Orientations and multiplicative structures of resolutions

By *Winfried Bruns* at Vechta

Let R be a commutative noetherian ring. It is well-known that a free resolution of a cyclic R-module can be given the structure of a strictly anticommutative algebra over R ([4], Proposition 1.1). We want to generalize this result, as far as possible, to arbitrary free resolutions, and to draw some consequences. Our results are based on the observation that a free resolution

$$\mathscr{F}: \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$$

very closely approximates complexes with cyclic augmentation, which are constructed from an orientation on $M_1 := \operatorname{Im} f_1$.

Definition. A finitely generated R-module M is called *orientable* with *orientation* μ if the following conditions hold: (i) M *is free in depth* 1, i.e. the localizations M_P, $P \in \operatorname{Spec} R$, $\operatorname{depth} R_P \leqq 1$, are free R_P-modules, (ii) of constant rank r, the *rank* of M, and (iii) there is a linear form $\mu: \bigwedge^r M \to R$ such that $\operatorname{grade} \operatorname{Im} \mu \geqq 2$.

We refer the reader to [6] for the notions of commutative algebra. Modules will be assumed finitely generated throughout. Examples of orientable modules:

(a) Free R-modules: the choice of a basis $e_1, \ldots, e_r$ determines an isomorphism

$$\mu: \bigwedge^r M \to R, \ \mu(e_1 \wedge \cdots \wedge e_u) = 1,$$

(b) projective R-modules H such that $\bigwedge^r H \cong R$,

(c) over a factorial ring every module free in depth 1,

(d) every module such that $\operatorname{grade} \operatorname{Ann} M \geqq 2$, and

(e) most important perhaps: every module M which is free in depth 1 and has a finite free resolution.

For (a), (b), and (d) this is trivial. Over a domain (i) includes (ii), and over a factorial domain every ideal is isomorphic to an ideal of grade $\geqq 2$. Orientability of the modules in (e) follows from the Buchsbaum-Eisenbud structure theorem to be stated below.

Let N be an arbitrary R-module and v a linear form $\bigwedge^{s} N \to R$. Then v induces linear maps

$$v^i\colon \bigwedge^{i} N \longrightarrow \left(\bigwedge^{s-i} N\right)^*, \quad i=0,\dots,s, \quad \text{and} \quad v^{s+1}\colon \bigwedge^{s+1} N \longrightarrow N$$

by

$$v^i(x)(y) := v(x \wedge y) \text{ for all } x \in \bigwedge^{i} N,\ y \in \bigwedge^{s-i} N,$$

and

$$v^{s+1}(x_1 \wedge \cdots \wedge x_{s+1}) := \sum_{j=1}^{s+1} (-1)^{j-1} v(x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{s+1})\, x_j \text{ resp.}$$

Proposition 1. *Let M be orientable with orientation μ, $\operatorname{rank} M = r$. Then the linear maps $(\mu^i)^*$, $(\mu^i)^{**}$ are isomorphisms for $i=0,\dots,r$. In particular, if $\bigwedge^{i} M$ is reflexive, then μ^i is an isomorphism itself.*

Proof. The maps $(\mu^i)^*$, $(\mu^i)^{**}$ are homomorphisms of reflexive modules. Therefore it is enough that for each prime ideal P, depth $R_P \leqq 1$, their localizations with respect to P are isomorphisms. Since the construction of μ^i and dualization commute with localization, the proposition is reduced to the case of a free module for which it is obvious.

Proposition 2. *Let M be torsionsfree and orientable with orientation μ, $\operatorname{rank} M = r$. Let $f\colon F \to M$ be a homomorphism from an arbitrary R-module F to M. Then, for the composition $\tilde{\mu} := \mu \circ \bigwedge^{r} f\colon \bigwedge^{r} F \to R$ one has $\tilde{\mu}^{r+1}\left(\bigwedge^{r+1} F\right) \subset \operatorname{Ker} f$.*

Proof. One immediately checks that $f \circ \tilde{\mu}^{r+1} = \mu^{r+1} \circ \bigwedge^{r+1} f$, and $\mu^{r+1} = 0$, since $\bigwedge^{r+1} M$ is a torsion module, whereas M is torsionsfree.

For the reader's convenience we now state the first structure theorem of Buchsbaum and Eisenbud ([3], Theorem 3.1) in the version of [2], Proposition (3.1) and Theorem (3.2).

Theorem 1. *Let*

$$\mathscr{F}\colon 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

be a finite free resolution. Let $r_k := \operatorname{rank} f_k$, and let φ_k denote an orientation on F_k. Then, for every k there exist uniquely determined linear forms $\mu_k \in \left(\bigwedge^{r_k} F_k\right)$ and $\hat{\mu}_k \in \left(\bigwedge^{r_k} F_{k-1}\right)^{**}$ such that:

(i) $\mu_n = \varphi_n$,

(ii) *for every k the diagram below is commutative, and*

(iii) $\mu_{k-1} = (\varphi_{k-1}^{r_{k-1}})^* (\hat{\mu}_k)$:

$$\begin{array}{ccc} \bigwedge^{r_k} F_k & \xrightarrow{\bigwedge^{r_k} f_k} & \bigwedge^{r_k} F_{k-1} \\ & \searrow^{\mu_k} \quad \nearrow_{\hat{\mu}_k^*} & \\ & R & \end{array}$$

Furthermore μ_k, $k = 0, \ldots, n$, *factors through* $\bigwedge^{r_k} M_k$, *and induces an orientation on* M_k *for* $k = 1, \ldots, n$. *If* $M_0 := \operatorname{Coker} f_1$ *is free in depth* 1, *then* μ_0 *induces an orientation on* M_0, *too.*

We now consider a finite or infinite free resolution

$$\mathscr{F}: \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0.$$

Let $M := \operatorname{Im} f_1$, $r := \operatorname{rank} M$, $s := \operatorname{rank} F_1 - r$. Suppose that M be oriented by μ, and choose a basis $e_1, \ldots, e_t$ of F_1. We consider the projection π: $F_1 \to G_1 := \sum_{i=1}^{s+1} Re_i$ with $\pi(e_j) = 0$ for $j > s+1$, and the linear form $\psi \in F_1^*$, $\psi(x) := \mu^{r-1}\left(\bigwedge^{r-1} f_1(\bar{e})\right)(f_1(x))$, where $\bar{e} = e_{s+2} \wedge \cdots \wedge e_t$. Since $\psi = \psi \circ \pi$ and ψ factors through f,

$$\mathscr{G}: \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{\pi \circ f_2} G_1 \xrightarrow{\psi} G_0, \quad G_0 := R,$$

is a complex. For $i \geqq 2$ we put $G_i := F_i$.

Theorem 2. *Let M be orientable with orientation μ. Then the complex $\mathscr{G}$ just constructed carries the structure of a strictly anticommutative R-algebra.*

Proof. We proceed as in [4], Proposition 1.1. The claim of the theorem amounts to the existence of a comparison map $\mathscr{G} \otimes \mathscr{G} \to \mathscr{G}$ factoring through the "symmetric square" $S^2(\mathscr{G})$, which is the quotient of $\mathscr{G} \otimes \mathscr{G}$ by the subcomplex generated by the elements $x \otimes y - (-1)^{ij} y \otimes x$ for all $x \in G_i$, $x \in G_j$, $i, j \in \mathbb{N}$, and $x \otimes x$, $x \in G_i$, $i \in \mathbb{N}$, i odd. Let T_i be the i-th homogeneous component of $\mathscr{G} \otimes \mathscr{G}$. We have to construct a comparison map

$$\begin{array}{ccccccc} F_3 & \xrightarrow{f_3} & F_2 & \xrightarrow{\pi \circ f_2} & G_1 & \xrightarrow{\psi} & G_0 \\ & & \uparrow{\scriptstyle \varkappa_2} & & \uparrow{\scriptstyle \varkappa_1} & & \|{\scriptstyle \varkappa_0} \\ T_3 & \xrightarrow{\tau_3} & T_2 & \xrightarrow{\tau_2} & T_1 & \xrightarrow{\tau_2} & T_0 \end{array}$$

such that $\varkappa_i$, $i = 0, 1, 2$, factor through the corresponding homogeneous component of $S^2(\mathscr{G})$ and $f_2 \circ \varkappa_2 \circ \tau_3 = 0$. The rest is automatic since $\mathscr{G}$ is acyclic in degree $\geqq 3$ and $S^2(\mathscr{G})$ is a free complex.

The maps $\mathscr{G} \otimes G_0 = \mathscr{G} \otimes R \to \mathscr{G}$ and $G_0 \otimes \mathscr{G} \to \mathscr{G}$ being taken as the natural ones, the only critical choice is the part of $\varkappa_2$ which maps $G_1 \otimes G_1$ to F_2. We define ξ: $G_1 \otimes G_1 \to F_1$ by $\xi(x \otimes y) := \tilde{\mu}^{r+1}(\bar{e} \wedge x \wedge y)$, $\tilde{\mu}$ denoting the composition $\mu \circ \bigwedge^r f_1$. Then $\xi(G_1 \otimes G_1) \subset \operatorname{Ker} f_1$ by virtue of Proposition 2. So we may take $\varkappa_2 | G_1 \otimes G_1$ as a lifting of ξ to F_2 factoring through the second component of $S^2(\mathscr{G})$. The latter condition can be satisfied since it is fulfilled for ξ.

Now we obtain

$$(\varkappa_1 \circ \tau_2)(x \otimes y) = \psi(x) y - x \psi(y) = \tilde{\mu}(\bar{e} \wedge x) y - \tilde{\mu}(\bar{e} \wedge y) x$$
$$= \pi(\tilde{\mu}^{r+1}(\bar{e} \wedge x \wedge y)) = ((\pi \circ f_2) \circ \varkappa_2)(x \otimes y).$$

Hence $\varkappa_2$ fits into a comparison map. Furthermore for $u \in G_1$, $v \in F_2$

$$(f_2 \circ \varkappa_2 \circ \tau_3)(u \otimes v) = \psi(u) f_2(v) - \tilde{\mu}^{r+1}(\bar{e} \wedge u \wedge f_2(v))$$
$$= \tilde{\mu}(\bar{e} \wedge u) f_2(v) - \tilde{\mu}(\bar{e} \wedge u) f_2(v) = 0,$$

for all the remaining terms in the evaluation of $\tilde{\mu}^{r+1}(\bar{e} \wedge u \wedge f_2(v))$ vanish:

$$\tilde{\mu}(z \wedge f_2(v)) = \mu\left(\bigwedge^r f_1(z \wedge f_2(v))\right) = 0 \text{ for all } z \in \bigwedge^{r-1} F.$$

Corollary 1. *Multiplication by an element $b \in \operatorname{Im} \psi$ is homotopic to zero on $\mathscr{G}$.*

Proof. Let $x \in G_1$ such that $b = \psi(x)$. Then the collection of the maps η_i: $G_i \to G_{i+1}$, $\eta_i(y) := \varkappa_{i+1}(x \otimes y)$ is a homotopy.

Corollary 2. *Let M be a torsionfree orientable R-module with orientation μ. Then $(\operatorname{Im} \mu) \cdot \operatorname{Ext}_R^i(M, N) = 0$ for all $i \geqq 1$ and all R-modules N.*

Proof. Let $\mathscr{F}$ be a free resolution with $M = \operatorname{Im} f_1$ as above. It suffices to prove $b \operatorname{Ext}_R^i(M, N) = 0$ for the elements b of a system of generators of $\operatorname{Im} \mu$. Hence we may assume $b \in \operatorname{Im} \psi$ for a suitable chosen complex $\mathscr{G}$ as above. As a consequence of Corollary 1, multiplication by b is zero on the homology of $\operatorname{Hom}_R(\mathscr{G}, N)$. Clearly $\operatorname{Ext}_R^i(M, N) = H^{i+1}(\operatorname{Hom}_R(\mathscr{G}, N))$ for $i \geqq 2$, and $\operatorname{Ext}_R^i(M, N)$ is a quotient of $H^2(\operatorname{Hom}_R(\mathscr{G}, N))$.

One of the roots of the Buchsbaum-Eisenbud structure theorem is the theorem of Hilbert-Burch, which is in fact the special case of the structure theorem for resolutions

$$0 \longrightarrow R^n \xrightarrow{f_2} R^{n+1} \xrightarrow{f_1} R.$$

In this case f_2 controls f_1 completely: the coefficients of f_1 are up to sign and a constant factor just the minors of f_2. The following corollary may be considered as a weak version of the Hilbert-Burch theorem:

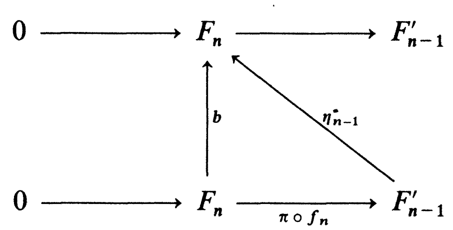
Corollary 3. *Let*

$$\mathscr{F}: 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

be a free resolution, and let μ_i be an orientation on $\operatorname{Im} f_i$, $i = 1, \ldots, n$. *(Such orientations exist by virtue of Theorem* 1.*) Suppose $e_1, \ldots, e_q$ is a basis of F_n. Then for every $b \in \operatorname{Im} \mu_i$, $i = 1, \ldots, n-1$ there is a linear form $\alpha \in (F_{n-1})^*$ such that*

$$\alpha(f_n(e_1)) = b \quad \textit{and} \quad \alpha(f_n(e_j)) = 0 \textit{ for } j = 2, \ldots, q.$$

Proof. We may certainly assume that $i=1$, and, as for Corollary 2, that multiplication by b is homotopic to zero on a suitable complex $\mathcal{G}$ as above. Then we have a commutative diagram



in which F'_{n-1} is a free direct summand of F_{n-1} (which is equal to F_{n-1} as soon as $n>2$) and π is the projection onto F'_{n-1} (the identity for $n>2$). One now takes $\alpha = e_1^* \circ \eta_{n-1} \circ \pi$.

Under stricter hypotheses we can approximate the Hilbert-Burch theorem somewhat closer. One more notation: For a homomorphism f: $F \to G$ of free R-modules let $\mathrm{I}_s(f)$ denote the ideal generated by the s-minors of a matrix representing f with respect to bases in F and G; it is independent of the bases.

Corollary 4. *Under the hypotheses of Corollary* 3 *let* $p = \operatorname{rank} F_{n-1}$ *and suppose that* $\operatorname{grade} \mathrm{I}_{q-1}(f_n) \geqq p-q+2$. *Then* $\operatorname{Im} \mu_k \subset \mathrm{I}_q(f_n)$, *for* $k=1, \ldots, n-1$.

Proof. Let $r=p-q$. By a general position argument (cf. [1]) one can find a basis $e_1, \ldots, e_q$ of F_n such that $\operatorname{grade} \mathrm{I}_{q-1}(f'_n) \geqq r+2$ for the induced embedding f'_n: $\sum_{i=2}^{q} Re_i \to F_{n-1}$. (Actually, $r+2$ is the maximal value for $\operatorname{grade} \mathrm{I}_{q-1}(f'_n)$.) Let $N := \operatorname{Coker} f'_n$. The linear form α associated to $b \in \operatorname{Im} \mu_k$ by Corollary 3 may be considered a linear form on N, since $\alpha(f'_n(e_i)) = 0$ for $i=2, \ldots, q$. According to Lebelt [5], $\bigwedge^r N$ is reflexive, whence for any orientation ν on N the map ν^r: $\bigwedge^r N \to N^*$ is an isomorphism by Proposition 1. We choose an orientation φ on F_{n-1}, and define ν by

$$\nu(x) := \varphi(\tilde{x} \wedge f_n(e_2) \wedge \cdots \wedge f_n(e_q)), \; x \in \bigwedge^{r+1} N, \; \tilde{x} \text{ a preimage of } x \text{ in } \bigwedge^{r+1} F_{n-1}.$$

It is easily checked that ν is well-defined and indeed an orientation on N. (It is exactly the orientation on N determined by Theorem 1 relative to φ and the choice of basis in F_n.) Let now $y \in \bigwedge^r N$ such that $\alpha = \nu^r(y)$. Then

$$b = \alpha(f_n(e_1)) = \nu^r(y)\,(f_n(e_1)) = \nu^{r+1}(y \wedge f_n(e_1)) = \varphi(\tilde{y} \wedge f_n(e_1) \wedge \cdots \wedge f_n(e_q))$$

is an element of $\mathrm{I}_r(f_n)$, since

$$\mathrm{I}_r(f_n) = \{\varphi(z \wedge f_n(e_1) \wedge \cdots \wedge f_n(e_q)) \colon z \in \bigwedge^r F_{n-1}\}.$$

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Universität Osnabrück, Abteilung Vechta, Driverstraße 22, D-2848 Vechta

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