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Products of Borel fixed ideals of maximal minors

Winfried Bruns [a], Aldo Conca [b,*]

[a] Universität Osnabrück, Institut für Mathematik, 49069 Osnabrück, Germany
[b] Dipartimento di Matematica, Università degli Studi di Genova, Italy

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ABSTRACT

We study a large family of products of Borel fixed ideals of maximal minors. We compute their initial ideals and primary decompositions, and show that they have linear free resolutions. The main tools are an extension of straightening law and a very uniform primary decomposition formula. We study also the homological properties of associated multi-Rees algebra which are shown to be Cohen–Macaulay, Koszul and defined by a Gröbner basis of quadrics.

1. Introduction

Let K be a field and $X = (x_{ij})$ be the $m \times n$ matrix whose entries are the indeterminates of the polynomial ring $R = K[x_{ij} : 1 \le i \le m,\ 1 \le j \le n]$, and assume that $m \le n$. The ideals $I_t(X)$, generated by the t-minors of X, and their varieties are classical objects of commutative algebra, representation theory and algebraic geometry. They are clearly invariant under the natural action of $\mathrm{GL}_m(K) \times \mathrm{GL}_n(K)$ on R. Their arithmeti-

* Corresponding author.
E-mail addresses: wbruns@uos.de (W. Bruns), conca@dima.unige.it (A. Conca).

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cal and homological properties are well-understood as well as their Gröbner bases and initial ideals with respect to diagonal (or antidiagonal) monomial orders, i.e., monomial orders under which the initial monomial of a minor is the product over its diagonal (or antidiagonal); see our survey [7]. Bruns and Vetter [13] and Miller and Sturmfels [25] are comprehensive treatments.

Among the ideals of minors the best-behaved is undoubtedly the ideal of maximal minors, namely the ideal $I_m(X)$. It has the following important features:

Theorem 1.1.

(1) *The powers of $I_m(X)$ have a linear resolution.*
(2) *They are primary and integrally closed.*
(3) *Computing initial ideals commutes with taking powers for diagonal or anti-diagonal monomial orders:* $\mathrm{in}(I_m(X)^k) = \mathrm{in}(I_m(X))^k$ *for all k, and the natural generators of $I_m(X)^k$ form a Gröbner basis.*
(4) *The Rees algebra of $I_m(X)$ is Koszul, Cohen–Macaulay and normal.*

In the theorem and throughout this article "resolution" stands for "minimal graded free resolution". The grading is always the standard grading on the polynomial ring.

Concerning the statements in (1), one knows that $I_m(X)$ itself is resolved by the Eagon–Northcott complex and the resolution for the powers is described by Akin, Buchsbaum and Weyman in [2]. References for assertions (2), (3) and (4) can be found in [6,7,10,13], and Eisenbud and Huneke [17]. Note also that the maximal minors form a universal Gröbner basis (i.e., a Gröbner basis with respect to every monomial order) as proved by Bernstein, Sturmfels and Zelevinsky in [4,27] and generalized by Conca, De Negri, Gorla [15]. But for $m > 2$ and $k > 1$ there are monomial orders $<$ such that $\mathrm{in}_<(I_m(X)^k)$ is strictly larger than $\mathrm{in}(I_m(X))^k$. In other words, the natural generators of $I_m(X)^k$ do not form a universal Gröbner basis. This is related to the fact that the maximal minors do not form a universal Sagbi basis for the coordinate ring of the Grassmannian, as observed, for example, by Speyer and Sturmfels [26, 5.6].

For $1 < t < m$ the ideal $I_t(X)$ does not have a linear resolution and its powers are not primary. The primary decomposition of the powers of $I_t(X)$ has been computed by De Concini, Eisenbud and Procesi [19] and in [13]. The Castelnuovo–Mumford regularity of $I_t(X)$ is computed by Bruns and Herzog [11]. Furthermore, the formation of initial ideals does not commute with taking powers, but $I_t(X)^k$ has a Gröbner basis in degree tk as the results in [6] show.

In our joint work with Berget [3], Theorem 1.1 was extended to arbitrary products of the ideals $I_t(X_t)$ where X_t is the submatrix of the first t rows of X. We proved the following results:

Theorem 1.2. *Let $1 \le t_1, \dots, t_w \le m$ and $I = I_{t_1}(X_{t_1}) \cdots I_{t_w}(X_{t_w})$.*

(1) *Then I has a linear resolution.*

(2) *I is integrally closed and it has a primary decomposition whose components are powers of ideals $I_t(X_t)$ for various values of t.*
(3) $\mathrm{in}(I) = \mathrm{in}(I_{t_1}(X_{t_1})) \cdots \mathrm{in}(I_{t_w}(X_{t_w}))$ *and the natural generators of I form a Gröbner basis with respect to a diagonal or anti-diagonal monomial order.*
(4) *The multi-Rees algebra associated to $I_1(X_1), \dots, I_m(X_m)$ is Koszul, Cohen–Macaulay and normal.*

Note that the ideals $I_t(X_t)$ are fixed by the natural action of the subgroup $B_m(K) \times \mathrm{GL}_n(K)$ of $\mathrm{GL}_m(K) \times \mathrm{GL}_n(K)$, where $B_m(K)$ denotes the subgroup of lower triangular matrices. For use below we denote the subgroup of upper triangular matrices in $\mathrm{GL}_n(K)$ by $B'_n(K)$.

Ideals of minors that are invariant under the Borel group $B_m(K) \times B'_n(K)$ have been introduced and studied by Fulton in [20]. They come up in the study of singularities of various kinds of Schubert subvarieties of the Grassmannians and flag varieties. Those that arise as Borel orbit closures of (partial) permutation matrices are called Schubert determinantal ideals by Knutson and Miller in the their beautiful paper [24] where they describe the associated Gröbner bases, as well as Schubert and Grothendieck polynomials.

The goal of this paper is to extend the results of Theorems 1.1 and 1.2 to a class of ideals that are fixed by the Borel group. Depending on whether one takes upper or lower triangular matrices on the left or on the right, one ends up with different "orientations", in the sense that for $B_m(K) \times B'_n(K)$ one gets ideals of minors that flock in the northwest corner of the matrix while for $B_m(K) \times B_n(K)$ the ideals of minors flock in the northeast corner, and so on. Of course, there is no real difference between the four cases, but because we prefer to work with diagonal monomial orders, we will choose the northeast orientation. Clearly, all the results we prove can be formulated in terms of the other three orientations as well.

Theorem 1.2 was motivated by the application to the description of a generating set of a certain equivariant Grothendieck group. Our motivations for proposing a further generalization are the following. Firstly, the study of the Castelnuovo–Mumford regularity has been a central topic in commutative algebra and algebraic geometry in the last forty years, and the identification of large families of ideals whose products have linear resolution play an important role in this study. Secondly, the Gorenstein property and factoriality of multi-Rees rings and fiber rings that we discuss in the last chapter generalize classical results on Grassmannians and flag varieties.

Let us define the *northeast ideals $I_t(a)$ of maximal minors*: $I_t(a)$ is generated by the t-minors of the $t \times (n - a + 1)$ *northeast submatrix*

$$X_t(a) = (x_{ij} : 1 \le i \le t,\ a \le j \le n).$$

The main results can be summed up as follows:

Theorem 1.3. *Let $I_{t_1}(a_1), \dots, I_{t_w}(a_w)$ be northeast ideals of maximal minors, and let I be their product. Then*

(1) *I has a linear resolution.*
(2) $\mathrm{in}(I) = \mathrm{in}(I_{t_1}(a_1)) \cdots \mathrm{in}(I_{t_w}(a_w))$*, and the natural generators of I form a Gröbner basis with respect to a diagonal monomial order.*
(3) *I is integrally closed, and it has a primary decomposition whose components are powers of ideals $I_t(a)$ for various values of t and a.*
(4) *The multi-Rees algebra associated to the family of ideals $I_t(a)$ with $t, a > 0$ and $t + a \le n + 1$ is Koszul, Cohen–Macaulay and normal.*

Statements (1), (3) and (4) hold analogously for the initial ideals, in particular the primary components of $\mathrm{in}(I)$ can be taken to be powers of ideals of variables.

The patters highlighted in Theorem 1.3, i.e. the linearity of the resolution of products, good homological properties of multi-Rees algebras and the existence of a primary decomposition whose components are powers of primes, are present also in other interesting situations that we discuss in [9]. In particular in [9] we highlight the analogy between the ideals that we discuss in this paper and the (monomial) Borel-fixed ideals pointing out that both families have a tendency to give linear resolutions. Unfortunately we do not have an explanation, not even an heuristic one, why the Borel action is related to the linearity of the resolution.

One could consider a more general definition of northeast ideals of maximal minors, allowing also more rows than columns. Unfortunately the results of Theorem 1.3 do not hold in this generality. Let $I'_t(a)$ denote the ideal of the t-minors in the submatrix formed by the last t columns and first a rows (with $a > t$). For example, one can check in a 3×3 matrix that the product of (Borel-fixed ideals of maximal minors) $I_1(2)I_2(1)I'_1(2)I'_2(3)$ does not have a linear resolution.

The proofs of the results of [3] are based on the classical straightening law for minors of Doubillet, Rota, and Stein. For generalities, historical remarks on the classical straightening law for minors and its applications to the study of determinantal ideals we refer the reader to [13].

The point is that the ideals considered in [3] have K-bases of classical standard monomials. This is no longer true for the ideals $I_t(a)$ in general, let alone for products of such ideals. Therefore we had to develop a more general notion of "normal form" that we call *northeast canonical*, see Section 4. Using this type of normal form we will prove the crucial description of the initial ideal $\mathrm{in}(I)$ as an intersection of powers of the ideals $\mathrm{in}(I_t(a))$.

The northeast canonical form allows us to prove that the multi-Rees algebra defined by all ideals $I_t(a)$ is a normal domain and is defined by a Gröbner basis of quadrics. A theorem of Blum [5] then implies that all our ideals have linear fee resolutions. The same statements have counterparts for the initial ideals and their multi-Rees algebra as well.

We conclude the paper with a discussion of the Gorenstein property of certain multigraded Rees rings and the factoriality of certain fiber rings that come up in connection with the northeast ideals. In particular, we prove that the multigraded Rees algebra associated to a strictly ascending chain of ideals $J_1 \subset J_2 \cdots \subset J_v$ is Gorenstein, provided each J_i belongs to the family of the $I_t(a)$ and has height i.

The results of this paper originated from extensive computations with the systems CoCoA [1], Macaulay 2 [22], Normaliz [12] and Singular [18].

2. Minors, diagonals and the straightening law

Let K be a field and $X = (x_{ij})$ an $m \times n$ matrix of indeterminates. The ideals we want to investigate live in

$$R = K[x_{ij} : 1 \le i \le m,\ 1 \le j \le n].$$

Let $X_t(a)$ be the submatrix of X that consists of the entries x_{ij} with $1 \le i \le t$ and $a \le j \le n$. We call it a *northeast submatrix* since it sits in the right upper corner of X. The ideal

$$I_t(a) = I_t(X_t(a))$$

is called a *northeast ideal of maximal minors*, or a *northeast ideal* for short. In the following "northeast" will be abbreviated by "NE". Since $I_t(a) = 0$ if $t + a > n + 1$, we will always assume that $t + a \le n + 1$.

We fix a monomial order on R that fits the NE ideals very well: the lexicographic order $>_{\text{lex}}$ (or simply $>$) in which x_{11} is the largest indeterminate, followed by the elements in the first row of X, then the elements in the second row from left to right, etc. More formally:

$$x_{ij} >_{\text{lex}} x_{uv} \quad \text{if} \quad i < u \quad \text{or} \quad i = u \text{ and } j < v.$$

The minor

$$\delta = \det(x_{ib_j} : i, j = 1, \dots, t), \qquad b_1 < \cdots < b_t,$$

is denoted by $[b_1 \dots b_t]$. The *shape* $|\delta|$ is the number t of rows. The initial monomial of δ is the diagonal

$$\langle b_1 \dots b_t \rangle = x_{1b_1} \cdots x_{tb_t}.$$

Therefore $<$ is a *diagonal* monomial order. All our theorems remain valid for an arbitrary diagonal monomial order $\prec$ since we will see that for our ideals I the initial ideals $\text{in}_{<_{\text{lex}}}(I)$ are generated by initial monomials of products of minors. Therefore

$\mathrm{in}_{<_{\mathrm{lex}}}(I) \subset \mathrm{in}_{\prec}(I)$, and the inclusion implies equality. In view of this observation we will suppress the reference to the monomial order in denoting initial *ideals*, always assuming that the monomial order is diagonal. However, when we compare single *monomials*, the lexicographic order introduced above will be used.

In the straightening law, Theorem 2.2, we need a partial order for the minors and also for their initial monomials:

$$\begin{aligned}[b_1 \dots b_t] \le_{\mathrm{str}} [c_1 \dots c_u] &\iff \langle b_1 \dots b_t \rangle \le_{\mathrm{str}} \langle c_1 \dots c_u \rangle \\ &\iff t \ge u \text{ and } b_i \le c_i,\ i = 1, \dots, u.\end{aligned}$$

It is easy to see that the minors as well as their initial monomials form a lattice with the meet and join operations defined as follows: if $t \ge u$,

$$[b_1 \dots b_t] \vee [c_1 \dots c_u] = [c_1 \dots c_u] \vee [b_1 \dots b_t] = [\max(b_1, c_1), \dots, \max(b_u, c_u)],$$

$$[b_1 \dots b_t] \wedge [c_1 \dots c_u] = [c_1 \dots c_u] \wedge [b_1 \dots b_t] = [\min(b_1, c_1), \dots, \min(b_u, c_u), b_{t+1}, \dots, b_t].$$

The meet and join of two diagonals are defined in the same way: just replace $[\cdots]$ by $\langle \cdots \rangle$.

A product

$$\Delta = \delta_1 \cdots \delta_p, \quad \delta_i = [b_{i1} \dots b_{it_i}],\ i = 1, \dots, p,$$

of minors is called a *tableau*. The *shape* of Δ is the p-tuple $|\Delta| = (t_1, \dots, t_p)$, provided $t_1 \ge \cdots \ge t_p$, a condition that does not restrict us in any way.

If

$$\delta_1 \le_{\mathrm{str}} \cdots \le_{\mathrm{str}} \delta_p$$

then we say that Δ is a *standard tableau*. In the context of determinantal ideals one usually has to deal with bitableaux, but in this paper the row indices are always fixed so that we only need to take care of the column indices. Since the product does not determine the order of its factors, one should distinguish the sequence of minors from the product if one wants to be formally correct; as usually, we tacitly assume that such products come with an order of their factors.

Proposition 2.1. *Let Δ be a tableau. Then there exists a unique standard tableau Σ such that* $\mathrm{in}(\Delta) = \mathrm{in}(\Sigma)$. *Furthermore Δ and Σ have the same shape.*

This is easy to see: if $\Delta = \delta_1 \cdots \delta_p$ is not standard, then there exist i and j such that δ_i and δ_j are not comparable. Since $\mathrm{in}(\delta_i\delta_j) = \mathrm{in}((\delta_i \wedge \delta_j)(\delta_i \vee \delta_j))$ we can replace $\delta_i\delta_j$ by an ordered pair of factors, and after finitely many such operations we reach a standard tableau. It is evidently unique.

That the row indices are not only fixed, but always given by $1, \dots, t$ for a t-minor simplifies the *straightening law*.

Theorem 2.2.

(1) *Let* $\delta = [b_1 \dots b_t]$ *and* $\sigma = [c_1 \dots c_u]$ *be minors. Then there exist uniquely determined minors* η_i, ζ_i *and coefficients* $\lambda_i \in K$*,* $i = 1, \dots, q$*,* $q \geq 0$*, such that*

$$\delta\sigma = (\delta \wedge \sigma)(\delta \vee \sigma) + \lambda_1\eta_1\zeta_1 + \dots + \lambda_q\eta_q\zeta_q,$$

$$\eta_i \leq_{\mathrm{str}} \delta \wedge \sigma, \ \delta_i \vee \sigma \leq_{\mathrm{str}} \zeta_i, \ i = 1, \dots, q,$$

$$\mathrm{in}(\delta\sigma) = \mathrm{in}((\delta \wedge \sigma)(\delta \vee \sigma)) > \mathrm{in}(\eta_1\zeta_1) > \dots > \mathrm{in}(\eta_q\zeta_q).$$

(2) *For every tableau* Δ *there exist standard tableaux* $\Sigma_0 \dots, \Sigma_q$ *of the same shape as* Δ *and uniquely determined coefficients* $\lambda_1, \dots, \lambda_q$*,* $q \geq 0$*, such that*

$$\Delta = \Delta_0 + \lambda_1\Sigma_1 + \dots + \lambda_p\Sigma_p, \quad \mathrm{in}(\Delta) = \mathrm{in}(\Sigma_0) > \mathrm{in}(\Sigma_1) > \dots > \mathrm{in}(\Sigma_q).$$

Note that the K-algebra generated by the t-minors of the first t-rows for $t = 1, \dots, m$ is the coordinate of the flag variety. Hence Theorem 2.2 can be deduced from [25, 14.11], and can also be derived from [13, (11.3) and (11.4)], taking into account Proposition 2.1.

3. Initial ideals and primary decomposition

The main objects of this paper are products of ideals $I_t(a)$. We will access them via the initial ideals

$$J_t(a) = \mathrm{in}(I_t(a)).$$

Our first goal is to determine the primary decompositions of such products along with their initial ideals. For the powers of a single ideal $I_t(a)$ the answer is well-known:

Theorem 3.1.

(1) *The powers of the prime ideal* $I_t(a)$ *are primary. In other words, the ordinary and the symbolic powers of* $I_t(a)$ *coincide.*
(2) $J_t(a)$ *is generated by the initial monomials* $\mathrm{in}(\delta)$ *of the* t*-minors of* $I_t(a)$*.*
(3) $\mathrm{in}(I_t(a)^k) = J_t(a)^k$ *for all* $k \geq 1$*.*

See [13, (9.18)] for the first statement and [14] for the remaining statements. The results just quoted are formulated for $a = 1$, but they immediately extend to general a since polynomial extensions of the ground ring are harmless.

The primary decompositions of the powers of $J_t(a)$ have been determined in [8, Prop. 7.2]. We specify the technical details only as far as they are needed in this article:

Theorem 3.2. *The ideal $J_t(a)$ is radical. It is the intersection $J_t(a) = \bigcap_i P_i$ of prime ideals P_i that are generated by $(n-a-t+2)$ indeterminates, and $J_t(a)^k = \bigcap_i P_i^k$ for all k. In particular, $J_t(a)^k$ has no embedded primes and it is integrally closed.*

For the precise description of the set of prime ideals P_i appearing in Theorem 3.2 we refer the reader to [8].

Now we introduce the main players formally:

Definition 3.3. A *NE-pattern* is a finite sequence $\big((t_1, a_1), \dots, (t_w, a_w)\big)$ of pairs of positive natural numbers with $t_i + a_i \le n+1$ for $i = 1, \dots, w$ and which is ordered according to the following rule: if $1 \le i < j \le w$, then

$$a_i \le a_j \quad \text{and} \quad t_i \ge t_j \text{ if } a_i = a_j.$$

Let $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$ be a NE-pattern. A *pure NE-tableau of pattern S* is a product of minors

$$\Delta = \delta_1 \cdots \delta_w, \qquad \text{such that } \delta_i \text{ is a } t_i\text{-minor of } X_{t_i}(a_i),\ i = 1, \dots, w.$$

An *NE-tableau* is a product $M\Delta$ of a monomial M in the indeterminates x_{ij} and a pure NE-tableau Δ.

The NE-*ideal* of pattern S is the ideal generated by all (pure) NE-tableaux of pattern S. In other words, it is the ideal

$$I_S = I_{t_1}(a_1) \cdots I_{t_w}(a_w).$$

Furthermore we set

$$J_S = \operatorname{in}(I_S).$$

So I_S is simply a product of ideals of type $I_t(a)$ where, by convention, the factors have been ordered according to the rule specified in 3.3.

For $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$ and a pair (u, b) we set

$$e_{ub}(S) = |\{i : b \le a_i \text{ and } u \le t_i\}|.$$

Note that $b \le a_i$ and $u \le t_i$ is indeed equivalent to $I_{t_i}(a_i) \subset I_u(b)$.

Theorem 3.4. *Let $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$ be a NE-pattern. Then the following hold:*

$$J_S = J_{t_1}(a_1) \cdots J_{t_w}(a_w); \tag{3.1}$$

$$J_S = \bigcap_{u,b} J_u(b)^{e_{ub}(S)}; \tag{3.2}$$

$$I_S = \bigcap_{u,b} I_u(b)^{e_{ub}(S)}. \tag{3.3}$$

Equation (3.3) gives a primary decomposition of I_S. The ideals I_S and J_S are integrally closed.

As soon as the equation (3.3) will have been proved, it indeed yields a primary decomposition of I_S since all the ideals $I_u(b)^e$ are primary, being powers of ideals of maximal minors. The intersection in (3.3) is almost always redundant. An irredundant decomposition will be described in Proposition 3.9. Together with Theorem 3.2, equation (3.2) gives a primary decomposition of J_S. The ideals I_S and J_S are integrally closed because the ideals appearing in their primary decomposition are symbolic powers of prime ideals and therefore integrally closed.

The special case of Theorem 3.4 in which all a_i are equal has been proved in [3, Corollary 2.3] and [3, Theorem 3.3]. It will be used in the proof of the theorem. (Note however that in [3] our ideal I_S is denoted by J_S.)

Proof of Theorem 3.4. By the definition of $e_{ub}(S)$ we have

$$I_S \subset \bigcap_{u,b} I_u(b)^{e_{ub}(S)}.$$

This implies the chain of inclusions

$$\prod_{i=1}^{w} J_{t_i}(a_i) \subset J_S \subset \operatorname{in}\Bigl(\bigcap_{u,b} I_u(b)^{e_{ub}(S)}\Bigr) \subset \bigcap_{u,b} \operatorname{in}\bigl(I_u(b)^{e_{ub}(S)}\bigr) = \bigcap_{u,b} J_u(b)^{e_{ub}(S)}$$

where we have used Theorem 3.1 for the equality of the two rightmost terms. If

$$\bigcap_{u,b} J_u(b)^{e_{ub}(S)} \subset \prod_{i=1}^{w} J_{t_i}(a_i)) \tag{3.4}$$

as well, then we have equality throughout, implying (3.1) and (3.2). Then (3.3) follows since two ideals with the same initial ideal must coincide if one is contained in the other. Therefore (3.4) is the crucial inclusion.

We prove it by induction on w. Let M be a monomial in $\bigcap_{u,b} J_u(b)^{e_{ub}(S)}$. Then M is contained in $J_{t_w}(a_w)$. This ideal is generated by all diagonals $\langle f_1 \dots f_{t_w}\rangle$ with $f_1 \geq a_w$ by Theorem 3.1(2). Among all these diagonals we choose the *lexicographically smallest* and call it F.

Set $T = \bigl((t_1, a_1), \dots, (t_{w-1}, a_{w-1})\bigr)$. It is enough to show that $M/F \in \bigcap_{u,b} J_u(b)^{e_{ub}(T)}$, and for this containment we must show $M/F \in J_u(b)^{e_{ub}(T)}$ for all u and b. Evidently

$$e_{ub}(T) = \begin{cases} e_{ub}(S), & b \le a_w,\ u > t_w, \\ e_{ub}(S) - 1, & b \le a_w,\ u \le t_w, \\ 0, & \text{else.} \end{cases}$$

If $e_{t_w b}(T) = 0$, there is nothing to show. If $b \le a_w$, $u > t_w$, we have $e_{t_w b}(S) \ge e_{ub}(S)+1$ because $I_{t_w}(a_w)$ contributes to $e_{t_w b}(S)$, but not to $e_{ub}(S)$. This observation is important for the application of Lemma 3.5 that covers this case. The case $b \le a_w$, $u \le t_w$ is Lemma 3.6. □

Lemma 3.5. *Let $k \in \mathbb{N}$. Let $b \le a$ and $u > t$. Let $M \in J_u(b)^k \cap J_t(b)^{k+1} \cap J_t(a)$ be a monomial and let F be the lexicographic smallest diagonal of length t that divides M. Then $M/F \in J_u(b)^k$.*

Proof. We can apply [3, Theorem 3.3] to $J_u(b)^k \cap J_t(b)^{k+1}$: M is divided by a product $D_1 \cdots D_k E$ where $D_1, \dots, D_k$ are diagonals of length u starting in column b or further right, and E is such a diagonal of length t. Even more: $J_u(b)^k \cap J_t(b)^{k+1}$ is generated by the initial monomials of the standard tableaux in $I_u(b) \cap I_t(b)^{k+1}$ (Proposition 2.1). Therefore we can assume that $D_1 \le_{\text{str}} \cdots \le_{\text{str}} D_k \le_{\text{str}} E$.

The greatest common divisor of F and $D_1 \cdots D_k E$ must divide E—if not we could replace F by $F \vee E$ and obtain a lexicographically smaller diagonal of length t. Therefore $D_1 \cdots D_k$ divides M/F. □

Lemma 3.6. *Let $k \in \mathbb{N}$. Let $b \le a$ and $u \le t$. Let $M \in J_u(b)^k \cap J_t(a)$ be a monomial and let F be the lexicographic smallest diagonal of length t that divides M. Then $M/F \in J_u(b)^{k-1}$.*

Proof. By Theorem 3.1 here exist diagonals $D_1, \dots, D_k$ such of length u such that $D_1 \cdots D_k$ divides M and $D_1 \le_{\text{str}} \cdots \le_{\text{str}} D_k$. Division by F can “destroy” more than one of these diagonals but, as we will see, the fragments can be joined to form $k-1$ diagonals of length u as desired.

We explain the argument first by an example: $M = x_{11}x_{12}x_{23}x_{24}x_{34} \in J_2(1)^2 \cap J_3(2)$. The lexicographically smallest diagonal of length 3 is $\langle 234 \rangle$. It intersects both 2-diagonals $\langle 13 \rangle$ and $\langle 24 \rangle$, but we can produce the new 2-diagonal $\langle 14 \rangle$ from the two fragments, and are done in this case: $M \in J_2(1)J_3(2)$.

Let $r_1 \le \cdots \le r_p$ be the rows in which F intersects one of the D_i, and choose g_i maximal such that F intersects D_{g_i} in row r_i. In view of the order of the D_i and by the choice of F as the lexicographically smallest t-diagonal dividing M, we must have $g_{i+1} \le g_i$ for $i = 1, \dots, p-1$.

Every time that F “jumps” to another diagonal, i.e., if $g_i > g_{i+1}$, we concatenate the entries in rows $1, \dots, r_i$ of $D_{g_{i+1}}$ with the entries in rows $r_i + 1, \dots, u$ of D_{g_i}, thus producing a new diagonal. (Note that F cannot return to $D_{g_{i+1}}$ in rows $\le r_i$ and has not touched D_{g_i} in the other rows.) Only one u-diagonal is lost this way. □

Our next goal is to identify the irredundant components in the primary decomposition of I_S described in Theorem 3.4. To this end we prove the following facts.

Lemma 3.7. *Let S be a NE-pattern and let D be a subsequence of S. Set $T = S \setminus D$. Then $I_S : I_D = I_T$.*

Proof. By induction on the cardinality of D, we may assume right away D is a singleton. Using Theorem 3.4 the desired equality boils down to the proof that for every $k > 0$ one has $I_u(b)^k : I_t(a) = I_u(b)^{k-1}$ if $(b, u) \le (t, a)$, and $I_u(b)^k : I_t(a) = I_u(b)^k$ otherwise. Both equalities follow from the fact that $I_u(b)$ has primary powers. □

Lemma 3.8. *Let $(t, a), (u, b) \in \mathbb{N}_+^2$ such that $t \le u$, $a \le b$ and $u + b \le n + 1$. Then $I_t(a)I_u(b) \subset I_t(b)I_u(a)$. Actually, $I_t(a)$ is an associated prime to $R/I_t(b)I_u(a)$.*

Proof. For the inclusion $I_t(a)I_u(b) \subset I_t(b)I_u(a)$, in view of Theorem 3.4 it is enough to show that $e_{vc}(S) \ge e_{vc}(T)$ for every (v, c) where $S = \{(t, a), (u, b)\}$ and $T = \{(t, b), (u, a)\}$, and this is easy.

The inclusion just proved shows one inclusion of the equality $\big(I_t(b)I_u(a)\big) : I_u(b) = I_t(a)$. The other follows from the fact that $I_t(a)$ is prime and contains $I_t(b)I_u(a)$. □

Now we are ready to prove:

Proposition 3.9. *Given S, let Y be the set of the elements $(t, a) \in \mathbb{N}_+^2$, $(t, a) \notin S$, such that there exists $(u, b) \in \mathbb{N}_+^2$ for which $(t, b), (u, a) \in S$ and $t < u$, $a < b$. Then we have the following primary decomposition:*

$$I_S = \bigcap_{(v,c)\in S\cup Y} I_v(c)^{e_{vc}(S)}$$

which is irredundant provided all the points (u, b) above can be taken so that $u+b \le n+1$. In particular, for fixed S, the given primary decomposition above is irredundant if n is sufficiently large, and in this case all powers I_S^k have the same associated prime ideals as I_S.

Proof. The equality holds because of Theorem 3.4 and because if $(v, c) \notin S \cup Y$ then $e_{vc}(S)$ is either equal to $e_{v+1,c}(S)$ or $e_{v,c+1}(S)$.

It remains to show that the decomposition is irredundant under the extra assumption. We can equivalently prove that every prime $I_t(a)$ with $(t, a) \in S \cup Y$ is associated to R/I_S. For $(t, a) \in S$ this follows from a general fact: for prime ideals $P_1, \dots, P_r \neq 0$ in a noetherian domain A each P_i is associated to A/I, $I = P_1 \cdots P_r$. This follows easily by localization; one only needs that $IA_{P_i} \neq 0$ for all i.

Now let $(t, a) \in Y$ and $(t, b), (u, a) \in S$ and such that $t < u$ and $a < b$ and $u+b \le n+1$. Set $D = S \setminus \{(t, b), (u, a)\}$. Then by 3.7 we have $I_S : I_D = I_t(b)I_u(a)$, and by 3.8 $I_t(a)$ is associated to $R/I_t(b)I_u(a)$. It follows that $I_t(a)$ is associated to R/I_S as well.

For the last statement we note that the set Y does not change if we pass from I_S to I_S^k. $\square$

Let us illustrate Theorem 3.4 and Proposition 3.9 by two examples.

Example 3.10. Let $n \geq 5$ and $S = \{(3,1),(3,3),(2,3),(1,4)\}$ so that

$$I_S = I_3(1)I_3(3)I_2(3)I_1(4).$$

The ideal and the values $e_{ub}(S)$ are given by the following tables:

			•		
		•			
•		•			

4	3	3	$\boxed{1}$	0	0
3	2	$\boxed{2}$	0	0	0
$\boxed{2}$	1	$\boxed{1}$	0	0	0

The boxed and circled values are the essential ones and give rise to the irredundant components. The boxed correspond to elements in S and the circled to elements in Y. Hence a irredundant primary decomposition of I_S is:

$$I_S = I_1(4) \cap I_1(3)^3 \cap I_2(3)^2 \cap I_3(3) \cap I_1(1)^4 \cap I_2(1)^3 \cap I_3(1)^2.$$

Example 3.11. If the criterion in Proposition 3.9 does not apply, $I_t(a)$ may nevertheless be associated to R/I_S. We choose $n = 4$.

First, let $S = \{(3,1),(2,2),(1,3)\}$. Then $(1,1) \in Y$ and the corresponding "(u,b)" is $(3,3)$ which does not satisfy $u + b \leq n + 1$. Unexpectedly, $I_1(1)$ is associated to I_S.

Second, let $S = \{(3,1),(1,3)\}$. Again $(1,1) \in Y$, and it has the same corresponding (u,b). But in this case $I_1(1)$ is not associated to I_S.

4. The northeast straightening law

It is now crucial to have a "normal form" for elements of I_S. For this purpose we select a K-basis that involves the natural system of generators, the NE-tableaux Δ of pattern S. It has already become apparent in the proof of Lemma 3.6 that we cannot simply require that Δ is a standard tableau, and the following example for $S = \big((1,2),(3,3)\big)$ shows this explicitly:

$$[14][234] = [134][24] - [124][34].$$

The difficulty is that the transformations occurring in the standard straightening procedure do not respect the bounds of the NE-ideals in general. However, they do so in an important special case to which we will come back.

Let $M\Delta$ be a NE-tableau. A monomial $\langle c_1, \dots, c_t \rangle$ is called a *diagonal of type (t,a) in* $M\Delta$ if $c_1 \geq a$ and $\langle c_1, \dots, c_t \rangle \mid \mathrm{in}(M\Delta) = M\,\mathrm{in}(\Delta)$.

Definition 4.1. Let $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$ be a NE-pattern. A NE-tableau $M\Delta$ of pattern S, $\Delta = \delta_1 \cdots \delta_w$, is *$S$-canonical* if $\operatorname{in}(\delta_j)$ is the lexicographically smallest diagonal of type (t_j, a_j) in the NE-tableau $M\delta_1 \cdots \delta_j$ of pattern $\big((t_1, a_1, \dots, (t_j, a_j)\big)$ for $j = 1, \dots, w$.

As an example we consider the monomial $M = x_{11}x_{12}x_{13}x_{23}x_{24}x_{25}x_{35}$, graphically symbolized by the following table:

•	•	•		
		•	•	•
				•

It depends on the pattern S which S-canonical tableau has M as its initial monomial.

(1) For $S = \big((2, 1), (3, 2), (2, 2)\big)$ the canonical tableau with initial monomial M is

$$[13][245][35].$$

(2) For $S = \big((2, 1), (2, 2), (3, 3)\big)$ it is

$$[13][25][345].$$

(3) For $S = \big((2, 1), (2, 2), (2, 3)\big)$ it is

$$x_{35}[13][24][35].$$

Note that different canonical NE-tableaux of the same pattern are K-linearly independent since they have different initial monomials: if the pattern S is fixed, then an S-canonical tableau is uniquely determined by its initial monomial. In fact, the diagonals that are split off successively are uniquely determined, and each diagonal belongs to a unique minor.

We can now formulate the *NE-straightening law*:

Theorem 4.2. *Let $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$ be a NE-pattern and $x \in I_S$. Then there exist uniquely determined S-canonical NE-tableaux $M_i\Gamma_i$, $i = 0, \dots, p$, and coefficients $\lambda_i \in K$ such that*

$$x = \lambda_0 M_0 \Gamma_0 + \lambda_1 M_1 \Gamma_1 + \cdots + \lambda_p M_p \Gamma_p$$

and

$$\operatorname{in}(x) = \operatorname{in}(M_0\Gamma_0) > \operatorname{in}(M_1\Gamma_1) > \cdots > \operatorname{in}(M_p\Gamma_p).$$

Proof. Let λ_0 be the initial coefficient of x. It is enough to show the existence of $M_0\Gamma_0$ since $\operatorname{in}(x - \lambda_0 M_0\Gamma_0) < \operatorname{in}(x)$, and we can apply induction.

Clearly $\mathrm{in}(x) \in \mathrm{in}(I_S)$. The factorization of the monomial $\mathrm{in}(x)$ constructed recursively in the proof of Theorem 3.4 is exactly the factorization that gives it the structure $M_0 \,\mathrm{in}(\Gamma_0)$ for an S-canonical NE-tableau of pattern S: it starts by extracting the lexicographically smallest diagonal D_w of length t_w from $\mathrm{in}(x)$, and applies the same procedure to $\mathrm{in}(x)/D_w$ recursively. As pointed out above, this factorization belongs to a unique S-canonical tableau. $\square$

We call Theorem 4.2 a straightening law, since it generalizes the "ordinary" straightening law to some extent:

Theorem 4.3. *Suppose the NE-pattern $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$ satisfies the condition $t_i \geq t_{i+1}$ for $i = 1, \dots, w-1$, and let Δ be pure NE-tableau of pattern S. Then the representation*

$$\Delta = \Delta_0 + \lambda_1 \Sigma_1 + \dots + \lambda_p \Sigma_p$$

of Theorem 2.2(2) is the S-canonical representation.

Proof. The only question that could arise is whether the representation is S-canonical. It is successively obtained from Δ by applying the straightening law to pairs of minors:

$$\delta_\sigma = (\delta \wedge \sigma)(\delta_\vee \sigma) + \lambda_1 \eta_1 \zeta_1 + \dots + \lambda_q \eta_q \zeta_q, \qquad \eta_i \leq_{\mathrm{str}} \delta_\wedge \sigma,\ \delta_i \vee \sigma \leq_{\mathrm{str}} \zeta_i,\ i = 1, \dots, q,$$

as in Theorem 2.2(1). Therefore it is enough to consider the case $w = 2$, $S = ((t, a), (u, b))$, $a \leq b$, $t \geq u$, $\delta = [d_1 \dots d_t]$, $\sigma = [s_1 \dots s_u]$. The smallest column index is $\min(d_1, s_1) \geq a$. So all minors ζ_i belong to $I_t(a)$. The minors η_i satisfy the inequalities $\eta_i \geq_{\mathrm{str}} \delta$ and $\eta_i \geq_{\mathrm{str}} \sigma$. Therefore they belong to $I_u(b)$. $\square$

For later use we single out two special cases of Theorem 4.2.

(1) For $\delta \in I_t(a)$, $|\delta| = t$, $\sigma \in I_u(b)$, $|\sigma| = u$, there is an equation

$$\delta_\sigma = \delta_0 \sigma_0 + \lambda_1 \delta_1 \sigma_1 + \dots \lambda_p \delta_p \sigma_p \tag{4.1}$$

with $\lambda_1, \dots, \lambda_p \in K$ and canonical NE-tableaux $\delta_0 \sigma_0, \dots \delta_p \sigma_p$ of pattern $((t.a), (u, b))$ and $\mathrm{in}(\delta\sigma) = \mathrm{in}(\delta_0 \sigma_0) > \mathrm{in}(\delta_1 \sigma_1) + \dots \mathrm{in}(\delta_p \sigma_p)$.

(2) With the same notation for δ, for every indeterminate x_{uv} there is an equation

$$x_{uv} \delta = x_{u_0 v_0} \delta_0 + \lambda_1 x_{u_1 v_1} \delta_1 + \dots + x_{u_p v_p} \delta_p \tag{4.2}$$

with $\lambda_1, \dots, \lambda_p \in K$, $x_{u_0 v_0} \delta_0, \dots, x_{u_p v_p} \delta_p$ canonical of pattern (t, a) and $\mathrm{in}(x_{uv} \delta) = \mathrm{in}(x_{u_0 v_0} \delta_0) > \mathrm{in}(x_{u_1 v_1} \delta_1) > \mathrm{in}(x_{u_p v_p} \delta_p)$.

Equation (4.2) is nothing but a linear syzygy of t-minors (unless it is a tautology). These syzygies have sneaked in through the use of the theorem that the t-minors form a Gröbner basis of $I_t(a)$.

We complement the discussion of canonical decompositions by showing that a non-canonical tableau can be recognized by comparing the factors pairwise.

Lemma 4.4. *Let S be a NE-pattern and $M\Delta$, $\Delta = \delta_1 \cdots \delta_w$, be a NE-tableau of pattern S. If $M\Delta$ is not S-canonical, then at least one of the following two cases occurs:*

(1) *there exist a divisor x_{ij} of M and an index k such that $x_{ij}\delta_k$ is not NE-canonical of pattern (t_k, a_k);*
(2) *there exist indices q and k, $q < k$, such that $\delta_q\delta_k$ is not $((t_q, a_q), (t_k, a_k))$-canonical.*

Proof. We choose k maximal with the property that $\operatorname{in}(\delta_k)$ is not the lexicographically smallest (t_k, a_k)-diagonal dividing $\operatorname{in}(M\delta_1 \dots \delta_k)$. Set $t = t_k$, $\delta = [d_1 \dots d_t]$ and let $\langle e_1 \dots e_t\rangle$ be the lexicographically smallest such diagonal. Then choose r maximal with the property that $d_r < e_r$. Since x_{re_r} divides $\operatorname{in}(M\delta_1 \cdots \delta_k)$, at least one of the following two cases must hold:

(1) $x_{re_r} \mid M$;
(2) $x_{re_r} \mid \operatorname{in}(\delta_q)$ for some $q < k$.

In the first case $x_{re_r}\delta_k$ is not (t_k, a_k)-canonical, and in the second case $\delta_q\delta_k$ fits case (2) of the lemma: $\langle d_1 \dots d_{r-1}e_r d_{r+1} \dots e_t\rangle$ is lexicographically smaller than $\langle d_1 \dots d_t\rangle$ and divides $\operatorname{in}(x_{re_r}\delta_k)$ or $\operatorname{in}(\delta_q\delta_k)$, respectively. $\square$

Lemma 4.4 and the equations (4.1) and (4.2) indicate that the S-canonical representation of an element $x \in I_S$ can be obtained by the successive application of quadratic relations. This is indeed true and will be formalized in the next section.

5. The multi-Rees algebra

The natural object for the simultaneous study of the ideals I_S is the multi-Rees algebra

$$\mathscr{R} = R[I_t(a)T_{ta} : 1 \le t, a \le n,\ t + a \le n + 1]$$

where the T_{ta} are new indeterminates It is a subalgebra of the polynomial ring

$$R[T_{ta} : 1 \le t, a \le n,\ t + a \le n + 1],$$

and the products of the ideals $I_t(a)$ appear as the coefficient ideals of the monomials in the indeterminates T_{ta}. These monomials are parametrized by the patterns S: for $S = ((t_1, a_1), \dots, (t_w, a_w))$ we set

$$T^S = T_{t_1 a_1} \cdots T_{t_w a_w}.$$

Then

$$\mathscr{R} = \bigoplus_S I_S T^S.$$

The monomial order on R is extended to $R[T_{ta} : 1 \le a \le n,\ 1 \le t + a \le n+1]$ in an arbitrary way. The extension will be denoted by $<_{\text{lex}}$ as well.

Alongside with $\mathscr{R}$ we consider the multi-Rees algebra $\mathscr{R}_{\text{in}}$ defined by the initial ideals $J_t(a)$:

$$\mathscr{R}_{\text{in}} = R[J_t(a) T_{ta} : 1 \le a \le n,\ 1 \le t + a \le n+1].$$

As always in this context, there is a second "initial" object that comes into play, namely the initial subalgebra of $\mathscr{R}$:

$$\operatorname{in}(\mathscr{R}) = \bigoplus_S J_S T^S.$$

(Recall that $J_S = \operatorname{in}(I_S)$ by definition.) From Theorem 3.4 one can easily derive a first structural result on $\mathscr{R}$ and $\mathscr{R}_{\text{in}}$.

Theorem 5.1.

(1) *With respect to any extension of the monomial order on R to $R[T_{ta} : 1 \le a \le n,\ 1 \le t + a \le n+1]$ one has $\operatorname{in}(\mathscr{R}) = \mathscr{R}_{\text{in}}$.*
(2) *$\mathscr{R}$ and $\mathscr{R}_{\text{in}}$ are normal Cohen–Macaulay domains.*

Proof. The equation $\operatorname{in}(\mathscr{R}) = \mathscr{R}_{\text{in}}$ is just equation (3.1) read simultaneously for all NE-patterns S.

The normality of $\mathscr{R}$ and $\mathscr{R}_{\text{in}}$ follows from the fact that all the ideals I_S and J_S are integrally closed by Theorem 3.4. We observe that $\mathscr{R}_{\text{in}}$ is a normal monoid domain, and therefore Cohen–Macaulay by Hochster's theorem. Finally we use the transfer of the Cohen–Macaulay property from $\operatorname{in}(\mathscr{R}) = \mathscr{R}_{\text{in}}$ to $\mathscr{R}$, see [16]. $\square$

In order to gain insight into the minimal free resolutions of the ideals I_S over R we must understand $\mathscr{R}$ as the residue class ring of a polynomial ring over K. To this end we introduce a variable $z_{a\delta}$ for every bound a and every t-minor $\delta \in I_t(a)$. Let

$$\mathscr{S} = R[z_{a\delta} : |\delta| = t,\ t + a \leq n + 1,\ \delta \in I_t(a)].$$

Viewed as a K-algebra, $\mathscr{S}$ needs also the indeterminates x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$. We want to study the surjective R-algebra homomorphism

$$\Phi : \mathscr{S} \to \mathscr{R}, \qquad \Phi(z_{a\delta}) = \delta T_{a|\delta|}, \quad \Phi|R = \mathrm{id}\,.$$

We introduce an auxiliary monomial order on $\mathscr{S}$ by first ordering the indeterminates. The x_{ij} are ordered as in R. Next we set

$$x_{ij} > z_{a\delta}$$

for all i, j, a, δ, and

$$z_{a\delta} > z_{b\sigma} \qquad \Longleftrightarrow \qquad a < b \quad \text{or } a = b \text{ and } \operatorname{in}(\delta) >_{\mathrm{lex}} \operatorname{in}(\sigma).$$

This order of the indeterminates is extended to the *reverse* lexicographic order $\leq_{\mathrm{revlex}}$ it induces on the monomials in $\mathscr{S}$.

Now we define the main monomial order on $\mathscr{S}$ as follows. For monomials Z_1 and Z_2 in x_{ij} and $z_{a\delta}$ we set

$$\begin{aligned} Z_1 \prec Z_2 \qquad \Longleftrightarrow \qquad & \operatorname{in}(\Phi(Z_1)) <_{\mathrm{lex}} \operatorname{in}(\Phi(Z_2)) \qquad \text{or} \\ & \operatorname{in}(\Phi(Z_1)) = \operatorname{in}(\Phi(Z_2)) \text{ and } Z_1 <_{\mathrm{revlex}} Z_2. \end{aligned}$$

In other words, we pull the monomial order on $\mathscr{R}$ back to $\mathscr{S}$ and then use our auxiliary order to separate monomials with the same image under Φ.

Theorem 5.2.

(1) *With respect to the monomial order $\prec$ the ideal $\mathscr{I} = \operatorname{Ker}\Phi$ has a Gröbner basis of quadrics, given by the equations (4.1) and (4.2) (interpreted as elements in $\mathscr{S}$).*
(2) *In particular $\mathscr{R}$ is a Koszul algebra.*
(3) *All the ideals I_S have linear minimal free resolutions over R.*

Proof. Equation (4.1) defines the polynomial

$$z_{a\delta} z_{b\sigma} - \left(z_{a\delta_0} z_{b\sigma_0} + \lambda_1 z_{a\delta_1} z_{b\sigma_1} + \cdots + \lambda_w z_{a\delta_w} z_{b\sigma_w}\right)$$

in $\mathscr{I}$. By the definition of $\prec$ we first observe that only $z_{a\delta} z_{b\sigma}$ or $z_{a\delta_0} z_{b\sigma_0}$ can be the initial monomial. But then the auxiliary reverse lexicographic oder makes $z_{a\delta} z_{b\sigma}$ the leading monomial. Similarly one sees that $x_{uv} z_{a\delta}$ is the leading monomial of the polynomial in $\mathscr{I}$ defined by (4.2).

Let $\mathscr{J}$ be the ideal generated by all monomials $Z = Mz_{a_1\delta_1}\cdots z_{a_w\delta_w}$, $M \in R$, for which $M\delta_1\cdots\delta_w$ is not canonical of pattern $S = ((|\delta_1|, a_1), \dots, (|\delta_1|, a_1))$. It follows from Lemma 4.4 that $MMz_{a_1\delta_1}\cdots z_{a_w\delta_w}$ then contains a factor $z_{a_i\delta_i}z_{j\delta_j}$ or a factor $x_{uv}z_{a_j\delta_j}$ that is not mapped to a canonical tableau of the associated pattern. In connection wit the argument above, this observation implies that $\mathscr{J} \subset \operatorname{in}_\prec(\mathscr{I})$.

On the other hand the set of monomials that do not belong to $\mathscr{J}$ form a K-basis of $\mathscr{S}/\mathscr{I} = \mathscr{R}$ by Theorem 4.2. This is only possible if $\mathscr{J} = \operatorname{in}(\mathscr{I})$.

Since $\mathscr{I}$ has a Gröbner basis of quadratic polynomials, $\mathscr{R} = \mathscr{S}/\mathscr{I}$ is a Koszul algebra. By (the multigraded version of) a theorem of Blum [5] the linearity of the resolutions follows from the Koszul property of the multi-Rees algebra. □

In addition to Φ, we have a surjective R-algebra homomorphism

$$\Psi : \mathscr{S} \to \mathscr{R}_{\mathrm{in}}, \qquad \Psi(z_{a,\delta}) = \operatorname{in}(\delta)T_{a|\delta|}, \quad \Phi|R = \mathrm{id}\,.$$

Theorem 5.3.

(1) *With respect to the monomial order $\prec$ the ideal $\mathscr{B} = \operatorname{Ker}\Psi$ has a Gröbner basis of quadrics, given by the binomials resulting from the equations* $\operatorname{in}(\delta\sigma) = \operatorname{in}(\delta_0\sigma_0)$ *in (4.1) and* $\operatorname{in}(x_{uv}\delta_)} = \operatorname{in}(x_{u_0v_0}\delta_0)$ *in (4.2) (interpreted as elements in $\mathscr{I}$).*
(2) *In particular $\mathscr{R}_{\mathrm{in}}$ is a Koszul algebra.*
(3) *All the ideals J_S have linear minimal free resolutions over R.*

Proof. The first statement is proved completely analogous the first statement in Theorem 5.2, and the second and third follow from it in the same way as for Theorem 5.2. □

6. Some Gorenstein Rees rings and some factorial fiber rings

In this section we will consider multi-Rees algebras defined by some of the ideals $I_t(a)$. More generally, if $I_1, \dots, I_p$ are ideals in R, we let

$$R(I_1, \dots, I_p) = R[I_iT_i : i = 1, \dots, p] \subset R[T_1, \dots, T_p]$$

denote the multi-Rees algebra defined by $I_1, \dots, I_t$. Note that we could as well have defined it by taking ordinary Rees algebras successively, since

$$R(I_1, \dots, I_p) = B\bigl(I_pB\bigr) \quad \text{where} \quad B = R(I_1, \dots, I_{p-1}).$$

Some of the Rees rings defined by NE-ideals of minors are Gorenstein. This is not true in general for the “total” multi-Rees rings of the last section: the first potential non-Gorenstein example is a 2×3-matrix, and the corresponding total multi-Rees ring is indeed not Gorenstein. Nevertheless, the multi-Rees rings defined by certain selections of the ideals $I_t(a)$ are Gorenstein, as we will see in the following.

The ideals $I_t(a)$ form a poset under inclusion. The minimal elements are the principal ideals $I_t(n-t+1)$ and the maximal element is $I_1(1)$, the ideal generated by the indeterminates in the first row of our matrix X. The next theorem states the Gorenstein property of the multi-Rees algebras defined by an unrefinable ascending chain in our poset that starts from a minimal element or a cover of a minimal element.

Theorem 6.1. *Let $I_{t_1}(a_1) \subset I_{t_2}(a_2) \subset \cdots \subset I_{t_p}(a_p)$ such that* $\operatorname{height} I_{t_1}(a_1) = 1$ *or* 2 *and* $\operatorname{height} I_{t_i}(a_i) = 1 + \operatorname{height} I_{t_{i-1}}(a_{i-1})$ *for $i = 2, \dots, p$. Equivalently,*

(1) $a_1 = n - t_1$ *or* $a_1 = n - t + 1$*;*
(2) $t_i = t_{i-1}$ *and* $a_i = a_{i-1} - 1$ *or* $t_i = t_{i-1} - 1$ *and* $a_i = a_{i-1}$ *for* $i = 2, \dots, p$.

Then the multi-Rees algebra $R\big(I_{t_1}(a_1), \dots, I_{t_p}(a_p)\big)$ is Gorenstein and normal with divisor class group $\mathbb{Z}^{p-1}$ or $\mathbb{Z}^p$, depending on whether $a_1 = n - t_1$ or $a_1 = n - t + 1$.

Proof. Note that the smallest ideal is a principal ideal if $a_1 = n - t_1 + 1$. In this case $R\big(I_{t_1}(a_1), \dots, I_{t_p}(a_{t_p})\big)$ is just (isomorphic to) a polynomial ring over $R\big(I_{t_2}(a_2), \dots, I_{t_p}(a_p)\big)$, and $a_2 = n - t_2$. Since polynomial extensions do not affect the Gorenstein property, we can assume that $a_1 = n - t_1$.

Let $\mathscr{R} = R\big(I_{t_1}(a_1), \dots, I_{t_p}(a_p)\big)$, $\mathscr{R}' = R\big(I_{t_1}(a_1), \dots, I_{t_{p-1}}(a_{p-1})\big)$, and $Q = I_{t_p}(a_p)\mathscr{R}'$. Then $\mathscr{R}$ is just the ordinary Rees algebra of the ideal Q of $\mathscr{R}'$, and by induction on p it is enough to understand the extension of $\mathscr{R}'$ to $\mathscr{R}$.

By the next lemma, Q is a prime ideal of height 2 such that $Q\mathscr{R}'_Q$ is generated by 2 elements. Moreover, $\mathscr{R}'$ and $\mathscr{R}$ are normal domains since they are retracts of the total multi-Rees algebra of the last section (or by Theorem 3.4). Under these conditions a theorem of Herzog and Vasconcelos [23, Theorem(c), p. 183] shows that the canonical module of $\mathscr{R}$ has the same divisor class as the canonical module of $\mathscr{R}'$ (extended to $\mathscr{R}$):

$$cl(\omega_{\mathscr{R}}) = cl(\omega_{\mathscr{R}'}) + (\operatorname{ht} Q - 2)cl(Q\mathscr{R}) = cl(\omega_{\mathscr{R}'}) \in \operatorname{Cl}(\mathscr{R}) = \operatorname{Cl}(\mathscr{R}') \oplus \mathbb{Z}.$$

By induction $\mathscr{R}'$ is Gorenstein, $cl(\omega_{\mathscr{R}'}) = 0$. Therefore $\mathscr{R}$ is Gorenstein as well, and we are done. The assertion on the divisor class group follows as well. $\square$

Lemma 6.2. *With the notation of the preceding proof, Q is a prime ideal of height 2 in $\mathscr{R}'$ such that $Q\mathscr{R}'_Q$ is generated by 2 elements.*

Proof. The most difficult claim is the primeness of Q. We show primeness of a larger class of ideals, namely all ideals $I_u(b)\mathscr{R}'$ such that $I_u(b) \supset I_{t_{p-1}}(a_{p-1})$. Set $P = I_u(b)$.

As an auxiliary ring we consider the multi-Rees algebra $\mathscr{S} = R(P, \dots, P)$ with $p-1$ "factors" P. For $\mathscr{R}'$ as above one has $\mathscr{S} \supset \mathscr{R}'$ since P contains all the ideals defining $\mathscr{R}'$. It follows from Equation (3.3) that

$$P\mathscr{R}' = P\mathscr{S} \cap \mathscr{R}'.$$

In fact, both algebras use the variables $T_1, \dots, T_{p-1}$. The coefficient ideal of $T_1^{e_1} \cdots T_{p-1}^{e_{p-1}}$ in $P\mathscr{S}$ is $P^{1+e_1+\cdots+e_{p-1}}$ and its coefficient ideal in $\mathscr{R}'$ is

$$I_{t_1}(a_1)^{e_1} \cdots I_{t_{p-1}}(a_{p-1})^{e_{p-1}}$$

whereas the coefficient ideal in $P\mathscr{R}'$ is $PI_{t_1}(a_1)^{e_1} \cdots I_{t_{p-1}}(a_{p-1})^{e_{p-1}}$. Equation (3.3) implies

$$PI_{t_1}(a_1)^{e_1} \cdots I_{t_{p-1}}(a_{p-1})^{e_{p-1}} = P^{1+e_1+\cdots+e_{p-1}} \cap I_{t_1}(a_1)^{e_1} \cdots I_{t_{p-1}}(a_{p-1})^{e_{p-1}},$$

and this is the desired equality.

The primeness of $P\mathscr{R}'$ follows if $P\mathscr{S}$ is a prime ideal. The algebra $\mathscr{S}$ is the Segre product of the polynomial ring in $p-1$ variables over K and the ordinary Rees algebra $S = R[PT]$. Consequently $\mathscr{R}'/P\mathscr{R}'$ is the Segre product of the same polynomial ring and the associated graded ring S/PS of P. But the latter is an integral domain [13, (9.17)].

The smallest choice for P is $I_{t_{p-1}}(a_{p-1})$. This nonzero prime ideal is properly contained in Q. Therefore $\operatorname{ht} Q \geq 2$. In order to finish the proof it remains to show that $Q\mathscr{R}'_Q$ is generated by 2 elements. The indeterminate x_{1n} in the right upper corner of X is not contained in Q if $t_p > 1$. We can invert it and, roughly speaking, reduce all minor sizes and n by 1. This is a standard localization argument; see [13, (2.4)] (where it is given for x_{11}). Therefore we can assume that $t_p = 1$.

If even $p = 1$, then $a_1 = n - 1$, and P is evidently generated by 2 elements. So suppose that $p > 1$. There are two cases left, namely $t_{p-1} = 1$, $a_{p-1} = a_p - 1$ or $t_{p-1} = 2$, $a_{p-1} = a_p$.

In the first case we use the equations

$$x_{1i}(x_{1n}T_{p-1}) = x_{1n}(x_{1i}T_{p-1}), \qquad i \geq a_{p-1} = a_p + 1.$$

The element $x_{1n}T_{p-1}$ does not belong to Q, and becomes a unit in $\mathscr{R}'_Q$. Thus $Q\mathscr{R}'_Q$ is generated x_{1a_p} and x_1n.

In the other case one uses the linear syzygies of the 2-minors in $I_2(a_{p-1})$ with coefficients from the first row of X in order to show that $Q\mathscr{R}'_Q$ is generated by x_{1n-1} and x_{1n}. $\square$

Remark 6.3. (a) An alternative proof of Theorem 6.1 can be given by toric methods. Using Theorem 3.2 one can describe the cone of the exponent vectors of $\operatorname{in}(\mathscr{R})$ ($\mathscr{R}$ as in Theorem 6.1) by inequalities. These inequalities have coefficients in $\{0, \pm 1\}$, and 1 occurs exactly one more time than -1. Therefore the exponent vector with all entries 1 generates the interior of the cone of exponent vectors, which is the set of exponent vectors of the canonical module of $\operatorname{in}(\mathscr{R})$ by theorem of Danilov and Stanley [11, 6.3.5]. It follows that $\operatorname{in}(\mathscr{R})$ is Gorenstein and therefore $\mathscr{R}$ is also Gorenstein.

The opposite implication also works for the Gorenstein property since $\operatorname{in}(\mathscr{R})$ is known to be Cohen–Macaulay. For Cohen–Macaulay domains the Gorenstein property only depends on the Hilbert series by a theorem of Stanley [11, 4.4.6].

(b) In general, extensions of the prime ideals $I_t(a)$ to Rees algebras defined by collections of the ideals $I_u(b)$ are not prime. However, by extending the intersection argument in the proof of Lemma 6.2 one can show that they are radical ideals.

A Cohen–Macaulay factorial domain is Gorenstein. So one may wonder whether the Rees rings discussed above can be factorial. But, apart from trivial exceptions, Rees rings cannot be factorial. On the other hand, the fiber rings have more chances to be factorial. The fiber ring $F(I_1, \dots, I_p)$ of associated to the multi-Rees ring of ideals I_i, $i = 1, \dots, p$ is defined as

$$F(I_1, \dots, I_p) = R(I_1, \dots, I_p)/\mathfrak{m}R(I_1, \dots, I_p)$$

where $\mathfrak{m}$ is the irrelevant maximal ideal of R. If each of the ideals I_i is generated by elements of the same degree, say d_i, the multi-fiber ring is a retract of the Rees ring, namely

$$F(I_1, \dots, I_p) = K[(I_i)_{d_i} T_i : i = 1, \dots, p]$$

where $(I_i)_{d_i}$ is the homogeneous component of degree d_i. It can of course be replaced by a system of degree d_i generators of I_i.

Let us consider a sequence $(t_1, a_1), \dots, (t_p, a_p)$ such that $t_1 < \dots < t_p$ and $I_i = I_{t_i}(a_i)$. In this case the multi-fiber ring can even be identified with the subalgebra

$$K[(I_i)_{t_i} : i = 1, \dots, p] \tag{6.1}$$

of R (it is only essential that the degrees t_i are pairwise different). Thus the multi-fiber ring is a subalgebra of the homogeneous coordinate ring of the flag variety of K^n. The latter is the subalgebra of $K[X]$ (where X is an $n \times n$-matrix) generated by the t-minors of the first t rows, $t = 1, \dots, n$. The coordinate ring of the flag variety is factorial. See [21, p. 138] for an invariant-theoretic argument; an alternative proof is given below.

Theorem 6.4. *Let $t_1 < \dots < t_p$ and $a_1 \geq \dots \geq a_p$ and $I_i = I_{t_i}(a_i)$ for $i = 1, \dots, p$. Then the multi-fiber ring $F(I_1, \dots, I_p)$ is factorial and therefore Gorenstein.*

Proof. Set $F = F(I_1, \dots, I_p)$. In the first step we reduce the claim to the special case in which $p = n$ and $t_i = i$ for $i = 1, \dots, n$. Starting from the given data, we augment X to have at least n rows. Changing the indeterminates for the embedding of F into a polynomial ring over R, we can assume that $F = K[(I_{t_i}(a_i))_{t_i} T_{t_i}]$. Then we let G be the multi-fiber ring defined by $(1, b_1), \dots, (n, b_n)$ where $b_i = a_1$ if $i < t_1$, $b_i = a_j$ if $t_j \leq i < t_{j+1}$, $b_j = a_t$ for $j > t_p$. Consider the R-endomorphism Φ of $R[T_1, \dots, T_n]$ that maps all indeterminates T_{t_i} to themselves and the other T_j to 0. Then Φ is the identity on F and maps G onto F. Thus F is a retract of G. Since retracts of factorial rings are factorial, it is enough to consider G, and we have reduced the general claim to the special case in which $p = n$ and $t_i = i$ for $i = 1, \dots, n$. Moreover, we can use the embedding (6.1) to simplify notation.

Using the NE straightening law for pure (!) NE-tableaux one sees that x_{1n} is a prime element in F. By the theorem of Gauß–Nagata, the passage to $F[x_{1n}^{-1}]$ does not affect factoriality.

We repeat the localization argument of the proof of Theorem 6.1. Note that the linear syzygies of the t-minors in $I_t(a_t)$ with coefficients x_{1i} are polynomial equations of the algebra generators of F since $a_1 \geq a_j$ for $j = 1, \dots, n$. It follows that $F[x_{1n}^{-1}]$ is a multi-fiber ring defined by minors of sizes $1, \dots, n-1$ over a Laurent polynomial ring. This does not harm us since we can replace K by a factorial ring of coefficients right from the start. This concludes the proof that F is factorial. As we will remark in 6.5, F is a Cohen–Macaulay domain, so we may conclude it is Gorenstein by virtue of Murthy's theorem [11, 3.3.19]. □

Note that the theorem covers the flag variety coordinate ring for which all the bounds a_i are equal to 1.

Remark 6.5. (a) In general the multi-fiber ring $F(I_{t_1}(a_1), \dots, I_{t_p}(a_p))$ is not factorial. For example, for $t + 2 \leq n$ factoriality fails for $F(I_t(1), I_t(2))$ because of the Segre-type relations $(fT_1)(gT_2) = (gT_1)(fT_2)$ for distinct t-minors f, g in $I_t(2)$.

(b) The multi-fiber ring $F(I_{t_1}(a_1), \dots, I_{t_p}(a_p))$ is a Cohen–Macaulay normal domain for every $(t_1, a_1), \dots, (t_p, a_p)$, as can be seen via deformation to the initial algebras.

(c) In general $F(I_{t_1}(a_1), \dots, I_{t_p}(a_p))$ is not Gorenstein, for example $F(I_1(1), I_1(2))$ is not Gorenstein when $n \geq 4$. On the other hand, there is strong experimental evidence that the multi-fiber rings defined by sequences $(t_1, a_1), \dots, (t_p, a_p)$ as in Theorem 6.1 are Gorenstein. In the case in which the t_i are all equal, say equal to t, this is clearly true because the initial algebra of $F(I_t(1), I_t(2), \dots, I_t(n+1-t))$ coincides with the initial algebra of the coordinate ring of the Grassmannian $G(t+1, n+1)$.

References

[1] J. Abbott, A.M. Bigatti, G. Lagorio, CoCoA-5: a system for doing Computations in Commutative Algebra, available at http://cocoa.dima.unige.it.

[2] K. Akin, D. Buchsbaum, J. Weyman, Resolutions of determinantal ideals: the submaximal minors, Adv. Math. 39 (1981) 1–30.

[3] A. Berget, W. Bruns, A. Conca, Ideals generated by superstandard tableaux, in: D. Eisenbud, et al. (Eds.), Commutative Algebra and Noncommutative Algebraic Geometry, in: MSRI Publications, vol. 67, Cambridge University Press, 2015, pp. 43–62.

[4] D. Bernstein, A. Zelevinsky, Combinatorics of maximal minors, J. Algebraic Combin. 2 (1993) 111–121.

[5] S. Blum, Subalgebras of bigraded Koszul algebras, J. Algebra 242 (2001) 795–809.

[6] W. Bruns, A. Conca, KRS and powers of determinantal ideals, Compos. Math. 111 (1998) 111–122.

[7] W. Bruns, A. Conca, Gröbner bases and determinantal ideals, in: J. Herzog, V. Vuletescu (Eds.), Commutative Algebra, Singularities and Computer Algebra, Kluwer, 2003, pp. 9–66.

[8] W. Bruns, A. Conca, KRS and determinantal rings, in: J. Herzog, G. Restuccia (Eds.), Geometric and Combinatorial Aspects of Commutative Algebra, in: Lect. Notes Pure Appl. Math., vol. 217, M. Dekker, 2001, pp. 67–87.

[9] W. Bruns, A. Conca, Linear resolutions of powers and products, in: Singularities and Computer Algebra, Festschrift for Gert-Martin Greuel on the Occasion of his 70th Birthday, Springer, 2017, pp. 47–69.
[10] W. Bruns, A. Conca, M. Varbaro, Maximal minors and linear powers, J. Reine Angew. Math. 702 (2015) 41–53.
[11] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Rev. ed., Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, 1998.
[12] W. Bruns, B. Ichim, R. Sieg, T. Römer, C. Söger, Normaliz. Algorithms for rational cones and affine monoids, available at http://normaliz.uos.de/normaliz.
[13] W. Bruns, U. Vetter, Determinantal Rings, Lecture Notes in Math., vol. 1327, Springer, 1988.
[14] A. Conca, Gröbner bases of powers of ideals of maximal minors, J. Pure Appl. Algebra 121 (1997) 223–231.
[15] A. Conca, E. De Negri, E. Gorla, Universal Gröbner bases for maximal minors, Int. Math. Res. Not. 11 (2015) 3245–3262.
[16] A. Conca, J. Herzog, G. Valla, Sagbi bases with applications to blow-up algebras, J. Reine Angew. Math. 474 (1996) 113–138.
[17] D. Eisenbud, C. Huneke, Cohen–Macaulay Rees algebras and their specializations, J. Algebra 81 (1983) 202–224.
[18] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, Singular 4-0-2 — a computer algebra system for polynomial computations, available at http://www.singular.uni-kl.de.
[19] C. De Concini, D. Eisenbud, C. Procesi, Young diagrams and determinantal varieties, Invent. Math. 56 (1980) 129–165.
[20] W. Fulton, Flags, Schubert polynomials, degeneracy loci, and determinantal formulas, Duke Math. J. 65 (1992) 381–420.
[21] W. Fulton, Young Tableaux, London Math. Soc. Stud. Texts, vol. 35, Cambridge University Press, 1997.
[22] D. Grayson, M. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.
[23] J. Herzog, W. Vasconcelos, On the divisor class group of Rees-algebras, J. Algebra 93 (1985) 182–188.
[24] A. Knutson, E. Miller, Gröbner geometry of Schubert polynomials, Ann. of Math. 161 (2005) 1245–1318.
[25] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, Grad. Texts in Math., vol. 227, Springer-Verlag, 2005.
[26] D. Speyer, B. Sturmfels, The tropical Grassmannian, Adv. Geom. 4 (2004) 389–411.
[27] B. Sturmfels, A. Zelevinsky, Maximal minors and their leading terms, Adv. Math. 98 (1993) 65–112.