uses Fourier-Motzkin elimination recursively and allows computations in cones with many support hyperplanes and big triangulations.
   (iii) The shelling algorithm [6] has been improved.
(3) We have added a *graphical interface* called jNormaliz [2]. This interface is written in Java. It allows us to combine the good portability (on different operating systems) of the graphical elements provided by Java with the computational advantages of the C++ implementation of Normaliz.
(4) The *input modes* have been augmented.
   (i) We have introduced a new format for the input files allowing us to combine systems of inequalities, equations and congruences. (In the previous versions only a system of inequalities or a system of equations was allowed as input). The new format is fully compatible with the old format.

(ii) We have also added a new type of input called "lattice ideal". The input is a matrix of vectors in $\mathbb{Z}^n$, representing binomial equations of the monoid generators (see [3] Section 4.C for details).

(5) Now a better and finer *specialization of the computations performed* is available.

(i) The user can now chose to compute only the height 1 elements of the Hilbert basis of a homogeneous monoid. This is much faster than computing all the elements of the Hilbert basis and allows the fast computation of the lattice points of an integral polytope.

(ii) The Hilbert basis can be computed using only a partial triangulation. This is very fast in some particular but interesting cases (like the one presented in Section 4).

(iii) Specific computations in cones with a large number of support hyperplanes and big triangulations can now be made.

4 One Computational Example

We call a monoid $M \subseteq \mathbb{Z}^d$ *normal* if $M = C \cap \text{lattice}(M)$ where C is the cone generated by M. After an identification lattice$(M) \cong Z^r$, checking normality of M amounts to verifying whether the Hilbert basis of C (with respect to $\mathbb{Z}^r$) is contained in M.

An $r_1 \times r_2 \times \cdots \times r_N$ *contingency table* is a function $T : \{1, \ldots, r_1\} \times \cdots \times \{1, \ldots, r_N\} \to \mathbb{Z}_+$ where $\mathbb{Z}_+$ denotes the nonnegative integers. The j-th $(N-1)$-*marginal* T_j of T is the $r_1 \times \cdots \times r_{j-1} \times r_{j+1} \times \cdots \times r_N$ contingency table defined by $T_j(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_N) = \sum_{k=1}^{r_j} T(i_1, \ldots, i_{j-1}, k, i_{j+1}, \ldots, i_N)$.

The $r_1 \times r_2 \times \cdots \times r_N$ contingency tables form the monoid $\mathcal{O}$ of integral points in the nonnegative orthant of $\mathbb{R}^D$ where $D = r_1 \cdots r_N$. The assignment $T \mapsto (T_1, \ldots, T_N)$ is a monoid homomorphism $\mathcal{M}$ from $\mathcal{O}$ into the monoid of nonnegative integer points in $\mathbb{R}^{d_1+\cdots+d_N}$ where $d_j = r_1 \cdots r_{j-1} r_{j+1} \cdots r_N$. The image $\mathcal{M}(\mathcal{O})$ is called the *monoid derived from* $r_1 \times r_2 \times \cdots \times r_N$ *contingency tables* (by taking line sums). For the role of these monoids and their normality in algebraic statistics we refer the reader to Drton, Sturmfels and Sullivant [10] and Sullivant [15].

Normality of monoids derived from $r_1 \times r_2 \times \cdots \times r_N$ contingency tables by taking $N-1$-marginals was settled almost completely by Ohsugi and Hibi [14]. Using Normaliz one can now show computationally, that the monoids of the missing cases $5 \times 5 \times 3$, $5 \times 4 \times 3$ and $4 \times 4 \times 3$ are normal. For details we refer the reader to [4]. Another (independent) computational solution to this problem can be found using the software LattE for tea, too [13].

Note that this normality problem cannot be settled directly by using previously available software such as Normaliz version 2.2 or `4ti2` version 1.3.2 [1]. Both codes fail to return an answer due to time and to memory requirements of intermediate computations.

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