EXAMPLES OF INFINITELY GENERATED KOSZUL ALGEBRAS

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Let K be a skew field and $A = K \oplus A_1 \oplus \cdots$ a graded K-algebra (both of them not necessarily commutative). We call A homogeneous (or standard) if it is generated by A_1 as a K-algebra. A homogeneous K-algebra A is *Koszul* if there exists a linear free resolution

$$F\colon \cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} A \xrightarrow{\partial_0} K \longrightarrow 0,$$

of the residue field $K \cong A/A_+$ as an A-module. Here $\partial_0\colon A \to K$ is the natural augmentation, the F_i are considered graded left free A-modules whose basis elements have degree 0, and that the resolution is linear means the boundary maps ∂_n, $n \geq 1$, are graded of degree 1 (unless $\partial_n = 0$).

The examples we will discuss in Section 1 are variants of the polytopal semigroup rings considered in Bruns, Gubeladze, and Trung [4]; in Section 1 the base field K is always supposed to be commutative. For the first class of examples we replace the finite set of lattice points in a bounded polytope $P \subset \mathbb{R}^n$ by the intersection of P with a c-divisible subgroup of $\mathbb{R}^n$ (for example $\mathbb{R}^n$ itself or $\mathbb{Q}^n$). It turns out that the corresponding semigroup rings $K[S]$ are Koszul, and this follows from the fact that $K[S]$ can be written as the direct limit of suitably re-embedded 'high' Veronese subrings of finitely generated subalgebras. The latter are Koszul according to a theorem of Eisenbud, Reeves, and Totaro [5]. To obtain the second class of examples we replace the polytope C by a cone with vertex in the origin. Then the intersection $C \cap U$ yields a Koszul semigroup ring R for every subgroup U of $\mathbb{R}^n$. In fact, R has the form $K + X\Lambda[X]$ for some K-algebra Λ, and it turns out that $K + X\Lambda[X]$ is always Koszul (with respect to the grading by the powers of X). Again we will use the 'Veronese trick'.

In Section 2 we treat the construction $K + X\Lambda[X]$ for arbitrary skew fields K and associative K-algebras Λ. (See Anderson, Anderson, and Zafrullah [1] and Anderson and Ryckeart [2] for the investigation of $K + X\Lambda[X]$ under a different aspect.) For them an explicit free resolution of the residue class field is constructed. This construction is of interest also when K and Λ are commutative, and may have further applications.

1. The commutative case

In this section we construct various examples of non-finitely generated commutative Koszul algebras. Our main tool is the following lemma. It holds without the assumption of commutativity.

Lemma 1.1 (Inductive limit lemma)**.** *Let K be a skew field. Assume we are given a directed diagram of Koszul K-algebras*

$$D\colon (A^\alpha \xrightarrow{f^{\alpha\gamma}} A^\gamma)$$

where all the homomorphisms $f^{\alpha\gamma}$ are graded of degree 0. Then the inductive limit $\varinjlim D$ is also Koszul.

Proof. For each index α we fix an exact graded resolution

$$\cdots \to F_2^\alpha \xrightarrow{\partial_2^\alpha} F_1^\alpha \xrightarrow{\partial_1^\alpha} A^\alpha \xrightarrow{\partial_0^\alpha} K \to 0$$

where ∂_j^α has degree 1 for all $j \in \mathbb{N}$ and ∂_0^α is the natural augmentation.

Claim. There exists a system of maps

$$\left\{g_j^{\alpha\gamma}\colon F_j^\alpha \to F_j^\gamma\right\}_{\alpha<\gamma,\ j\in\mathbb{N}}$$

which satisfies the conditions
(i) for any $\alpha < \gamma$ and any $j \in \mathbb{N}$ the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_2^\alpha & \xrightarrow{\partial_2^\alpha} & F_1^\alpha & \xrightarrow{\partial_1^\alpha} & A^\alpha & \xrightarrow{\partial_0^\alpha} & K & \longrightarrow & 0 \\
 & & {\scriptstyle g_2^{\alpha\gamma}}\big\downarrow & & {\scriptstyle g_1^{\alpha\gamma}}\big\downarrow & & {\scriptstyle f^{\alpha\gamma}}\big\downarrow & & \big\| & & \\
\cdots & \longrightarrow & F_2^\gamma & \xrightarrow{\partial_2^\gamma} & F_1^\gamma & \xrightarrow{\partial_1^\gamma} & A^\gamma & \xrightarrow{\partial_0^\gamma} & K & \longrightarrow & 0
\end{array}$$

commutes,
(ii) for any $\alpha < \gamma < \delta$ and any $j \in \mathbb{N}$

$$g_j^{\gamma\delta} \circ g_j^{\alpha\gamma} = g_j^{\alpha\delta},$$

(iii) $g_j^{\alpha\gamma}$ is a graded homomorphism of A^α-modules of degree 0 for any $\alpha < \gamma$ and any $j \in \mathbb{N}$ (A^γ, and more generally a free A^γ-module, is considered as a graded A^α-module via $f^{\alpha\gamma}$).

In order to prove the claim we choose $g_0^{\alpha\gamma} = f^{\alpha\gamma}$ for all $\alpha < \gamma$. Assume we have constructed a system $\{g_j^{\alpha\gamma}\}_{\alpha<\gamma,\ j\le I}$ for some $I \in \mathbb{Z}_+$ that satisfies the desired conditions. In order to prove the claim it suffices to show the existence of mappings

$$\left\{g_{I+1}^{\alpha\gamma}\colon F_{I+1}^\alpha \to F_{I+1}^\gamma\right\}_{\alpha<\gamma},$$

such that the extended system $\{g_j^{\alpha\gamma}\}_{\alpha<\gamma,\ j\le I+1}$ also satisfies (i) – (iii). Let

$$F_n^\alpha = \bigoplus_{i=0}^{\infty} (F_n^\alpha)_i$$

be the decomposition of F_n^α into its graded components. Furthermore we denote the K-linear subspace

$$\partial_{I+1}^\alpha (F_{I+1}^\alpha)_0 \subset (F_I^\alpha)_1$$

by V^α. Observe that the exactness of our complexes and the commutativity of the squares

$$\begin{array}{ccc} F_I^\alpha & \xrightarrow{\partial_I^\alpha} & F_{I-1}^\alpha \\ {\scriptstyle g_I^{\alpha\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle g_{I-1}^{\alpha\gamma}} \\ F_I^\gamma & \xrightarrow{\partial_I^\gamma} & F_{I-1}^\gamma \end{array}$$

imply $g_I^{\alpha\gamma}(V^\alpha) \subset V^\gamma$. (We assume $F_{-1}^\alpha = K$ and $g_{-1}^{\alpha\gamma} = \mathrm{id}_K$.)

For any index α we put

$$W^\alpha = (F_{I+1}^\alpha)_0 \quad \text{and} \quad \partial^\alpha = \partial_{I+1}^\alpha | W^\alpha.$$

Since all the maps ∂^α are surjective and K-linear, standard arguments in linear algebra imply the existence of K-linear homomorphisms

$$h^{\alpha\gamma} \colon W^\alpha \to W^\gamma$$

for all $\alpha < \gamma$ which make the squares

$$\begin{array}{ccc} W^\alpha & \xrightarrow{\partial^\alpha} & V^\alpha \\ {\scriptstyle h^{\alpha\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle g_I^{\alpha\gamma}} \\ W^\gamma & \xrightarrow{\partial^\gamma} & V^\gamma \end{array}$$

commutative and satisfy the condition $h^{\gamma\delta} \circ h^{\alpha\gamma} = h_{\alpha\delta}$ whenever $\alpha < \gamma < \delta$. One has just to fix an isomorphism of K-linear spaces

$$W^\alpha \cong V^\alpha \oplus \tilde{V}^\alpha$$

for each index α in such a way that ∂^α is the projection on V^α, and then let $h^{\alpha\gamma}$ be the composite map

$$W^\alpha \cong V^\alpha \oplus \tilde{V}^\alpha \xrightarrow{g_I^{\alpha\gamma} \oplus 0} V^\gamma \oplus \tilde{V}^\gamma \cong W^\gamma.$$

The mappings $h^{\alpha\gamma}$ give rise to the desired homomorphisms

$$g_{I+1}^{\alpha\gamma} \colon F_{I+1}^\alpha \to F_{I+1}^\gamma.$$

The claim is proved.

We now turn to the proof of the lemma itself. We use two basic facts: tensor product commutes with direct limits and inductive limits preserve exactness. Put $A = \varinjlim D$. Then A is a graded homogeneous K-algebra, since $A = K \oplus A_1 \oplus \cdots$ with

$$A_i = \varinjlim \Big(A_i^\alpha \xrightarrow{f^{\alpha\gamma}} A_i^\gamma \Big).$$

Let $g_j^{\alpha\gamma}$ be chosen as in the claim. The inductive limit of our fixed resolutions (with respect to the mappings $g_j^{\alpha\gamma}$) is exact, consists of graded A-modules and the corresponding boundary homomorphisms are all graded and of degree 1. It only remains to show that all terms of this limit resolution are A-free. But this follows from the observation that the j-th term (for any $j \in \mathbb{N}$) is computed as follows:

$$\varinjlim \Big(F_j^\alpha \xrightarrow{g_j^{\alpha\gamma}} F_j^\gamma \Big) = A \otimes \varinjlim \Big((F_j^\alpha)_0 \xrightarrow{g_j^{\alpha\gamma}} (F_j^\gamma)_0 \Big),$$

where $\varinjlim\Big((F_j^\alpha)_0 \xrightarrow{g_j^{\alpha\gamma}} (F_j^\gamma)_0\Big)$ is a limit of K-linear spaces and, hence, K-free. □

In the rest of this section all rings, especially the base field K, are assumed to be commutative.

In Bruns, Gubeladze, and Trung [4] the Koszul property of polytopal semigroups has been investigated. Let $P \subset \mathbb{R}^n$ be the polytope spanned by finitely many lattice points, i. e. points belonging to $\mathbb{Z}^n$. Then one considers the subsemigroup S_P of $\mathbb{Z}^{n+1}$ generated by the set

$$\Big\{(x,1)\colon x \in P \cap \mathbb{Z}^n\Big\}$$

and the semigroup ring $K[S_P]$ where K is an arbitrary field. Here we derive examples of *infinitely* generated Koszul algebras that, in a sense, are related to polytopal algebras similarly as dense sets are related to discrete ones.

Let $n, c \in \mathbb{N}$ and $c > 1$. Let H be any c-divisible subgroup of $\mathbb{R}^n$ (H may coincide with $\mathbb{R}^n$). The c-divisibility of H means that for all h $\in H$ there exists $t \in H$ with $ct = h$.

Suppose $W \subset \mathbb{R}^n$ is a convex n-dimensional subset (not neceessarily closed or bounded). Then $S(H,W)$ denotes the subsemigroup of $\mathbb{R}^{n+1}$ generated by

$$\Big\{(h,1)\colon h \in H \cap W\Big\} \subset \mathbb{R}^{n+1}.$$

It is clear that for any field K the semigroup algebra $K[S(H,W)]$ naturally carries a graded homogeneous K-algebra structure:

$$k[S(H,W)] = K \oplus A_1 \oplus A_2 \oplus \dots,$$

where A_i is the K-vector space spanned by those $x \in S(H,W)$ whose $(n+1)$-th coordinate is i.

It has been shown in [4] that for every lattice polytope $P \subset \mathbb{R}^n$ the semigroup ring $K[S_{cP}]$ defined by the multiple $cP = \{cx\colon x \in P\}$ is Koszul for all $c \geq n$. In view of the next theorem this result can be interpreted as saying that $P \cap (\mathbb{Z}^n/c)$ approximates P well enough to ensure the Koszul property for the discrete object $K[S_{cP}]$.

Theorem 1.2. *Let H be a c-divisible subsemigroup of $\mathbb{R}^n$ for some $c > 1$. Then the algebra $K[S(H,W)]$ is Koszul for every field K.*

Proof. H is an inductive limit of c-divisible hulls of finitely generated groups. By Lemma 1.1 we can assume H itself is a c-divisible hull of some finitely generated (actually free) abelian group G.

Similarly W is a inductive limit (a filtered union) of finite convex polytopes. Again, by Lemma 1.1 we can assume W itself is a finite polytope. Next we can assume all the vertices of W belong to G, because $H \cap W$ is an inductive limit of intersections $H \cap W^\alpha$, where the W^α are finite polytopes whose vertices belong to G, and Lemma 1.1 can again be applied.

For any $i \in \mathbb{N}$ we let, as defined above,

$$S(G, c^iW) \quad \text{and} \quad S((G/c^i), W)$$

denote the subsemigroups of $\mathbb{R}^{n+1}$ generated by

$$\{(g,1)\colon g \in G \cap c^i W\} \subset \mathbb{R}^{n+1} \quad \text{and} \quad \{(g,1)\colon g \in (G/c^i) \cap W\} \subset \mathbb{R}^{n+1}.$$

respectively. We have $S(G, c^i W) \cong S(G/c^i, W)$ and $H \cap W$ is a filtered union of the $(G/c^i) \cap W$. So by Lemma 1.1 it suffices to show that $K[S(G, c^i W)]$ is Koszul for i sufficiently large. But the theorem of [4] quoted above implies $K[S(G, c^i W)]$ is Koszul for all i such that $c^i \geq n$. □

Our next example is derived from polytopal semigroups in a different way. If the polytope P in the definition of a polytopal semigroup is substituted by a cone C (with vertex at the origin) then the generators of $K[S_C]$ constitute a semigroup – the semigroup of all lattice points in C. This situation fits into a more general picture.

In fact, let S be any commutative (not necessarily cancellative or torsionfree) semigroup and X be a variable. We denote by $K[S]_*$ the K-subalgebra (K is again a field) of $K[S][X] = K[S \oplus \mathbb{Z}_+]$ consisting of those polynomials with coefficients in $K[S]$ whose constant term belongs to K. Evidently $K[S]_*$ is a graded homogeneous K-algebra with respect to the grading defined by the powers of X. The algebra $K[S_C]$ mentioned above is naturally isomorphic to $K[S]_*$ where S it the semigroup of all lattice points in the cone C.

The construction of $K[S]_*$ makes sense for arbitrary (not necessarily commutative) semigroups. One easily shows that

$$K[S]_* \cong K[\{X_s\}_{s\in S}]/(X_{s_1}X_{s_2} - X_{t_1}X_{t_2})$$

where $K[\{X_s\}_{s\in S}]$ is a polynomial ring in the variables X_s labelled by the elements of S and the quotient is considered with respect to the ideal generated by all binomials $X_{s_1}X_{s_2} - X_{t_1}X_{t_2}$ where $s_1, s_2, t_1, t_2 \in S$ satisfy the equation $s_1 s_2 = t_1 t_2$. The same isomorphism takes place for general, not necessarily commutative, semigroups – one just considers noncommuting variables and two-sided ideals.

There is a still more general framework including all these constructions. let Λ be any commutative ring and X a variable. Assume K is a field contained in Λ. By $K + X\Lambda[X]$ we denote the K-subalgebra of $\Lambda[X]$ containing the polynomials with constant term in K. In the case in which $\Lambda = K[S]$ we have $K[S]_* = K + X\Lambda[X]$. Again, we may consider $K + X\Lambda[X]$ for a not necessarily commutative K-algebra Λ. These algebras are graded homogeneous K-algebras with respect to the degree in X. (Observe that $K + X\Lambda[X]$ is Noetherian if and only if $\dim_K \Lambda < \infty$.)

It turns out that all these constructions give examples of Koszul algebras. The commutative case can be covered by the 'Veronese trick' used in the proof of 1.3 while the general case needs direct arguments (see the next section).

Theorem 1.3. *Let Λ be a commutative K-algebra for some field K and X a variable commuting with the elements of Λ. Then the graded homogeneous K-algebra*

$$\Lambda_* = K + X\Lambda[X]$$

is Koszul.

Proof. Λ is a filtered union of finitely generated K-algebras, say $\Lambda = \varinjlim \Lambda^\alpha$. Then $\Lambda_* = \varinjlim (K + X\Lambda^\alpha[X])$. By the inductive limit lemma we can assume Λ itself is finitely generated. Let Γ be a finite generating set of the K-algebra Λ containing $1 \in K$. Let $\langle \Gamma^i \rangle$ denote the K-vector subspace of Λ generated by the products $\gamma_1 \cdots \gamma_i$ with $\gamma_i \in \Gamma$. Then $\langle \Gamma^i \rangle \subset \langle \Gamma^j \rangle$ whenever $i \le j$, and $\varinjlim \langle \Gamma^i \rangle = \Lambda$. Consider the K-subalgebra B of Λ_* generated by $X\langle \Gamma^1 \rangle$. B is a graded homogeneous finitely generated K-algebra. By Eisenbud, Reeves, and Totaro [5] or Aramova, Barcanescu, and Herzog [3] the Veronese subrings $B_{(d)} \subset B$ are Koszul for d sufficiently large. Observe that $B(d)$ is isomorphic as a graded K-algebra to the K-subalgebra C_d of Λ_* generated by $X\langle \Gamma^d \rangle$. On the other hand, $\Lambda_* = \varinjlim C_d$. Since C_d is Koszul for d sufficiently large, the inductive limit lemma completes the proof. $\square$

2. The non-commutative case

In this section we show that $K + X\Lambda[X]$ is a Koszul algebra for an arbitrary skew field K and every associative K-algebra Λ. The proof is based on the construction of an 'explicit' free resolution of K.

Theorem 2.1. *Let K be a skew field, Λ an arbitrary K-algebra and X a variable, commuting with the elements of Λ. Then the homogeneous graded K-algebra $\Lambda_* = K + X\Lambda[X]$ (considered as a graded K-algebra with respect to the degree in X) is Koszul.*

That Λ_* is homogeneous is clear. We first define a chain complex of free Λ_*-modules

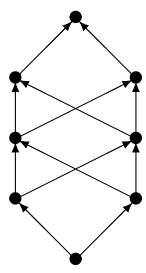
$$\mathbb{T}\colon \cdots \longrightarrow T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} \Lambda_* \xrightarrow{d_0} K \longrightarrow 0$$

for which d_0 is the natural augmentation and d_n is graded of degree 1 for all $n \in \mathbb{N}$. (As above, the basis elements of T_n have degree 0.)

To this end we introduce the notion of a tower. Let $n \in \mathbb{N}$. A *tower graph of height n* is an oriented graph G with vertex set

$$V(G) = \bigcup_{i=0}^{n} L_i,$$

such that $L_i \cap L_j = \emptyset$ for distinct $i, j \in [0, n]$, $\#L_0 = \#L_n = 1$ and $\#L_i = 2$ for all $i \in [1, n-1]$; further, for each pair of elements $x \in L_{i-1}$, $y \in L_i$ for $i \in [1, n]$ there exists exactly one edge of G connecting x and y, and the orientation of this edge is from x to y; we also require that G has no other edges; and finally, each of the sets L_i for $i \in [1, n-1]$ is linearly ordered. The following figure presents a tower graph of height 4:



Later on $E(G)$ will denote the set of edges of a tower graph G. A *represented tower of height* n is a pair $\tau = (G, t)$, where G is a tower graph of height n and $t\colon E(G) \to \Lambda$ is a function, satisfying the following condition:

For all oriented paths $[l_1 \dots l_k]$ and $[l'_1 \dots l'_k]$ in G of the same length k (where $l_i, l'_i \in E(G)$ and the orientations are from left to right) and having the same origin vertex in L_i and the same ending vertex in L_{i+k} the equality

$$t(l_k)t(l_{k-1}) \dots t(l_1) = t(l'_k)t(l'_{k-1}) \dots t(l'_1)$$

holds. We shall say that two represented towers $\tau = (G, t)$ and $\tau' = (G', t')$ are *isomorphic* if there exists an isomorphism of oriented graphs $\psi\colon G \to G'$ that respects the orderings of L_i and L'_i for all i and such that $t(l) = t'(\psi(l))$ for all $l \in E(G)$.

Finally, a *tower of height* n is defined as the isomorphism class of a represented tower of height n. We write $[\tau]$ for the isomorphism class of τ.

Let $\tau = (G, t)$ be a height n represented tower. The elements of L_i, $i \in [0, n]$, will be called vertices of τ of level i. If $i = 0$ or $i = n$ then the corresponding vertices will be called the bottom and the top vertex of τ, respectively.

Let $n \geq 2$ and $[\tau]$ be a height n tower, $\tau = (G, t)$. Then we have the two naturally determined height $n-1$ towers $[\tau_1]$ and $[\tau_2]$. Namely, the subgraph of G, spanned by the vertex subset $V(G) \setminus L_n(G)$ uniquely determines two height $n-1$ tower graphs, which coincide up to the $(n-2)$-th level; we denote these height $n-1$ tower graphs by G_1 and G_2, respectively, and consider the represented towers of height $n-1$: $\tau_1 = (G_1, t_1)$ and $\tau_2 = (G_2, t_2)$, where $t_1 = t|E(G)$ and $t_2 = t|E(G_2)$. These represented towers in their turn define the height $n-1$ towers $[\tau_1]$ and $[\tau_2]$.

Remark 2.2. Clearly, τ_1 and τ_2 are different (as their supporting vertex sets differ). But the towers $[\tau_1]$ and $[\tau_2]$ can coincide. For example, suppose that $n = 2$, $L_0 = \{v_0\}$, $L_1 = \{v_{11}, v_{12}\}$, $L_2 = \{v_2\}$, $t(v_0, v_{11}) = a$, $t(v_0, v_{12}) = a$, $t(v_{11}, v_2) = b$, $t(v_{12}, v_2) = b$.

Now we are ready to define the complex $\mathbb{T}$. For $n \in \mathbb{N}$ we put $T_n = \bigoplus_{\mathrm{Tow}_n} \Lambda_*$, where Tow_n denotes the set of height n towers. In what follows Tow_n will be identified with the standard Λ_*-basis of T_n. Each of T_n naturally inherits a graded structure from that of Λ_*. Correspondigly, we shall write

$$T_n = \bigoplus_{i=0}^{\infty} T_{ni}.$$

In particular $T_{n0} = \bigoplus_{\mathrm{Tow}_n} K$.

The homomorphism $d_1\colon T_1 \to T_0 = \Lambda_*$ is defined by $d_1([\tau]) = t(l)X$ for every $[\tau] \in \mathrm{Tow}_1$, $\tau = (G, t)$, and the unique element l of $E(G)$.

Now assume $n \geq 2$ and $[\tau] \in \mathrm{Tow}_n$ for some height n represented tower $\tau = (G, t)$. Assume $L_{n-1} = \{v_1, v_2\}$ and $v_1 > v_2$ (notation as above). As mentioned above, $[\tau]$ defines in a natural way two height $n-1$ towers $[\tau_1]$ and $[\tau_2]$. We can assume that the top vertex of τ_1 is v_1 and that of τ_2 is v_2. Under this enumeration of the towers we put

$$d_n([\tau]) = t(l_1)X[\tau_1] - t(l_2)X[\tau_2],$$

where l_1 and l_2 are the edges of G emerging from v_1 and v_2 respectively.

Observe, that the definition of d_n is correct (it does not depend on the representatives).

Claim 1. $\mathbb{T}$ is a complex.

The equation $d_0 d_1 = 0$ is obvious. Now choose $[\tau] \in \mathrm{Tow}_n$ for some $n \geq 3$. We have $d_n([\tau]) = t(l_1)X[\tau_1] - t(l_2)X[\tau_2]$, by the definition of d_n. Let the objects $l_{11}, l_{12}, [\tau_{11}], [\tau_{12}]$ and $l_{21}, l_{22}, [\tau_2 1], [\tau_{22}]$ relate to the towers $[\tau_1]$ and $[\tau_2]$ respectively in the same way as $l_1, l_2, [\tau_1], [\tau_2]$ relate to τ. Then one obtains

$$\begin{aligned} d_{n-1}d_n([\tau]) &= d_{n-1}\Big(t(l_1)X[\tau_1] - t(l_2)X[\tau_2]\Big) \\ &= t(l_1)X\Big(t(l_{11})X[\tau_{11}] - t(l_{12})X[\tau_{12}]\Big) \\ &\quad - t(l_2)X\Big(t(l_{21})X[\tau_{21}] - t(l_{22})X[\tau_{22}]\Big). \end{aligned}$$

On the other hand it is clear that $[\tau_{11}] = [\tau_{21}]$ and $[\tau_{12}] = [\tau_{22}]$. Therefore,

$$d_{n-1}d_n([\tau]) = X^2\Big(t(l_1)t(l_{11}) - t(l_2)t(l_{21})\Big)[\tau_{11}] - X^2\Big(t(l_1)t(l_{12}) - t(l_2)t(l_{22})\Big)[\tau_{12}].$$

Now the oriented paths $[l_{11}l_1]$ and $[l_{21}l_2]$ (in G) have the same origin and the same end. The same holds for the pair of paths $[l_{12}l_1]$ and $[l_{22}l_2]$. So, by the definition of a tower graph,

$$t(l_1)t(l_{11}) = t(l_2)t(l_{21}), \quad t(l_1)t(l_{12}) = t(l_2)t(l_{22}).$$

It follows that $d_{n-1}d_n([\tau]) = 0$. The verification of the equaility $d_1 d_2([\tau]) = 0$ for any $[\tau] \in \mathrm{Tow}_2$ can be carried out similarly.

Observe, that the d_n are graded and of degree 1 for all $n \geq 1$.

Now we define the desired graded free resolution of K with the natural augmentation ∂_0 (i. e. $d_0 = \partial_0$)

$$\mathbb{F}\colon \cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \Lambda_* \xrightarrow{\partial_0} K \longrightarrow 0.$$

We shall construct F_n and ∂_n inductively.

Since $\deg(d_1) = 1$ we have the inclusion of K-vector spaces $d_1(T_{10}) \subset A_1$, where we adopt the notation

$$\Lambda_* = K \oplus A_1 \oplus A_2 \oplus \dots$$

Since K is a skew field there exists a subset $B_1 \subset \mathrm{Tow}_1$, satisfying the conditions

(a) $d_1|B_1\colon B_1 \to d_1(T_{10})$ is injective;

(b) $d_1(B_1)$ is a K-basis of $d_1(T_{10})$.

We fix one such subset B_1 and put

$$F_1 = \bigoplus_{B_1} \Lambda_*;$$

F_1 will be identified with the corresponding graded direct summand of T_1. Next we define a surjective graded Λ_*-homomorphism $f_1\colon T_1 \to F_1$ of degree 0, which is split by the inclusion $F_1 \hookrightarrow T_1$, as follows:

$$f_1([\tau]) = \sum_{[\rho]\in B_1} a_{[\rho]}[\rho], \qquad [\tau] \in \mathrm{Tow}_1,$$

where the elements $a_{[\rho]} \in K$ (all but a finite number of exceptions of them are zero) are determined by the equation

$$d_1([\tau]) = \sum_{[\rho]\in B_1} a_{[\rho]} d_1([\rho]).$$

By the construction it is clear that there exists a graded degree 1 homomorphism ∂_1 of Λ_*-modules, making the square

$$\begin{array}{ccc} T_1 & \xrightarrow{d_1} & \Lambda_* \\ {\scriptstyle f_1}\big\downarrow & & \big\| \\ F_1 & \xrightarrow{\partial_1} & \Lambda_* \end{array}$$

commutative (of course we choose $\partial_1 = d_1|F_1$).

Remark 2.3. f_1 can send a basis element of T_1 to $0 \in F_1$. Indeed, if $\tau = (G, t)$ is the height 1 represented tower whose single edge $l \in E(G)$ is mapped to zero by t, then $f_1([\tau]) = 0$.

Next we proceed further and define F_2 and ∂_2 as follows. Again, we can fix a subset $B_2 \subset \mathrm{Tow}_2$ in such a way that

$$f_1 d_2|B_2 \colon B_2 \to F_1$$

is injective and $f_1 d_2(B_2)$ is a K-basis of the K-linear subspace $f_1 d_2(T_{20}) \subset T_{11}$, where F_{11} is the degree 1 component of the graded free Λ_*-module $F_1 = \bigoplus_{i=0}^{\infty} F_{1i}$. Thereafter we put

$$F_2 = \bigoplus_{B_2} \Lambda_*;$$

F_2 will be identified with the corresponding graded direct summand of T_2. We define a surjective graded degree 0 homomorphism $f_2 \colon T_2 \to F_2$ of Λ_*-modules, split by the inclusion $F_2 \hookrightarrow T_2$, by setting

$$f_2([\tau]) = \sum_{[\rho]\in B_2} a_{[\rho]}[\rho]$$

where the coefficients $a_{[\rho]} \in K$ are given by the equation

$$f_1 d_2([\tau]) = \sum_{[\rho]\in B_2} a_{[\rho]}\Big(f_1 d_2([\rho])\Big).$$

Again, there exists a unique graded degree 1 homomorphism ∂_2, for which the square

$$\begin{array}{ccc} T_2 & \xrightarrow{d_2} & T_1 \\ {\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_1}\big\downarrow \\ F_2 & \xrightarrow{\partial_2} & F_1 \end{array}$$

commutes. Continuing in this spirit we will obtain the infinite commutative diagram of graded free Λ_*-modules with graded degree 1 horizontal and graded degree 0 vertical Λ_*-homomorphisms

$$\begin{array}{ccccccccccccccc}
\mathbb{T}\colon \cdots & \longrightarrow & T_n & \xrightarrow{d_n} & \cdots & \longrightarrow & T_2 & \xrightarrow{d_2} & T_1 & \xrightarrow{d_1} & \Lambda_* & \xrightarrow{d_0} & K & \longrightarrow & 0 \\
 & & {\scriptstyle f_n}\downarrow & & & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & \| & & \| & & \\
\mathbb{F}\colon \cdots & \longrightarrow & F_n & \xrightarrow{\partial_n} & \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \xrightarrow{\partial_1} & \Lambda_* & \xrightarrow{\partial_0} & K & \longrightarrow & 0.
\end{array}$$

In particular we see that $\mathbb{F}$ is complex.

Remark 2.4. The construction of $\mathbb{F}$ is the 'minimization of $\mathbb{T}$ with respect to K-linear relations'. In particular, all the towers mapped to 0 by the boundary map of $\mathbb{T}$ had to be 'killed' (or, more generally, all the K-linear combinations of towers which were cycles in $\mathbb{T}$ had to be 'killed' in the minimalized complex $\mathbb{F}$).

Now the following claim completes the proof of the theorem.

Claim 2. $\mathbb{F}$ is acyclic.

The exactness at Λ_* is clear. Assume $c \in F_n$ is a cycle for some $n \geq 2$. We write

$$c = \sum_i \lambda_i^* [\rho_i]$$

for some $\lambda_i^* \in \Lambda_*$ and $[\rho_i] \in B_n$. Since ∂_n is graded, we can assume without loss of generality that the λ_i^* are homogeneous and of the same degree. The case $\deg(\lambda_i^*) = 0$ happens only if $c = 0$, for otherwise we would obtain a nontrivial K-linear dependence of the $\partial_n([\rho_i])$ which is impossible by the definition of B_n. Set $d = \deg(\lambda_i^*) > 0$. Then $\lambda_i^* = \lambda_i X^d$ for each i with $\lambda_i \in \Lambda \setminus \{0\}$. We have

$$\partial_n([\rho_i]) = t_i(l_{i1} X f_{n-1}([\tau_{i1}]) - t_i(l_{i2}) X f_{n-1}([\tau_{i2}])$$

for certain height $n-1$ towers $[\tau_{i1}]$ and $[\tau_{i2}]$ ($\rho_i = (G_i, t_i)$, and $l_{i1}, l_{i2} \in E(G_i)$ are the 'top' edges).

Now we define new represented towers ρ_i' of height n as follows. We let the corresponding tower graphs G_i' be defined by the vertex set $\bigl(V(G_i) \setminus L_n(G_i)\bigr) \cup \{v_i'\}$, where we assume $v_i' \notin L_0(G_i) \cup \cdots \cup L_n(G_i)$ (i. e. $V(G_i')$ differs from $V(G_i)$ at the highest level only). For the functions $t_i'\colon E(G_i') \to \Lambda$ we put

$$t_i' | E(G_i' \setminus \mathrm{top}) = t | E(G_i \setminus \mathrm{top}),$$

where $G_i' \setminus \mathrm{top}$ denotes the subgraph of G_i' spanned by the vertex subset $V(G_i') \setminus L_n(G_i)$ and $G_i \setminus \mathrm{top}$ is defined similarly . So $G_i' \setminus \mathrm{top} = G_i \setminus \mathrm{top}$. We also preserve the orderings of the sets $L_1(G_i') = L_1(G_i), \dots, L_{n-1}(G_i') = L_{n-1}(G_i)$. To complete the definition of ρ_i' it remains to fix the values of t_i' at the two edges of G_i' emerging from $L_{n-1}(G_i)$ and ending in v_i' respectively. Assume $L_n(G_i) = \{v_i\}$ and $L_{n-1}(G_i) = \{v_{1i}, v_{2i}\}, v_{2i} < v_{1i}$. We put

$$t_i'\colon (v_{1i} v_i') \mapsto \lambda_i t(v_{1i} v_i) = \lambda_i t(l_{i1}) \quad \text{and} \quad t_i'\colon (v_{2i} v_i') \mapsto \lambda_i t(v_{2i} v_i) = \lambda_i t(l_{i2}).$$

The maps t_i' obviously satisfy the required conditions and, thus, ρ_i' is well defined. Observe that if $\lambda_i = 1$, then $[\rho_i] = [\rho_i']$.

For each index i we now define a represented tower μ_i of height $n+1$ as follows. The corresponding graph $\tilde{G}_i$ contains as subgraphs both of G_i and G'_i so that

$$L_j(\tilde{G}_i) = L_j(G_i)\Big(= L_0(G'_i)\Big) \quad \text{as ordered sets for} \quad j = 0, \dots, n-1,$$
$$L_n(\tilde{G}_i) = \{v_i, v'_i\}, \quad L_{n+1}(\tilde{G}_i) = \{w_i\},$$

where $w_i \notin L_0(\tilde{G}_i) \cup \dots \cup L_n(\tilde{G}_i)$. Further, we fix the ordering of $L_n(\tilde{G}_i)$ by putting $v_i < v'_i$. Now we have to define the function $\tilde{t}_i \colon E(\tilde{G}_i) \to \Lambda$. If $l \in E(G_i)$, we put $\tilde{t}_i(l) = t_i(l)$, and if $l \in E(G'_i)$, then $\tilde{t}_i(l) = t'_i(l)$. There remain the edges $(v_i w_i)$ and $(v'_i w_i)$ of $\tilde{G}_i$; we set

$$\tilde{t}_i(v_i w_i)\lambda_i \quad \text{and} \quad \tilde{t}_i(v'_i w_i) = 1.$$

The verification that all the objects are well-defined is straightforward.

For all i we clearly have the equations

$$d_{n+1}([\mu_i]) = X[\rho'_i] - \lambda_i X[\rho_i].$$

Therefore $\partial_{n+1} f_{n+1}([\mu_i]) = X\Big(f_n([\rho'_i])\Big) - \lambda_i X[\rho_i]$. This implies

$$0 = \partial_n \partial_{n+1} f_{n+1}([\mu_i]) = X\Big(\partial_n f_n([\rho'_i])\Big) - \lambda_i X \partial_n([\rho_i]).$$

It follows that

$$\begin{aligned} 0 = \partial_n(c) &= \partial_n\Big(\sum_i \lambda_i^*[\rho_i]\Big) = \partial_n\Big(\sum_i X^d \lambda_i[\rho_i]\Big) \\ &= X^{d-1}\partial_n\Big(\sum_i \lambda_i X[\rho_i]\Big) = X^{d-1}\Big(\sum_i X \partial_n f_n([\rho'_i])\Big) \\ &= X^d \sum_i \partial_n f_n([\rho'_i]). \end{aligned}$$

On the other hand X^d is not a zero-divisor on Λ_*. So it is not a zero-divisor of the free Λ_*-module F_{n-1}. Consequently

$$\sum_i \partial_n f_n([\rho'_i]) = 0.$$

But $\sum_i f_n([\rho'_i]) \in F_{n0}$ and by the construction of the complex $\mathbb{F}$ the homomorphism $\partial_n | F_{n0} \colon F_{n0} \to F_{n-1\,1}$ is injective for all $n \geq 1$. Therefore $\sum_i f_n([\rho'_i]) = 0$. Finally we get

$$\begin{aligned} c = \sum_i \lambda_i^*[\rho_i] &= X^{d-1}\Big(\sum_i \lambda_i X[\rho_i]\Big) = X^{d-1}\Big(\sum_i (\lambda_i X[\rho_i] - X f_n[\rho'_i])\Big) \\ &= X^{d-1}\Big(\sum_i \partial_{n+1} f_{n+1}(-[\mu_i])\Big) \in \operatorname{Im}(\partial_{n+1}). \end{aligned}$$

This shows $\operatorname{Ker}(\partial_n) = \operatorname{Im}(\partial_{n+1})$ for $n \geq 2$. The verification of the equality $\operatorname{Ker}(\partial_1) = \operatorname{Im}(\partial_2)$ makes no difference. One has to consider $1 \in \Lambda$ instead of $f_{n-1}([\tau_{i1}])$ and $f_{n-1}([\tau_{i2}])$ and the very same arguments apply.

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