

# Orientations and multiplicative structures of resolutions.

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# Orientations and multiplicative structures of resolutions

By *Winfried Bruns* at Vechta

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Let  $R$  be a commutative noetherian ring. It is well-known that a free resolution of a cyclic  $R$ -module can be given the structure of a strictly anticommutative algebra over  $R$  ([4], Proposition 1.1). We want to generalize this result, as far as possible, to arbitrary free resolutions, and to draw some consequences. Our results are based on the observation that a free resolution

$$\mathcal{F}: \dots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \dots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$$

very closely approximates complexes with cyclic augmentation, which are constructed from an orientation on  $M_1 := \text{Im } f_1$ .

**Definition.** A finitely generated  $R$ -module  $M$  is called *orientable* with orientation  $\mu$  if the following conditions hold: (i)  $M$  is free in depth 1, i.e. the localizations  $M_P$ ,  $P \in \text{Spec } R$ ,  $\text{depth } R_P \leq 1$ , are free  $R_P$ -modules, (ii) of constant rank  $r$ , the rank of  $M$ , and (iii) there is a linear form  $\mu: \bigwedge^r M \rightarrow R$  such that  $\text{grade } \text{Im } \mu \geq 2$ .

We refer the reader to [6] for the notions of commutative algebra. Modules will be assumed finitely generated throughout. Examples of orientable modules:

- (a) Free  $R$ -modules: the choice of a basis  $e_1, \dots, e_r$  determines an isomorphism

$$\mu: \bigwedge^r M \rightarrow R, \quad \mu(e_1 \wedge \dots \wedge e_r) = 1,$$

- (b) projective  $R$ -modules  $H$  such that  $\bigwedge^r H \cong R$ ,

- (c) over a factorial ring every module free in depth 1,

- (d) every module such that  $\text{grade } \text{Ann } M \geq 2$ , and

- (e) most important perhaps: every module  $M$  which is free in depth 1 and has a finite free resolution.

For (a), (b), and (d) this is trivial. Over a domain (i) includes (ii), and over a factorial domain every ideal is isomorphic to an ideal of grade  $\geq 2$ . Orientability of the modules in (e) follows from the Buchsbaum-Eisenbud structure theorem to be stated below.

Let  $N$  be an arbitrary  $R$ -module and  $v$  a linear form  $\bigwedge^s N \rightarrow R$ . Then  $v$  induces linear maps

$$v^i: \bigwedge^i N \longrightarrow \left( \bigwedge^{s-i} N \right)^*, \quad i=0, \dots, s, \quad \text{and} \quad v^{s+1}: \bigwedge^{s+1} N \longrightarrow R$$

by

$$v^i(x)(y) := v(x \wedge y) \quad \text{for all } x \in \bigwedge^i N, y \in \bigwedge^{s-i} N,$$

and

$$v^{s+1}(x_1 \wedge \dots \wedge x_{s+1}) := \sum_{j=1}^{s+1} (-1)^{j-1} v(x_1 \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_{s+1}) x_j \quad \text{resp.}$$

**Proposition 1.** *Let  $M$  be orientable with orientation  $\mu$ ,  $\text{rank } M = r$ . Then the linear maps  $(\mu^i)^*$ ,  $(\mu^i)^{**}$  are isomorphisms for  $i=0, \dots, r$ . In particular, if  $\bigwedge^i M$  is reflexive, then  $\mu^i$  is an isomorphism itself.*

*Proof.* The maps  $(\mu^i)^*$ ,  $(\mu^i)^{**}$  are homomorphisms of reflexive modules. Therefore it is enough that for each prime ideal  $P$ ,  $\text{depth } R_P \leq 1$ , their localizations with respect to  $P$  are isomorphisms. Since the construction of  $\mu^i$  and dualization commute with localization, the proposition is reduced to the case of a free module for which it is obvious.

**Proposition 2.** *Let  $M$  be torsionfree and orientable with orientation  $\mu$ ,  $\text{rank } M = r$ . Let  $f: F \rightarrow M$  be a homomorphism from an arbitrary  $R$ -module  $F$  to  $M$ . Then, for the composition  $\tilde{\mu} := \mu \circ \bigwedge^r f: \bigwedge^r F \rightarrow R$  one has  $\tilde{\mu}^{r+1} \left( \bigwedge^{r+1} F \right) \subset \text{Ker } f$ .*

*Proof.* One immediately checks that  $f \circ \tilde{\mu}^{r+1} = \mu^{r+1} \circ \bigwedge^{r+1} f$ , and  $\mu^{r+1} = 0$ , since  $\bigwedge^{r+1} M$  is a torsion module, whereas  $M$  is torsionfree.

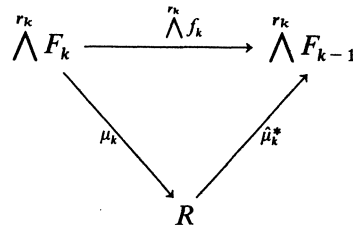
For the reader's convenience we now state the first structure theorem of Buchsbaum and Eisenbud ([3], Theorem 3.1) in the version of [2], Proposition (3.1) and Theorem (3.2).

**Theorem 1.** *Let*

$$\mathcal{F}: 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

*be a finite free resolution. Let  $r_k := \text{rank } f_k$ , and let  $\varphi_k$  denote an orientation on  $F_k$ . Then, for every  $k$  there exist uniquely determined linear forms  $\mu_k \in \left( \bigwedge^{r_k} F_k \right)^*$  and  $\hat{\mu}_k \in \left( \bigwedge^{r_k} F_{k-1} \right)^{**}$  such that:*

- (i)  $\mu_n = \varphi_n$ ,
- (ii) for every  $k$  the diagram below is commutative, and
- (iii)  $\mu_{k-1} = (\varphi_{k-1}^{r_{k-1}})^* (\hat{\mu}_k)$ :



Furthermore  $\mu_k, k = 0, \dots, n$ , factors through  $\bigwedge^{r_k} M_k$ , and induces an orientation on  $M_k$  for  $k = 1, \dots, n$ . If  $M_0 := \text{Coker } f_1$  is free in depth 1, then  $\mu_0$  induces an orientation on  $M_0$ , too.

We now consider a finite or infinite free resolution

$$\mathcal{F} : \dots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{f_1} F_0.$$

Let  $M := \text{Im } f_1, r := \text{rank } M, s := \text{rank } F_1 - r$ . Suppose that  $M$  be oriented by  $\mu$ , and choose a basis  $e_1, \dots, e_t$  of  $F_1$ . We consider the projection  $\pi : F_1 \rightarrow G_1 := \sum_{i=1}^{s+1} R e_i$  with  $\pi(e_j) = 0$  for  $j > s + 1$ , and the linear form  $\psi \in F_1^*, \psi(x) := \mu^{r-1} \left( \bigwedge_{i=1}^{r-1} f_1(\bar{e}) \right) (f_1(x))$ , where  $\bar{e} = e_{s+2} \wedge \dots \wedge e_t$ . Since  $\psi = \psi \circ \pi$  and  $\psi$  factors through  $f$ ,

$$\mathcal{G} : \dots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \dots \longrightarrow F_2 \xrightarrow{\pi \circ f_2} G_1 \xrightarrow{\psi} G_0, \quad G_0 := R,$$

is a complex. For  $i \geq 2$  we put  $G_i := F_i$ .

**Theorem 2.** *Let  $M$  be orientable with orientation  $\mu$ . Then the complex  $\mathcal{G}$  just constructed carries the structure of a strictly anticommutative  $R$ -algebra.*

*Proof.* We proceed as in [4], Proposition 1.1. The claim of the theorem amounts to the existence of a comparison map  $\mathcal{G} \otimes \mathcal{G} \rightarrow \mathcal{G}$  factoring through the ‘‘symmetric square’’  $S^2(\mathcal{G})$ , which is the quotient of  $\mathcal{G} \otimes \mathcal{G}$  by the subcomplex generated by the elements  $x \otimes y - (-1)^{ij} y \otimes x$  for all  $x \in G_i, y \in G_j, i, j \in \mathbb{N}$ , and  $x \otimes x, x \in G_i, i \in \mathbb{N}, i$  odd. Let  $T_i$  be the  $i$ -th homogeneous component of  $\mathcal{G} \otimes \mathcal{G}$ . We have to construct a comparison map

$$\begin{array}{ccccccc}
 F_3 & \xrightarrow{f_3} & F_2 & \xrightarrow{\pi \circ f_2} & G_1 & \xrightarrow{\psi} & G_0 \\
 & & \uparrow \kappa_2 & & \uparrow \kappa_1 & & \parallel \kappa_0 \\
 T_3 & \xrightarrow{\tau_3} & T_2 & \xrightarrow{\tau_2} & T_1 & \xrightarrow{\tau_2} & T_0
 \end{array}$$

such that  $\kappa_i, i = 0, 1, 2$ , factor through the corresponding homogeneous component of  $S^2(\mathcal{G})$  and  $f_2 \circ \kappa_2 \circ \tau_3 = 0$ . The rest is automatic since  $\mathcal{G}$  is acyclic in degree  $\geq 3$  and  $S^2(\mathcal{G})$  is a free complex.

The maps  $\mathcal{G} \otimes G_0 = \mathcal{G} \otimes R \rightarrow \mathcal{G}$  and  $G_0 \otimes \mathcal{G} \rightarrow \mathcal{G}$  being taken as the natural ones, the only critical choice is the part of  $\kappa_2$  which maps  $G_1 \otimes G_1$  to  $F_2$ . We define  $\xi: G_1 \otimes G_1 \rightarrow F_1$  by  $\xi(x \otimes y) := \tilde{\mu}^{r+1}(\bar{e} \wedge x \wedge y)$ ,  $\tilde{\mu}$  denoting the composition  $\mu \circ \bigwedge^r f_1$ . Then  $\xi(G_1 \otimes G_1) \subset \text{Ker } f_1$  by virtue of Proposition 2. So we may take  $\kappa_2|_{G_1 \otimes G_1}$  as a lifting of  $\xi$  to  $F_2$  factoring through the second component of  $S^2(\mathcal{G})$ . The latter condition can be satisfied since it is fulfilled for  $\xi$ .

Now we obtain

$$\begin{aligned} (\kappa_1 \circ \tau_2)(x \otimes y) &= \psi(x)y - x\psi(y) = \tilde{\mu}(\bar{e} \wedge x)y - \tilde{\mu}(\bar{e} \wedge y)x \\ &= \pi(\tilde{\mu}^{r+1}(\bar{e} \wedge x \wedge y)) = ((\pi \circ f_2) \circ \kappa_2)(x \otimes y). \end{aligned}$$

Hence  $\kappa_2$  fits into a comparison map. Furthermore for  $u \in G_1$ ,  $v \in F_2$

$$\begin{aligned} (f_2 \circ \kappa_2 \circ \tau_3)(u \otimes v) &= \psi(u)f_2(v) - \tilde{\mu}^{r+1}(\bar{e} \wedge u \wedge f_2(v)) \\ &= \tilde{\mu}(\bar{e} \wedge u)f_2(v) - \tilde{\mu}(\bar{e} \wedge u)f_2(v) = 0, \end{aligned}$$

for all the remaining terms in the evaluation of  $\tilde{\mu}^{r+1}(\bar{e} \wedge u \wedge f_2(v))$  vanish:

$$\tilde{\mu}(z \wedge f_2(v)) = \mu\left(\bigwedge^r f_1(z \wedge f_2(v))\right) = 0 \text{ for all } z \in \bigwedge^{r-1} F.$$

**Corollary 1.** *Multiplication by an element  $b \in \text{Im } \psi$  is homotopic to zero on  $\mathcal{G}$ .*

*Proof.* Let  $x \in G_1$  such that  $b = \psi(x)$ . Then the collection of the maps  $\eta_i: G_i \rightarrow G_{i+1}$ ,  $\eta_i(y) := \kappa_{i+1}(x \otimes y)$  is a homotopy.

**Corollary 2.** *Let  $M$  be a torsionfree orientable  $R$ -module with orientation  $\mu$ . Then  $(\text{Im } \mu) \cdot \text{Ext}_R^i(M, N) = 0$  for all  $i \geq 1$  and all  $R$ -modules  $N$ .*

*Proof.* Let  $\mathcal{F}$  be a free resolution with  $M = \text{Im } f_1$  as above. It suffices to prove  $b \text{Ext}_R^i(M, N) = 0$  for the elements  $b$  of a system of generators of  $\text{Im } \mu$ . Hence we may assume  $b \in \text{Im } \psi$  for a suitable chosen complex  $\mathcal{G}$  as above. As a consequence of Corollary 1, multiplication by  $b$  is zero on the homology of  $\text{Hom}_R(\mathcal{G}, N)$ . Clearly  $\text{Ext}_R^i(M, N) = H^{i+1}(\text{Hom}_R(\mathcal{G}, N))$  for  $i \geq 2$ , and  $\text{Ext}_R^1(M, N)$  is a quotient of  $H^2(\text{Hom}_R(\mathcal{G}, N))$ .

One of the roots of the Buchsbaum-Eisenbud structure theorem is the theorem of Hilbert-Burch, which is in fact the special case of the structure theorem for resolutions

$$0 \longrightarrow R^n \xrightarrow{f_2} R^{n+1} \xrightarrow{f_1} R.$$

In this case  $f_2$  controls  $f_1$  completely: the coefficients of  $f_1$  are up to sign and a constant factor just the minors of  $f_2$ . The following corollary may be considered as a weak version of the Hilbert-Burch theorem:

**Corollary 3.** *Let*

$$\mathcal{F}: 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

*be a free resolution, and let  $\mu_i$  be an orientation on  $\text{Im } f_i$ ,  $i = 1, \dots, n$ . (Such orientations exist by virtue of Theorem 1.) Suppose  $e_1, \dots, e_q$  is a basis of  $F_n$ . Then for every  $b \in \text{Im } \mu_i$ ,  $i = 1, \dots, n-1$  there is a linear form  $\alpha \in (F_{n-1})^*$  such that*

$$\alpha(f_n(e_1)) = b \text{ and } \alpha(f_n(e_j)) = 0 \text{ for } j = 2, \dots, q.$$

*Proof.* We may certainly assume that  $i=1$ , and, as for Corollary 2, that multiplication by  $b$  is homotopic to zero on a suitable complex  $\mathcal{G}$  as above. Then we have a commutative diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & F_n & \longrightarrow & F'_{n-1} \\
 & & \uparrow b & \nearrow \eta_{n-1}^* & \\
 0 & \longrightarrow & F_n & \xrightarrow{\pi \circ f_n} & F'_{n-1}
 \end{array}$$

in which  $F'_{n-1}$  is a free direct summand of  $F_{n-1}$  (which is equal to  $F_{n-1}$  as soon as  $n > 2$ ) and  $\pi$  is the projection onto  $F'_{n-1}$  (the identity for  $n > 2$ ). One now takes  $\alpha = e_1^* \circ \eta_{n-1} \circ \pi$ .

Under stricter hypotheses we can approximate the Hilbert-Burch theorem somewhat closer. One more notation: For a homomorphism  $f: F \rightarrow G$  of free  $R$ -modules let  $I_s(f)$  denote the ideal generated by the  $s$ -minors of a matrix representing  $f$  with respect to bases in  $F$  and  $G$ ; it is independent of the bases.

**Corollary 4.** *Under the hypotheses of Corollary 3 let  $p = \text{rank } F_{n-1}$  and suppose that  $\text{grade } I_{q-1}(f_n) \geq p - q + 2$ . Then  $\text{Im } \mu_k \subset I_q(f_n)$ , for  $k = 1, \dots, n-1$ .*

*Proof.* Let  $r = p - q$ . By a general position argument (cf. [1]) one can find a basis  $e_1, \dots, e_q$  of  $F_n$  such that  $\text{grade } I_{q-1}(f'_n) \geq r + 2$  for the induced embedding  $f'_n: \sum_{i=2}^q R e_i \rightarrow F_{n-1}$ . (Actually,  $r + 2$  is the maximal value for  $\text{grade } I_{q-1}(f'_n)$ .) Let  $N := \text{Coker } f'_n$ . The linear form  $\alpha$  associated to  $b \in \text{Im } \mu_k$  by Corollary 3 may be considered a linear form on  $N$ , since  $\alpha(f'_n(e_i)) = 0$  for  $i = 2, \dots, q$ . According to Lebelt [5],  $\overset{r}{\bigwedge} N$  is reflexive, whence for any orientation  $v$  on  $N$  the map  $v': \overset{r}{\bigwedge} N \rightarrow N^*$  is an isomorphism by Proposition 1. We choose an orientation  $\varphi$  on  $F_{n-1}$ , and define  $v$  by

$$v(x) := \varphi(\tilde{x} \wedge f_n(e_2) \wedge \dots \wedge f_n(e_q)), \quad x \in \overset{r+1}{\bigwedge} N, \quad \tilde{x} \text{ a preimage of } x \text{ in } \overset{r+1}{\bigwedge} F_{n-1}.$$

It is easily checked that  $v$  is well-defined and indeed an orientation on  $N$ . (It is exactly the orientation on  $N$  determined by Theorem 1 relative to  $\varphi$  and the choice of basis in  $F_n$ .) Let now  $y \in \overset{r}{\bigwedge} N$  such that  $\alpha = v'(y)$ . Then

$$b = \alpha(f_n(e_1)) = v'(y)(f_n(e_1)) = v^{r+1}(y \wedge f_n(e_1)) = \varphi(\tilde{y} \wedge f_n(e_1) \wedge \dots \wedge f_n(e_q))$$

is an element of  $I_r(f_n)$ , since

$$I_r(f_n) = \{\varphi(z \wedge f_n(e_1) \wedge \dots \wedge f_n(e_q)): z \in \overset{r}{\bigwedge} F_{n-1}\}.$$

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