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Polyhedral K_2

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Abstract. Using elementary graded automorphisms of polytopal algebras (essentially the coordinate rings of projective toric varieties) polyhedral versions of the group of elementary matrices and the Steinberg and Milnor groups are defined. They coincide with the usual K -theoretic groups in the special case when the polytope is a unit simplex and can be thought of as compact/polytopal substitutes for the tame automorphism groups of polynomial algebras. Relative to the classical case, many new aspects have to be taken into account. We describe these groups explicitly when the underlying polytope is 2-dimensional. Already this low-dimensional case provides interesting classes of groups.

1. Introduction

Polyhedral K -groups are associated to groups of graded automorphisms of algebras $R[P]$ associated with lattice polytopes P and their arrangements. These are essentially the homogeneous coordinate rings of projective toric varieties if the ring R of coefficients is an algebraically closed field. For the special case of a unit simplex P the algebra $R[P]$ is a polynomial ring and the group is just the general linear group. Thus polyhedral K -theory contains ordinary (algebraic) K -theory as a special case. However, for more general polytopes many new aspects have to be taken into account and new K -groups appear.

There are several sources of motivation for the “polyhedrization” of K -theory. They can be described as follows.

In the series of papers [BrG1]–[BrG4] we have investigated polytopal algebras $k[P]$ (k a field) as generalizations of k -vector spaces. Recall that for a lattice polytope $P \subset \mathbb{R}^n$ the associated polytopal algebra is the semigroup algebra $k[P] = k[S_P]$ of the (additive) sub-semigroup $S_P \subset \mathbb{Z}^{n+1}$ generated by $\{(x, 1) \mid x \in L_P\}$, where $L_P = P \cap \mathbb{Z}^n$. Equivalently, $k[P]$ is generated over k by the lattice points of P which are subject to the binomial relations reflecting the affine dependencies

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inside P . Polytopal algebras and their graded k -homomorphisms ($\deg(x) = 1$ for $x \in L_P$) define the *polytopal linear category* $\text{Pol}(k)$. The category $\text{Vect}(k)$ of finitely generated k -vector spaces is a full subcategory of $\text{Pol}(k)$: just consider the subclass of polytopal algebras associated with unimodular simplices (including the empty one).

In addition to the natural interest in the polytopal generalization of linear algebra the motivation also comes from applications to toric geometry – in [BrG1] we have derived strengthened versions (for projective toric varieties) of the well known results on automorphism groups by Demazure [D] and Cox [Cox]. In [BrG2] this description has been generalized to arrangements of projective toric varieties.

These results emphasize the analogy between the groups $\text{GL}_n(k)$ and the groups $\text{gr.aut}(k[P])$ of graded automorphisms of polytopal algebras $k[P]$. A crucial rôle in the description of $\text{gr.aut}(k[P])$ is played by *elementary automorphisms* of polytopal algebras which are just ordinary elementary matrices in the case the underlying polytope is a unimodular simplex. They made their first appearance (in the setting of toric varieties) in [D], the work that initiated the theory of toric varieties in the 1970s.

The objects studied in our previous papers include the *polytopal Picard group* of $\text{Pol}(k)$, automorphism groups of polytopal algebras and retracts of polytopal algebras. Retracts of free modules are projective modules. Therefore the study of algebra retracts can be considered as a non-linear variant of studying the group K_0 of a ring. The group K_1 compares automorphisms of free modules to the elementary ones, as does our theorem on the automorphisms in $\text{Pol}(k)$ (Theorem 2.2 below). Therefore the latter is a non-linear analogue of K_1 , and it is natural to push the analogy between $\text{Vect}(k)$ and $\text{Pol}(k)$ further into higher K -theory.

Algebraic K -theory (for rings) can to some extent be thought of as a theory of higher syzygies between elementary matrices. Accordingly, our goal is to develop the theory of higher syzygies between elementary automorphisms of polytopal algebras.

However, there is yet another source of interest in polyhedral K -theory. The group of automorphisms of polynomial algebras (over a field) has been a big challenge for researchers for several decades. The tame generation conjecture asserts that every automorphism is a composite of linear and triangular automorphisms. Only the case of two variables has been settled (Jung, van der Kulk, in the 40s and 50s [J], [K]). There was an attempt in the 70s (Bass, Connell and Wright [BaW], [Con1], [Con2], [ConW]) to develop a *non-linear* K -theory based on these groups as non-linear analogues of GL_n , and on retracts of polynomial rings as analogues of projective modules. But progress in this area seems to be blocked.

The relationship between polyhedral and non-linear polynomial K -theories looks as follows:

$$\begin{array}{ccc} \text{arbitrary automorphisms} & \xrightarrow{\text{restricted to}} & \text{graded automorphisms,} \\ \text{polynomial rings} & \xrightarrow{\text{generalized to}} & \text{polytopal rings} \longrightarrow \text{polyhedral algebras.} \end{array}$$

Moreover, one easily observes that the tame generation conjecture is equivalent to the generalization of Theorem 2.2 to not necessarily bounded lattice polytopes (actually the case of positive orthants $\Pi = \mathbb{R}_+^n$, $n \in \mathbb{N}$ would already suffice, see Proposition 2.3). The same relationship exists with the classical conjecture on retracts of polynomial algebras.

The polytopal linear groups interpolate between the linear and the ‘infinite dimensional’ groups (see Shafarevich [Sh]). Thus they provide a possibility for studying similarities of the ‘infinite dimensional’ objects and the linear ones within a finite dimensional framework.

The automorphism groups $\text{gr.aut}(k[P])$ and their subgroups of elementary automorphisms $\mathbb{E}_k(P)$ are linear groups in a natural way. But unlike $\text{GL}_n(k)$ they are essentially never reductive groups, as documented by the list in Theorem 10.2, which describes the corresponding *stable* groups $\mathbb{E}(R, P)$ when $\dim P = 2$ (and $k = R$). The upper triangular subgroups of the $\mathbb{E}(R, P)$ are contributions of the unipotent radicals of the corresponding *unstable* groups $\mathbb{E}_k(P_i)$, $i \in \mathbb{N}$. (For $\dim P > 2$ a complete classification as in Theorem 10.2 seems to be very difficult.) Further, the groups $\mathbb{E}_k(P)$ are very often non-perfect. Even more is true: it follows from Demazure’s work [D] that the groups $\text{gr.aut}(k[P])$ in the important special case of a smooth variety $\text{Proj}(K[P])$ are always semidirect products of unipotent groups and reductive groups with root system of type A_l . Therefore, the theory developed here is to some extent complementary to the theory of Chevalley groups and their universal central extensions. Milnor’s definition of K_2 was motivated by Steinberg’s fundamental work [Stb] who considered such groups over fields. This was further extended to groups over arbitrary commutative rings by Stein [St] (see also [KaSt]).

One has to reveal the hidden polytopal geometry behind the Steinberg relations when elementary matrices are generalized to elementary automorphisms. This leads us to introduce the class of *balanced* polytopes (Section 5). They allow various K -theoretic constructions (further confirmation of which is provided in [BrG5]).

One of the main features of the polyhedral theory (and a difference to Chevalley groups) is that one has to work with *stable* groups, rather than with the groups $\text{gr.aut}_R(R[P])$. This is explained by the absence of inductive homomorphisms between them (Remark 7.6). The recipe for the definition of stable groups is encoded in a purely polytopal construction called *doubling along a facet* (Section 4), which is minimal with respect to several natural properties (Remark 4.4(a)). The elementary automorphisms extend to the “doubled” polytopal algebras and the final outcome is a perfect group. In the special case of unit simplices one recovers the familiar sequence

$$E_2(R) \subset \cdots \subset E_n(R) \subset E_{n+1}(R) \subset \cdots, \quad * \mapsto \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}.$$

Using such *doubling spectra* for a commutative ring R and a lattice polytope P we define the stable group of elementary automorphisms $\mathbb{E}(R, P)$ (Section 7) and the stable Steinberg group $\text{St}(R, P)$ (Section 8). However, the situation for general polytopes is more complicated than for unit simplices – as remarked, $\mathbb{E}(R, P)$ is not the inductive limit of the corresponding unstable groups and one has to encode the inductive process into the underlying polytopes themselves, rather than into the

automorphism groups. Actually, the groups obtained depend bivariantly on R and P (more precisely, on the normal fan of P ; Section 9).

The main results of this paper are:

- (i) The existence of the universal central extension of polytopal groups

$$1 \rightarrow K_2(R, P) \rightarrow \mathbb{S}t(R, P) \rightarrow \mathbb{E}(R, P) \rightarrow 1$$

provided P is balanced (Theorem 8.4). Unlike in the situation of unit simplices, the fact that $K_2(R, P)$ coincides with the center of $\mathbb{S}t(R, P)$ is no longer trivial. It follows from the triviality of the center of $\mathbb{E}(R, P)$ (Theorem 7.7).

- (ii) The visualization of these groups in the first non-trivial case when P is a balanced polygon (i. e. $\dim P = 2$). We will see that there exist essentially 6 different types (Section 10).

Most of the results below generalize to the bigger class of *polyhedral algebras* [BrG3], which are composed of polytopal algebras in the same way as Stanley-Reisner rings are composed of polynomial algebras. However, we do not pursue this level of generality; we only use the attribute “polyhedral” to indicate the possibility of generalization. The results of this paper are essential for our study [BrG5] of higher polyhedral K -groups.

2. Column vectors and elementary automorphisms

In this section we introduce the notion of elementary automorphisms which is crucial for the sequel. To this end we recall the relevant part of [BrG1]. The only difference to [BrG1] is that we will consider arbitrary commutative rings instead of fields of coefficients. We do not include any arguments because they are completely parallel to those for fields. Throughout the paper R denotes an arbitrary commutative ring.

A polytope in \mathbb{R}^n is always assumed to be finite and convex. Further, we will only consider lattice polytopes, i. e. those with vertices in \mathbb{Z}^n . The objects to be defined below depend only on the affine structure of the set $L_P = P \cap \mathbb{Z}^n$ of lattice points of P . Therefore we can always assume that \mathbb{R}^n is the smallest affine space containing P and that \mathbb{Z}^n is the smallest affine sublattice of \mathbb{R}^n containing L_P . If necessary, we simply replace \mathbb{R}^n by the affine hull $\text{Aff}(P)$ of P and \mathbb{Z}^n by $z_0 + \sum_{z \in L_P} \mathbb{Z}(z - z_0)$ for some $z_0 \in L_P$.

Under this assumption let F be a facet of P and choose a point $z_F \in F$. Then the subgroup

$$F_{\mathbb{Z}} := (-z_F + \text{Aff}(F)) \cap \mathbb{Z}^n \subset \mathbb{Z}^n$$

is isomorphic to \mathbb{Z}^{n-1} . Moreover, there is a unique group homomorphism $\langle F, - \rangle : \mathbb{Z}^n \rightarrow \mathbb{Z}$, written as $x \mapsto \langle F, x \rangle$, such that $\text{Ker}(\langle F, - \rangle) = F_{\mathbb{Z}}$, $\text{Coker}(\langle F, - \rangle) = 0$, and $\langle F, - \rangle$ attains its minimum b_F on the set L_P at the lattice points of F .

The homomorphism $\langle F, - \rangle$ can be naturally extended to \mathbb{R}^n . On $\text{Aff}(F)$ it has the constant value $b_F \in \mathbb{Z}$. With this notation we obtain the representation

$$P = \{x \in \mathbb{R}^n : \langle F, x \rangle \geq b_F \text{ for all facets } F\},$$

of P as an intersection of closed halfspaces.

We can use $\langle F, - \rangle$ to measure the *height* of a lattice point x above the facet F by setting

$$\text{ht}_F(x) = \langle F, x \rangle - b_F.$$

The function ht_F counts the number of hyperplanes between F and x (in the direction of P) that are parallel to, but different from $\text{Aff}(F)$ and pass through lattice points.

The polytopal algebra $R[P]$ is by definition the R -algebra generated by the lattice points of P which are subject to the binomial relations reflecting the affine dependencies inside P . Equivalently, $R[P]$ is the semigroup ring $R[S_P]$ of the additive subsemigroup $S_P \subset \mathbb{Z}^{n+1}$ generated by

$$\{(x, 1) \mid x \in L_P\} \subset \mathbb{Z}^{n+1}.$$

For a polytope $P \subset \mathbb{R}^n$ the group $\Gamma_R(P) = \text{gr.aut}_R(R[P])$ coincides with $\text{GL}_n(R)$ in the case of the unit $(n - 1)$ -simplex

$$P = \Delta_{n-1} = \text{conv}((0, 0, \dots, 0), (1, 0, \dots, 0), \dots, (0, 0, \dots, 1)) \subset \mathbb{R}^{n-1}.$$

The groups $\Gamma_R(P)$ are called *polytopal linear groups* – they consist of the R -points of the corresponding group schemes (which are defined over \mathbb{Z} , see [BrG4, §2]).

An element $v \in \mathbb{Z}^n$, $v \neq 0$, is a *column vector* (for P) if there is a facet $F \subset P$ such that $x + v \in P$ for every lattice point $x \in P \setminus F$. The facet F is called *the base facet* of v . While the points $x \in P \cap \mathbb{Z}^n$ are identified with $(x, 1) \in \mathbb{Z}^{n+1}$, the vector v is to be identified with $(v, 0) \in \mathbb{Z}^{n+1}$.

The set of column vectors of P is denoted by $\text{Col}(P)$. A pair (P, v) , $v \in \text{Col}(P)$, is called a *column structure* (see Figure 1).

The function ht_F , defined above on $\mathbb{Z}^n \approx \mathbb{Z}^n \times \{1\}$ for each facet F of P , has a unique extension to a linear form on \mathbb{R}^{n+1} . The extension is again denoted by ht_F . For ht_{P_v} we simply write ht_v . It is an easy observation that $x + \text{ht}_v(x) \cdot v \in S_{P_v} \subset S_P$ for any $x \in S_P$.

Column vectors are the dual objects to the roots of the *normal fan* $\mathcal{N}(P)$ in the sense of Demazure (see [D], [Cox]; for normal fans see Section 9).

Let (P, v) be a column structure and $\lambda \in R$. Then the assignment

$$x \mapsto (1 + \lambda v)^{\text{ht}_v(x)} x.$$

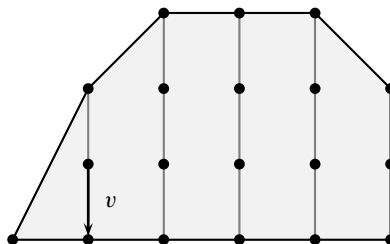


Fig. 1. A column structure

gives rise to a graded R -algebra automorphism e_v^λ of $R[P]$. (At this point we need the commutativity of R .) Its inverse is $e_v^{-\lambda}$. Observe that e_v^λ becomes an elementary matrix in the special case when $P = \Delta_{n-1}$, after the identifications $R[\Delta_{n-1}] = R[X_1, \dots, X_n]$ and $\Gamma_R(P) = \text{GL}_n(R)$. Accordingly e_v^λ is called an *elementary automorphism*.

The following alternative description of elementary automorphisms shows that they are always restrictions of elementary matrices if we interpret the latter as automorphisms of polynomial algebras.

We may assume $\dim P = n$ and $\text{gp}(S_P) = \mathbb{Z}^{n+1}$. By a suitable integral unimodular change of coordinates we may further assume that $v = (0, -1, 0, \dots, 0)$ and that P_v lies in the subspace \mathbb{R}^{n-1} (thus P is in the upper halfspace). Consider the standard unimodular simplex Δ_n (i. e. the one with vertices at the origin and the standard coordinate unit vectors). Clearly, P is contained in a parallel integral shift of $c\Delta_n$ for a sufficiently large natural number c . Then we have a graded R -embedding $R[P] \rightarrow R[c\Delta_n]$, the latter ring being just the c th Veronese subring of the polynomial ring $R[\Delta_n] = R[X_0, X_1, \dots, X_n]$. Moreover, $v = X_0/X_1$. Now the automorphism of $R[X_0, X_1, \dots, X_n]$ mapping X_1 to $X_1 + \lambda X_0$ and leaving all the other variables fixed induces an automorphism of $R[c\Delta_n]$ which restricts to an automorphism of $R[P]$. It is nothing but the elementary automorphism e_v^λ above.

Proposition 2.1. *Let v_1, \dots, v_s be pairwise different column vectors for P with the same base facet $F = P_{v_i}, i = 1, \dots, s$. The mapping*

$$\varphi : R^s \rightarrow \Gamma_R(P), \quad (\lambda_1, \dots, \lambda_s) \mapsto e_{v_1}^{\lambda_1} \circ \dots \circ e_{v_s}^{\lambda_s},$$

is an injective group homomorphism from the additive group R^s . In particular, $e_{v_i}^{\lambda_i}$ and $e_{v_j}^{\lambda_j}$ commute for all $i, j \in \{1, \dots, s\}$ and $(e_{v_1}^{\lambda_1} \circ \dots \circ e_{v_s}^{\lambda_s})^{-1} = e_{v_1}^{-\lambda_1} \circ \dots \circ e_{v_s}^{-\lambda_s}$.

In what follows $\mathbb{E}_R(P)$ denotes the subgroup of $\Gamma_R(P)$ generated by the elementary automorphisms. In particular, $\mathbb{E}_R(\Delta_{n-1}) = E_n(R)$ – the n th unstable group of elementary matrices.

The initial motivation for developing the theory of higher syzygies between the elementary automorphisms is the main result of [BrG1] (Theorem 2.2 below), which establishes a complete polytopal generalization of the standard linear algebra fact that any invertible matrix over a field can be reduced to a diagonal matrix using elementary transformations on columns (or rows). Moreover, we have normal forms for such reductions since the elementary transformations can be carried out in an increasing order of the column indices.

Let k be a field. Put $d = \dim P + 1$. The d -torus $\mathbb{T}_d = (k^*)^d$ acts naturally on $k[\text{gp}(S_P)]$ via the substitution

$$(\xi_1, \dots, \xi_d)(e_i) = \xi_i e_i, \quad \xi_i \in k^*, \quad i \in [1, d].$$

Here e_i is the i -th element of a fixed basis of $\text{gp}(S_P) = \mathbb{Z}^d$. Since the action restricts to $k[P]$, one has an algebraic embedding $\mathbb{T}_d \subset \Gamma_k(P)$, whose image we denote by $\mathbb{T}_k(P)$. It consists precisely of those automorphisms of $k[P]$ which multiply each monomial by a scalar from k^* .

The (finite) automorphism group $\Sigma(P)$ of the semigroup S_P is also a subgroup of $\Gamma_k(P)$. It is exactly the group of automorphisms of P as a lattice polytope.

The embedding φ , given by Lemma 2.1, is an embedding of algebraic groups over k . Denote by $\mathbb{A}(F)$ the image of φ . It is an affine space over k . Of course, $\mathbb{A}(F)$ may consist only of the identity map of $k[P]$ – namely, if there is no column vector with base facet F .

Theorem 2.2. *Let P be a n -polytope and k a field. Every element $\gamma \in \Gamma_k(P)$ has a (not uniquely determined) presentation*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r \circ \tau \circ \sigma,$$

where $\sigma \in \Sigma(P)$, $\tau \in \mathbb{T}_k(P)$, and $\alpha_i \in \mathbb{A}(F_i)$ such that the facets F_i are pairwise different and $\#L_{F_i} \leq \#L_{F_{i+1}}$, $i \in [1, r - 1]$.

We have $\dim \Gamma_k(P) = \#\text{Col}(P) + d$ (the left hand side is the Krull dimension of the affine group scheme $\Gamma_k(P)$), and $\mathbb{T}_k(P)$ is a maximal torus in $\Gamma_k(P)$, provided k is infinite.

The convex hull in \mathbb{R}^n of a subset $X \subset \mathbb{Z}^n$ will be called a *lattice polyhedron* if it is an intersection of a *finite* family of rational affine halfspaces in \mathbb{R}^n . (‘Rational affine’ means that the hyperplane bounding the halfspace is given by a rational linear equation.) For polyhedra (in contrast to polytopes) we do no longer require compactness.

Let $P \subset \mathbb{R}^n$ be a convex lattice polyhedron. As in the case of polytopes, there exists a height function ht_F for each facet F of P . It is the unique surjective homomorphism $\mathbb{Z}^n \rightarrow \mathbb{Z}$ that maps L_F to 0 and L_P to \mathbb{Z}_+ . Thus one can define the notion of column vector for P : an element $v \in \mathbb{Z}^n \setminus \{0\}$ is a column vector if there exists a facet $F \subset P$ such that (i) $x + v \in P$ whenever $x \in L_P \setminus F$ and (ii) $\text{ht}_F(v) = -1$. Observe that condition (i) implies (ii) if P is a (bounded) polytope, but this is not true for (unbounded) polyhedra in general.

One can define the ‘non-compact’ analogues of polytopal algebras and their elementary automorphisms, using this notion of column vectors. Clearly, Proposition 2.1 generalizes completely to the general polyhedral setting.

The positive orthant and the tameness conjecture. Consider the positive orthant $\Pi = \mathbb{R}_+^n$. One observes that $\Gamma_k(\Pi)$ contains a copy of the torus $(k^*)^{n+1}$, just as in the case of finite polytopes.

We want to relate $k[\Pi]$ to the polynomial ring $k[X_1, \dots, X_n]$. To this end we set $\mathbf{e}_0 = (0, 1) \in S_\Pi$ and $\mathbf{e}_i = (e_i, 1) \in S_\Pi$, $i \in [1, n]$. Then the assignment $X_i \mapsto \mathbf{e}_i$ embeds $k[X_1, \dots, X_n]$ into $k[\Pi]$. (This is just the embedding coming from the identification of Δ_{n-1} with the simplex spanned by e_1, \dots, e_n .) We identify the polynomial ring with its image. Then clearly $k[\Pi][\mathbf{e}_0^{-1}] \approx k[X_1, \dots, X_n][[\mathbf{e}_0^{\pm 1}]]$, and every automorphism of the polynomial ring extends to one of $k[\Pi]$ that leaves \mathbf{e}_0 invariant.

Conversely, one has a natural isomorphism $k[\Pi]/(\mathbf{e}_0 - 1) \approx k[X_1, \dots, X_n]$ given by dehomogenization, making $k[X_1, \dots, X_n]$ a retract of $k[\Pi]$. Therefore every automorphism of $k[\Pi]$ that leaves \mathbf{e}_0 fixed induces an automorphism of $k[X_1, \dots, X_n]$.

The analogue of Theorem 2.2 for the infinite polyhedron Π is equivalent to the tameness conjecture for the group of k -automorphisms of $k[X_1, \dots, X_n]$, as the next proposition shows.

Recall that an automorphism $\alpha : k[X_1, \dots, X_n] \rightarrow k[X_1, \dots, X_n]$ is called tame if it is a composite of affine (i. e. linear + constant) automorphisms and triangular automorphisms, i. e. those of type

$$X_1 \mapsto X_1, \quad X_i \mapsto X_i + \varphi_i, \quad \varphi_i \in k[X_1, \dots, X_{i-1}], \quad i \in [2, n].$$

Furthermore let $\mathbb{T}_k(\Pi)'$ denote the subgroup of toric automorphisms that leaves \mathbf{e}_0 invariant.

Proposition 2.3. *For a field k the subgroup of $\Gamma_k(\Pi)$ generated by $\mathbb{E}_k(\Pi)$ and $\mathbb{T}_k(\Pi)'$ $\approx (k^*)^n$ is naturally isomorphic to the group of tame automorphisms of the polynomial ring $k[X_1, \dots, X_n]$. Moreover, under this isomorphism $\mathbb{E}_k(\Pi)$ corresponds to the group of tame automorphisms of Jacobian 1.*

The proposition is easily proved (we leave the details to the reader).

Notice that Proposition 2.3 treats the augmented and non-augmented automorphisms in a uniform way. Moreover, there is no need to consider the affine and triangular automorphisms separately (as it is usually done in the literature).

3. The partial product operation on column vectors

For a polytope P we denote by $\mathbb{F}(P)$ the set of all facets of P . The following lemma is easily derived from the definition of column vector and the description of P as the intersection of closed halfspaces.

Lemma 3.1. *A vector $v \in \mathbb{Z}^n$ belongs to $\text{Col}(P)$ if and only if there exists $F \in \mathbb{F}(P)$ such that*

$$\langle F, v \rangle = -1 \quad \text{and} \quad \langle G, v \rangle \geq 0 \quad \text{for all } G \in \mathbb{F}(P), \quad G \neq F.$$

Clearly, this facet F is just the base facet P_v .

Now we introduce the notion of a partial product operation on column vectors and investigate its basic properties. Fix a polytope P and let (P, u) and (P, v) be column structures on it.

Definition 3.2. *We say that the product uv exists if $u + v \neq 0$ and $x + u \notin P_v$ for every point $x \in L_P \setminus P_u$. If uv exists, we put $uv = u + v$.*

Figure 2 shows a polytope with all its column vectors and the two existing products $w = uv$ and $u = w(-v)$.

Proposition 3.3.

(a) *The following conditions are equivalent:*

1. uv exists;
2. $u + v \in \text{Col}(P)$ and $P_{u+v} = P_u$;

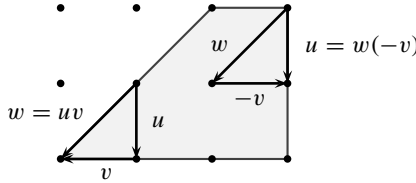


Fig. 2. The product of two column vectors

3. $u + v \neq 0$ and $\langle P_v, u \rangle > 0$.

In particular, $P_{uv} = P_u$ and v is parallel to P_u (i. e. $\langle P_u, v \rangle = 0$).

(b) $u + v \in \text{Col}(P)$ if and only if exactly one of the products uv and vu exists.

(c) Suppose $u, v \in \text{Col}(P)$, $u \neq -v$, and $\langle P_v, u \rangle > 0$. Then $u + iv \in \text{Col}(P)$ for all $i \in [1, \langle P_v, u \rangle]$ and $u + iv \notin \text{Col}(P)$ for all $i > \langle P_v, u \rangle$.

(d) The following are equivalent for $v \in \text{Col}(P)$:

1. $-v \in \text{Col}(P)$;
2. there exist facets F, G of P such that $\langle F, v \rangle = -1$, $\langle G, v \rangle = 1$, and $\langle H, v \rangle = 0$ for every facet $H \neq F, G$;
3. there exists $w \in \text{Col}(P)$ and facets F, G such that $\langle F, v \rangle = -1$, $\langle G, v \rangle = 1$, $\langle F, w \rangle = 1$, $\langle G, w \rangle = -1$.

(e) If $w = uv$ and $-w \in \text{Col}(P)$, then $-u, -v \in \text{Col}(P)$ as well.

Proof. (a) The equivalence of (1) and (2) as well as the implication (3) \implies (2) are straightforward. Now assume uv exists and $\langle P_v, u \rangle \leq 0$. If $\langle P_v, u \rangle < 0$, then $P_u = P_v$ according to Lemma 3.1 and therefore $\langle F_u, uv \rangle = -2$, which is impossible (again by Lemma 3.1). If $\langle P_v, u \rangle = 0$ then $P_u \neq P_v$ and, hence, there is an element $x \in L_{P_v} \setminus P_u$. But then the point $x + u + v$ is outside P – a contradiction.

(b) Assume $u + v \in \text{Col}(P)$. According to (a) we have to show that exactly one of the inequalities $\langle P_v, u \rangle > 0$ and $\langle P_u, v \rangle > 0$ holds. The opposite (strict) inequalities are excluded by the same reasons as in the proof of (a).

Next we exclude that simultaneously $\langle P_v, u \rangle = 0$ and $\langle P_u, v \rangle = 0$. Assume that $\langle P_v, u \rangle = 0$ and $\langle P_u, v \rangle = 0$. In particular $P_u \neq P_v$ and, therefore, one of these base facets differs from P_{u+v} . If $P_u \neq P_{u+v}$, then there is a point $x \in L_{P_u} \setminus P_{u+v}$ and by the equality $\langle P_u, v \rangle = 0$ we get a contradiction because $\text{ht}_u(x + u + v) = -1$, that is, $x + u + v$ is outside P .

It only remains to show that $\langle P_v, u \rangle > 0$ and $\langle P_u, v \rangle > 0$ can not hold simultaneously. But if this were the case, then for every lattice point $x \in L_P \setminus P_u$ we would have

$$x, x + u, x + u + v, x + u + v + u, x + u + v + u + v, \dots \in L_P.$$

This is a contradiction, because $x + c(u + v) \notin P$ for $c \gg 0$.

(c) The proof is straightforward.

(d) Suppose $-v \in \text{Col}(P)$. Then condition (2) is satisfied for $F = P_v, G = P_{-v}$: note that $\langle H, v - v \rangle = 0$ and $\langle H, v \rangle, \langle H, -v \rangle \geq 0$ for $H \neq F, G$. Similarly (3) holds with $w = -v$.

If condition (2) holds, then $-v$ is clearly a column vector with base facet G : $x + (-v) \in P$ for all $x \in L_P \setminus G$.

If (3) holds, then $\langle H, v + w \rangle \geq 0$ for all facets H . The only vector satisfying this condition is 0.

(e) Let $F = P_w = P_u, G = P_{-w}$. One has $\langle F, v \rangle = 0$, and there exists a facet E with $\langle E, v \rangle > 0$. If $E \neq G$, then $\langle E, w \rangle = \langle E, u + v \rangle > 0$, and this is impossible for $E \neq G$. Thus $E = G$, and since $\langle G, w \rangle = 1$, it follows that $\langle G, u \rangle = 0, \langle G, v \rangle = 1$.

Now consider the facet P_v . As seen already, it is different from F and G . So $\langle P_v, w \rangle = 0$ forces $\langle P_v, u \rangle = 1$. Moreover, for all facets $H \neq F, G, P_v$ we must have $\langle H, u \rangle = \langle H, v \rangle = 0$. In view of (d.2) both u and v are invertible column vectors. \square

Corollary 3.4. *Let $u, v, w \in \text{Col}(P)$.*

(a) *If both uv and vw exist and $u + v + w \neq 0$ then the products $(uv)w$ and $u(vw)$ also exist and, clearly, $(uv)w = u(vw)$.*

(b) *If vw and $u(vw)$ exist and $u + v \neq 0$ then uv also exists, while the existence of uv and $(uv)w$ does in general not imply the existence of vw , even if $v + w \neq 0$*

Proof. (a) That $(uv)w$ exists follows from the definition of the product and the equality $P_{uv} = P_u$. Thus, by Proposition 3.3(a) we only need to show that $\langle P_{vw}, u \rangle > 0$. But the same proposition implies $\langle P_{vw}, u \rangle = \langle P_v, u \rangle > 0$.

(b) That the existence of vw and $u(vw)$ implies that of uv follows from $\langle P_v, u \rangle = \langle P_{vw}, u \rangle > 0$.

The example (see Figure 3)

$$P = \text{conv}\{(0, 0, 0), (0, 0, 1), (1, 0, 0), (0, 1, 0), (1, 1, 0)\} \subset \mathbb{R}^3,$$

$$u = (0, 0, -1), v = (1, 0, 0), w = (0, 1, 0) \in \text{Col}(P),$$

completes the proof because uv and $(uv)w$ exist and vw does not exist. \square

4. Doubling along a facet

We now introduce an operator on the set of lattice polytopes, called *doubling along a facet*.

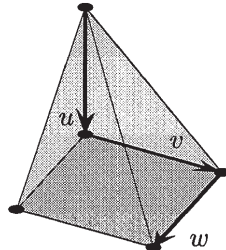


Fig. 3. The pyramid over the unit square

Let $P \subset \mathbb{R}^n$ be a polytope and $F \subset P$ be a facet. For simplicity we assume that $0 \in F$, a condition that can be satisfied by a parallel translation of P . Denote by $H \subset \mathbb{R}^{n+1}$ the n -dimensional linear subspace that contains F and whose normal vector is perpendicular to that of $\mathbb{R}^n = \mathbb{R}^n \oplus 0 \subset \mathbb{R}^{n+1}$ (with respect to the standard scalar product on \mathbb{R}^{n+1}). Then the upper half space $H \cap (\mathbb{R}^n \times \mathbb{R}_+)$ contains a congruent copy of P which differs from P by a 90° rotation. Denote the copy by P^{\perp_F} , or just by P^\perp if there is no danger of confusion.

Note that P^\perp is not always a lattice polytope with respect to the standard lattice \mathbb{Z}^{n+1} . However, it is so with respect to the sublattice $(\mathbb{Z}^n)^{\perp_F}$ which is the image of \mathbb{Z}^n under the 90° rotation.

The operator of doubling along a facet is then defined by

$$P^{\perp_F} = \text{conv}(P, P^\perp) \subset \mathbb{R}^{n+1}.$$

For typographical reasons we will sometimes write $\perp(P, F)$ instead of P^{\perp_F} .

The doubled polytope is a lattice polytope with respect to the subgroup $(\mathbb{Z}^n)^{\perp_F} = \mathbb{Z}^n + (\mathbb{Z}^n)^{\perp_F}$. After a change of basis in \mathbb{R}^{n+1} that does not affect \mathbb{R}^n we can replace $(\mathbb{Z}^n)^{\perp_F}$ by \mathbb{Z}^{n+1} , and consider P^{\perp_F} as an ordinary lattice polytope in \mathbb{R}^{n+1} . In what follows, whenever we double a lattice polytope $P \subset \mathbb{R}^n$ along a facet F , the lattice of reference in \mathbb{R}^{n+1} is always $\mathbb{Z}^n + (\mathbb{Z}^n)^{\perp_F}$. For simplicity of notation this lattice will be denoted by \mathbb{Z}^{n+1} . (We are grateful to the referee for pointing out an incorrectness in the previous version of this article.)

The construction is illustrated by Figure 4.

Sometimes P will be referred to by P^{-F} , or just by P^- .

In the special case when $F = P_v$ for some $v \in \text{Col}(P)$ we will use the self explanatory notations P^{\perp_v} , $\perp(P, v)$, P^{-v} , P^{\perp_v} , v^- , v^\perp , $\text{Col}(P)^-$ and $\text{Col}(P)^\perp$.

It is clear from the definition that $\dim P^{\perp_F} = \dim P + 1$ and that the correspondence

$$\Psi_P : \mathbb{F}(P) \setminus \{F\} \rightarrow \mathbb{F}(P^{\perp_F}), \quad \Psi(G) = \text{conv}(G^-, G^\perp),$$

extends to a bijection

$$(\mathbb{F}(P) \setminus \{F\}) \cup \{P^-, F\} \approx \mathbb{F}(P^{\perp_F}),$$

for which $P^- \mapsto P^-$ and $F \mapsto P^\perp$. (Here G^\perp denotes the facet of P^\perp that corresponds to $G = G^-$.) This bijection will again be denoted by Ψ_P .

The polytope P^\perp has two distinguished column vectors δ_P^+ and δ_P^- , which are the lattice unit vectors in \mathbb{Z}^{n+1} parallel to the lines connecting the points $x^- \in$

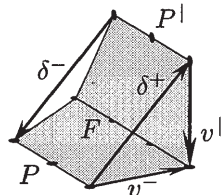


Fig. 4. Doubling along the facet F

$L_{P^-} \setminus F$ with the corresponding points $x^\perp \in L_{P^\perp}$. We choose δ_P^+ in such a way that $(P^\perp)_{\delta_P^+} = P^\perp$ and $(P^\perp)_{\delta_P^-} = P^-$. Clearly $\delta_P^- = -\delta_P^+$. For formal correctness we should write $\delta_{F,P}^+$ and $\delta_{F,P}^-$ or $\delta^+(F, P)$ and $\delta^-(F, P)$, (as we will do in Sections 7 and 8). But in most of the cases it will be clear from the context which facet has been used for the doubling, and we will simply write δ_P^+ and δ_P^- , or just δ^+ and δ^- .

In almost all cases doublings will be carried out along base facets of column vectors v . Then we simplify the notation $\delta_{P_v,P}^+$ to $\delta_{v,P}^+$ etc.

The following equations are easily observed:

$$\langle \Psi(G), \delta^+ \rangle = \langle \Psi(G), \delta^- \rangle = 0 \quad \text{for all } G \in \mathbb{F}(P) \setminus \{F\}, \tag{4_1}$$

$$\langle P^-, \delta^+ \rangle = \langle P^\perp, \delta^- \rangle = 1, \tag{4_2}$$

$$\langle G, z \rangle = \langle \Psi(G), z \rangle \quad \text{for all } z \in \mathbb{Z}^n, G \in \mathbb{F}(P). \tag{4_3}$$

(In (4₃) the pairings are considered for P and P^\perp respectively and \mathbb{Z}^n is thought of as the subgroup $\mathbb{Z}^n \oplus 0 \subset \mathbb{Z}^{n+1}$.)

In view of Lemma 3.1 the equations (4₃) imply

Lemma 4.1. *Let $F \subset P$ be a facet and $v \in \text{Col}(P)$ be a column vector. Then $v \in \text{Col}(P^{\perp F})$.*

Of crucial importance for what follows is

Corollary 4.2. *For any $v \in \text{Col}(P)$ the following equations hold in $\text{Col}(P^{\perp v})$:*

$$v = v^- = \delta^+ v^\perp, \quad v^\perp = \delta^- v^-.$$

Let $\{F_1, \dots, F_m\} \subset \mathbb{F}(P)$ be a system of facets. Then we get the system of polytopes (P_0, P_1, \dots, P_m) and facets G_{ij} defined recursively as follows

$$\begin{aligned} P_0 &= P, & G_{0j} &= F_j, \quad j \in [1, m] \\ P_i &= P_{i-1}^{\perp G_{i-1,i}}, & G_{ij} &= \Psi_{G_{i-1,i}}(G_{i-1,j}), \quad j \in [1, m], \quad i \in [2, m]. \end{aligned}$$

We will use the notation

$$P_m = P^{\perp(F_1, \dots, F_m)}.$$

In other words, $P^{\perp(F_1, \dots, F_m)}$ is the polytope we get after m successive doublings, starting from P , along the facets corresponding to F_1, \dots, F_m .

For any permutation $\sigma \in \Sigma_m$ we can form the polytope

$$P^{\perp(F_{\sigma(1)}, \dots, F_{\sigma(m)})}.$$

The point is that the resulting polytope is independent of σ . This will be important in our definition of polyhedral K -groups:

Proposition 4.3. *The polytopes $P^{\perp(F_1, \dots, F_m)}$ and $P^{\perp(F_{\sigma(1)}, \dots, F_{\sigma(m)})}$ are naturally isomorphic as lattice polytopes.*

Proof. We define the mapping

$$\Theta : P^{-\perp(F_1, \dots, F_m)} \rightarrow P^{-\perp(F_{\sigma(1)}, \dots, F_{\sigma(m)})}$$

as follows. For every lattice point $x \in P^{-\perp(F_1, \dots, F_m)}$ consider the recursively given sequence of nonnegative integers:

$$h_m = \langle P_{m-1}, x \rangle, h_{m-1} = \langle P_{m-2}, x + h_m \delta_{P_m}^- \rangle, \dots, \\ h_1 = \langle P_0, x + h_m \delta_{P_m}^- + \dots + h_3 \delta_{P_3}^- + h_2 \delta_{P_2}^- \rangle.$$

We also have the lattice point

$$y = x + \sum_{i=1}^m h_i \delta_{P_i}^- \in P_0 = P.$$

To the data (h_m, \dots, h_1) and y we now associate the lattice point

$$\Theta(x) = y + \sum_{i=1}^m h_{\sigma(i)} \delta_{P_i}^+ \in P_m^*,$$

where the sequence $(P_0^*, P_1^*, \dots, P_m^*)$ is related to $P^{-\perp(F_{\sigma(1)}, \dots, F_{\sigma(m)})}$ in the same way as (P_0, P_1, \dots, P_m) to $P^{-\perp(F_1, \dots, F_m)}$. It is straightforward to see that we do not get outside the source polytope and that the mapping Θ is an affine isomorphism respecting the lattice structures. \square

Remark 4.4. (a) Informally, the polytope $P^{-\perp v}$, $v \in \text{Col}(P)$ is the universal solution (i. e. minimal and applicable to all polytopes) to the problem posed by the following 3 conditions:

- (1) $P \subset P^{-\perp v}$ is a facet,
- (2) $\text{Col}(P) \subset \text{Col}(P^{-\perp v})$,
- (3) v is decomposed into a product of two column vectors in $P^{-\perp v}$.

These three properties are crucial for defining the stable group of elementary automorphisms $\mathbb{E}(R, P)$ which will turn out to be perfect (Section 7).

(b) In the special case of an algebraically closed field $R = k$ the ring $k[P^{-\perp F}]$ admits an algebro-geometric characterization (for simplicity we assume P is normal, i. e. S_P is a normal semigroup): $\text{Spec}(k[P^{-\perp F}])$ is the normalization of X in the pull-back diagram of toric varieties

$$\begin{array}{ccc} X & \longrightarrow & \text{Spec}(k[P]) \\ \downarrow & & \downarrow \\ \text{Spec}(k[P]) & \longrightarrow & \text{Spec}(k[F]), \end{array}$$

where $\text{Spec}(k[P]) \rightarrow \text{Spec}(k[F])$ is induced by the identity embedding $k[F] \rightarrow k[P]$. It is a split embedding whose left inverse is the k -homomorphism

$$\pi : k[P] \rightarrow k[F], \quad \pi(x) = 0 \text{ if } x \in L_P \setminus F \text{ and } \pi(x) = x \text{ if } x \in F.$$

The resulting closed embedding $\text{Spec}(k[F]) \subset \text{Spec}(k[P])$ is an equivariant divisor with respect to the big torus $(k^*)^{\dim P+1} \subset k[P]$. Conversely, any equivariant divisor is always associated to a facet of P .

5. Balanced polytopes

A polytope P is called *balanced* if $\langle F_u, v \rangle \leq 1$ for all $u, v \in \text{Col}(P)$. By Lemma 3.1 P is balanced if and only if $|\langle F_u, v \rangle| \leq 1$ for all $u, v \in \text{Col}(P)$.

The next lemma shows that column vectors behave well with doublings along facets.

Lemma 5.1. *Let P be a balanced polytope and F be one of its facets. Then*

$$\text{Col}(P^{\perp F}) = \text{Col}(P)^{-} \cup \text{Col}(P)^{\perp} \cup \{\delta^+, \delta^-\}.$$

Proof. Pick $v \in \text{Col}(P^{\perp F})$. First notice that it is impossible to have simultaneously $\langle P^{\perp}, v \rangle < 0$, $\langle P^-, v \rangle > 0$ and $v \neq \delta^+$. In fact, if these inequalities held, then the product δ^-v existed because P^{\perp} would be the base facet for v , and Proposition 3.3(a) applies. But similar arguments show that the product $v\delta^-$ existed as well – a contradiction with Proposition 3.3(b).

By symmetry the inequalities $\langle P^{\perp}, v \rangle > 0$, $\langle P^-, v \rangle < 0$ for $v \neq \delta^-$ are also impossible. It only remains to exclude the case in which $\langle P^{\perp}, v \rangle > 0$ and $\langle P^-, v \rangle > 0$. Then we can conclude that v is parallel to either P^{\perp} or P^- , as claimed.

But if $\langle P^{\perp}, v \rangle > 0$ and $\langle P^-, v \rangle > 0$ then the product $v\delta^-$ exists and, simultaneously,

$$\langle P^{\perp}, v\delta^- \rangle = \langle P^{\perp}, v \rangle + \langle P^{\perp}, \delta^- \rangle \geq 2$$

which is impossible because P is balanced. \square

Remark 5.2. (a) Clearly, the union in Lemma 5.1 may not be disjoint as

$$\text{Col}(P)^{-} \cap \text{Col}(P)^{\perp} = \{v \in \text{Col}(P) \mid \langle F, v \rangle = 0\}.$$

(b) The essential data which determine the partial product structure of column vectors are the heights $\langle F, v \rangle$ where F runs through the facets and v through the column vectors of P . Therefore it is useful to clarify how these data change under doubling. Using the equations (4₁), (4₂) and (4₃), and the lemma above we can describe the product structure on $\text{Col}(P^{\perp F})$: for $u, v \in \text{Col}(P)$ one has

$$w = uv \iff w^- = u^-v^- \iff w^{\perp} = u^{\perp}v^{\perp},$$

$$F = P_v \implies v^- = \delta^+v^{\perp} \text{ and } v^{\perp} = \delta^-v^-,$$

$$-v \in \text{Col}(P) \text{ and } F = P_v \implies v^-(-v)^{\perp} = \delta^+ \text{ and } v^{\perp}(-v^-) = \delta^-,$$

and there exist no other products in $\text{Col}(P^{\perp F})$.

(c) If P is not balanced then $P^{\perp F}$ can have essential new column vectors, even in the special case when $F = P_v$ for some $v \in \text{Col}(P)$. For instance, consider the triangle $\text{conv}((-2, 0), (0, 0), (0, 1))$, which is not balanced, and its column vector $v = (1, 0)$. Then the column vector $w = (-1, -1, 1) \in \text{Col}(P^{\perp v})$ is not of the type mentioned in Lemma 5.1; see Figure 5.

Corollary 5.3. *If P is a balanced polytope and $F \subset P$ a facet, then $P^{\perp F}$ is balanced as well.*

This follows immediately from the equations (4₁) and (4₂) and Lemma 5.1.

Corollary 5.3 shows that balanced polytopes exist in abundance. One just starts from a single such polytope and doubles it successively along arbitrary facets. In §10 we will see that there are infinite families of balanced polytopes even in the plane.

6. Commutators of elementary automorphisms

The main result of this section is the following

Theorem 6.1. *Let R be a ring, P be a polytope, $\lambda, \mu \in R$ and $u, v \in \text{Col}(P)$. Assume $u + v \neq 0$. Then for the commutator of the automorphisms $e_u^\lambda, e_v^\mu \in \mathbb{E}_R(P)$ one has*

$$[e_u^\lambda, e_v^\mu] = \begin{cases} \prod_{i=1}^n e_{u+iv}^{-\binom{n}{i}\lambda\mu^i} & \text{if } uv \text{ exists, } n = \langle P_v, u \rangle, \\ \prod_{i=1}^n e_{v+iu}^{\binom{n}{i}\mu\lambda^i} & \text{if } vu \text{ exists, } n = \langle P_u, v \rangle, \\ 1 & \text{if } u + v \notin \text{Col}(P). \end{cases}$$

The expressions in the first two equations make sense due to Proposition 3.3(c).

Corollary 6.2. *Assume P is a balanced polytope and $u, v \in \text{Col}(P)$, $u + v \neq 0$. Then for all $\lambda, \mu \in R$ we have*

$$[e_u^\lambda, e_v^\mu] = \begin{cases} e_{uv}^{-\lambda\mu} & \text{if } uv \text{ exists,} \\ e_{vu}^{\mu\lambda} & \text{if } vu \text{ exists,} \\ 1 & \text{if } u + v \notin \text{Col}(P). \end{cases}$$

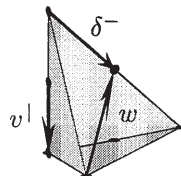


Fig. 5. A new column vector

Remark 6.3. Corollary 6.2 is the generalization of Steinberg’s relations between elementary matrices to balanced polytopes. In order to find the classical Steinberg relation $[e_{ij}^\lambda, e_{jk}^\mu] = e_{ik}^{\lambda\mu}$ in the corollary one must observe that in our setting the configuration $e_{ij}e_{jk}$ corresponds to the existence of vu if we associate with e_{ij} the column vector $v_i - v_j$ where v_1, \dots, v_n are the vectors of the canonical basis of \mathbb{R}^n , and simultaneously the vertices of the unit n -simplex.

That we associate $v_i - v_j$ with e_{ij} (and not $v_j - v_i$) is forced by our notation in which we add column vectors on the right. Thus the successive addition of first u and then v corresponds to the product uv .

Proof of Theorem 6.1. First consider the case when uv exists. As a lattice polytope P decomposes into lattice polytopal layers parallel to the affine plane spanned by u, v and uv , i. e. we consider the *maximal lattice* polytopes in the sections of P with 2-planes parallel to $\mathbb{R}u + \mathbb{R}v$.

Clearly, the automorphisms $\varepsilon_1 = [e_u^\lambda, e_v^\mu]$ and $\varepsilon_2 = \prod_{i=1}^n e_{u+iv}^{-\binom{n}{i}\lambda\mu^i}$ restrict to automorphisms of $R[P']$ for each of these layers P' and we have to check that ε_1 and ε_2 are the same layer by layer.

If a layer $P' \subset P$ has dimension < 2 , then at least two of the vectors u, v and uv do not belong to $\text{Col}(P')$. On the other hand, if $u \in \text{Col}(P')$ or $uv \in \text{Col}(P')$, then one easily deduces from Definition 3.2 that all the three vectors u, v and uv belong to $\text{Col}(P')$. Therefore, $u, uv \notin \text{Col}(P')$. But this means that $P' \subset P_u = P_{u+iv}$, $i \in [1, n]$. In particular, the automorphisms e_u^λ and $e_{u+iv}^{-\binom{n}{i}\lambda\mu^i}$ restrict to the identity automorphism of $R[P']$ and the commutator equality becomes trivial on the layer P' .

So we only need to consider the 2-dimensional layers P' . Then $u, v, uv \in \text{Col}(P')$ and uv exists as a product in $\text{Col}(P')$.

Since $\text{gp}(S_P) = \mathbb{Z}v \oplus \text{gp}(S_{P_v})$ we have $\langle P'_v, u \rangle = \langle P_v, u \rangle$. Without loss of generality we can therefore assume $P \subset \mathbb{R}^2$ and $\text{gp}(S_P) = \mathbb{Z}^3$.

By a suitable affine integral unimodular change of the coordinates in \mathbb{R}^2 one can achieve that $P \subset \mathbb{R} \times \mathbb{R}_+$, $P \cap (\mathbb{R}, 0)$ is an edge of P , and $u = (0, -1)$, $v = (1, 0)$. Moreover, by a parallel shift we can also assume that $(\langle P_v, u \rangle, 0)$ is the lowest right vertex of P .

Consider the triangle

$$\Delta = \text{conv}((0, 0), (\langle P_v, u \rangle, 0), (0, 1)) \subset P$$

(see Figure 6). We have $\text{gp}(S_P) = \text{gp}(S_\Delta)$. On the other hand $L_{P_u} \subset k[P]$ is point-wise fixed under automorphisms of type e_{u+iv}^* , $i \in [0, \langle P_v, u \rangle]$ and $[e_u^*, e_v^*]$ ($* \in R$). Therefore we only need to check that the two automorphisms

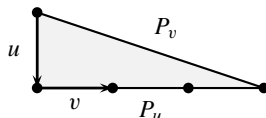


Fig. 6. The essential triangle

$$[e_u^\lambda, e_v^\mu] \quad \text{and} \quad \prod_{i=1}^n e_{u+iv}^{-\lambda \binom{n}{i} \mu^i} \in \mathbb{E}_R(P)$$

coincide at the vertex $(0, 1)$ of Δ .

Observe that all the automorphisms involved restrict to elements of $\mathbb{E}_R(\Delta)$. Therefore, after the natural identification

$$k[\Delta] = k[X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n][Z], \quad n = \langle P_v, u \rangle,$$

in such a way that $e_{u+iv}^\lambda(Z) = Z + \lambda X^{n-i} Y^i$ and $e_v^\lambda(X^{n-i} Y^i) = (X + \lambda Y)^{n-i} Y^i$ for $i \in [0, n]$ the problem has been reduced to the equation

$$[e_u^\lambda, e_v^\mu](Z) = \prod_{i=1}^n e_{u+iv}^{-\lambda \binom{n}{i} \mu^i}(Z)$$

the verification of which is accomplished by a routine computation.

The case when vu exists is just a consequence of the previous case.

Now assume $u + v \notin \text{Col}(P)$. In this situation, too, we could check the desired equality layer by layer. In fact, using Proposition 3.3(a) it is easy to show that $u + v \notin \text{Col}(P')$ for any of the layers P' as above. But the arguments below do not get simplified by the consideration of layers.

By Proposition 3.3(c) none of the products uv and vu exists. By Lemma 3.1 and Proposition 3.3(a) there are only two possibilities: either $P_u = P_v$ and the claim follows from Proposition 2.1 or $\langle P_v, u \rangle = \langle P_u, v \rangle = 0$, what we will assume.

Since v is parallel to P_u , we get $v \in \text{Col}(P_u)$. Then by Lemma 3.1 there is exactly one facet $g \subset P_u$ such that $\langle g, v \rangle = -1$ and for all the other facets $g' \subset P_u$ we have $\langle g, v \rangle \geq 0$ (the pairing is considered on P_u). Now g extends to a unique facet $G \subset P$, $G \neq P_u$, and we see that $\langle G, v \rangle < 0$. Again by Lemma 3.1 $G = P_v$. Using similar arguments for u and P_v we arrive at the conclusion that the two facets $P_u, P_v \subset P$ meet along a common facet (a codimension 2 face of P) and, moreover, $P_u \cap P_v$ is a base facet both for $u \in \text{Col}(P_v)$ and $v \in \text{Col}(P_u)$.

Assume for the moment that $P_u \cap P_v$ contains a lattice point x in its relative interior. Then $x - v \in L_{P_u}$ and $x - u \in L_{P_v}$. Assume in addition that $x - v$ is in the relative interior of P_u . Then $x - v - u \in L_P$. The automorphism $[e_u^\lambda, e_v^\mu]$ fixes pointwise both L_{P_u} and L_{P_v} . On the other hand $\text{gp}(S_P)$ is generated by L_{P_u} and $x - v - u$ (or, similarly, by L_{P_v} and $x - v - u$). Therefore, it suffices to show that the elementary automorphisms e_u^λ and e_v^μ commute at $x - v - u$. It is clear that both automorphisms restrict to an automorphism of the subalgebra $k[\square] \subset k[P]$ where $\square \subset P$ is the lattice 2 dimensional parallelepiped with vertices $x, x - v, x - u, x - u - v$. (Clearly, \square is a unit lattice square up to unimodular transformation.) Now the desired commutativity follows from the easily checked equation

$$(e_u^\lambda \circ e_v^\mu)(xu^{-1}v^{-1}) = xu^{-1}v^{-1} + \lambda(xv^{-1}) + \mu(xu^{-1}) + \lambda\mu x = (e_v^\mu \circ e_u^\lambda)(xu^{-1}v^{-1})$$

where the semigroup operation is written multiplicatively.

Consider the general case. Since S_P consists of non-zero-divisors it is enough to show that $e_u^\lambda \circ e_v^\mu$ and $e_v^\mu \circ e_u^\lambda$ coincide on Veronese subalgebras $k[P]_{(c)}$ for all

sufficiently big natural numbers c . It is in general not true that $k[P]_{(c)} = k[cP]$ (unless P is a normal polytope, that is the semigroup S_P is normal, see [BrGT] for a detailed study of these properties). But the monomials of degree c in $k[P]_{(c)}$ constitute a subset of $L(cP)$ such that all the arguments above that we have used for P apply to it. It only remains to notice that for c big enough the existence of the desired lattice points in the relative interiors as above is guaranteed. \square

7. The stable group of elementary automorphisms

From now on we assume that the polytopes being considered have at least one column vector.

For a polytope P the group of elementary automorphisms $\mathbb{E}_R(P)$ may not be perfect. For instance, P can have only one column structure (like the example $P = \text{conv}((0, 0), (2, 0), (1, 2))$), and then $\mathbb{E}_R(P)$ is isomorphic to the additive group R . Observe that such a polytope is automatically balanced.

We resolve this difficulty by the use of *doubling spectra* of polytopes.

Definition 7.1. *An ascending infinite chain of polytopes $\mathfrak{P} = (P = P_0 \subset P_1 \subset \dots)$ is called a doubling spectrum if the following conditions hold:*

- (i) *for every $i \in \mathbb{Z}_+$ there exists a column vector $v \in \text{Col}(P_i)$ such that $P_{i+1} = P_i^{-v}$,*
- (ii) *for all $i \in \mathbb{Z}_+$ and every $v \in \text{Col}(P_i)$ there is an index $j \geq i$ such that $P_{j+1} = P_j^{-v}$.*

Here we have used Lemma 4.1 which (together with condition (i)) allows one to consider v as an element of $\text{Col}(P_j)$. The second part of the definition simplifies the construction of doubling spectra. For example, if $v, -v \in \text{Col}(P)$, then it does not matter whether we double along P_v or P_{-v} since there exist an automorphism of P exchanging v and $-v$.

We say that $v \in \text{Col}(P_i)$ is *decomposed* at the j th step in \mathfrak{P} for some $j \geq i$ if $P_{j+1} = P_j^{-v}$. We will need the fact that every column vector, showing up in a doubling spectrum, gets decomposed infinitely many times:

Lemma 7.2. *For all $i \in \mathbb{Z}_+$ every vector $v \in \text{Col}(P_i)$ is decomposed at infinitely many steps in \mathfrak{P} .*

Proof. The desired infinite series of decompositions is derived as follows. First fix some j_1 such that $P_{j_1+1} = P_{j_1}^{-v}$ and then choose $j_2 < j_3 < \dots$ recursively such that

$$P_{j_{i+1}+1} = \lrcorner \left(P_{j_i+1}, \delta^+(v, P_{j_i}) \right).$$

The vector v gets decomposed at each of the indices j_i because $\delta^+(v, P_{j_i})$ and v share the base facet in P_{j_i+1} . \square

Associated to a doubling spectrum \mathfrak{P} is the “infinite polytopal” algebra

$$R[\mathfrak{P}] = \lim_{i \rightarrow \infty} R[P_i]$$

and the filtered union

$$\text{Col}(\mathfrak{P}) = \lim_{i \rightarrow \infty} \text{Col}(P_i).$$

The embeddings meant here are induced by the facet embeddings $P_i \subset P_{i+1}$ which, by Lemma 4.1 imply natural embeddings $\text{Col}(P_i) \subset \text{Col}(P_{i+1})$. For convenience of notation we will often identify P_i and $\text{Col}(P_i)$ with their images under these embeddings, without mentioning this explicitly.

All elements $v \in \text{Col}(\mathfrak{P})$ and $\lambda \in R$ give rise to a graded automorphism of $R[\mathfrak{P}]$ as follows: we choose an index i big enough so that $v \in \text{Col}(P_i)$. Then the elementary automorphisms $e_v^\lambda \in \mathbb{E}_R(P_j)$, $j \geq i$ constitute a compatible system and, therefore, define a graded automorphism of $R[\mathfrak{P}]$. This automorphism will again be called ‘elementary’ and it will also be denoted by e_v^λ .

Let $\mathbb{E}(R, \mathfrak{P})$ denote the subgroup of the automorphism group of $R[\mathfrak{P}]$ that is generated by the elementary automorphisms. We will call $\mathbb{E}(R, \mathfrak{P})$ the *stable group of elementary automorphisms* over P . Clearly, there are many doubling spectra starting from P , but all the resulting stable groups are pairwise naturally isomorphic.

Proposition 7.3. *Let $\mathfrak{P} = (P \subset P_1 \subset P_2 \subset \dots)$ and $\mathfrak{Q} = P \subset Q_1 \subset Q_2 \subset \dots$ be two doubling spectra. Then the groups $\mathbb{E}(R, \mathfrak{P})$ and $\mathbb{E}(R, \mathfrak{Q})$ are naturally isomorphic.*

Proof. Consider the ‘infinite lattice polytopes’

$$\mathcal{P} = \bigcup_{\mathbb{Z}_+} P_i \subset \bigoplus_{\mathbb{Z}_+} \mathbb{R} \quad \text{and} \quad \mathcal{Q} = \bigcup_{\mathbb{Z}_+} Q_i \subset \bigoplus_{\mathbb{Z}_+} \mathbb{R},$$

where we mean the filtered unions of polytopes and their ambient Euclidean spaces. It is enough to show that there is a global affine (i. e. linear + constant) transformation

$$\Theta : \bigoplus_{\mathbb{Z}_+} \mathbb{R} \rightarrow \bigoplus_{\mathbb{Z}_+} \mathbb{R}, \quad \Theta\left(\bigcup_{\mathbb{Z}_+} L_{P_i}\right) = \bigcup_{\mathbb{Z}_+} L_{Q_i},$$

such that $\Theta(\mathcal{P}) = \mathcal{Q}$ and $\Theta(\text{Col}(\mathfrak{P})) = \text{Col}(\mathfrak{Q})$.

The existence of such Θ is established as follows. By Lemma 5.1 (and Definition 7.1) for every index $i \in \mathbb{Z}_+$ there exists $j_i \geq i$ such that $\text{Col}(P_i)$ is a subset of $\text{Col}(Q_{j_i})$. Consider the minimal subpolytope $Q' \subset Q_{j_i}$ for which $P \subset Q'$ and $\text{Col}(P_i) \subset \text{Col}(Q')$. It is clear from Proposition 4.3 and Definition 7.1 that P_i and Q' are naturally isomorphic as lattice polytopes and, moreover, Q' is a face of Q_{j_i} . In particular we obtain an isomorphism between P_i and a face of Q_{j_i} . Let Θ_i denote the affine continuation of this isomorphism to the ambient Euclidean spaces. Then the Θ_i constitute a compatible system and, therefore, induce a global affine transformation $\Theta : \bigoplus_{\mathbb{Z}_+} \mathbb{R} \rightarrow \bigoplus_{\mathbb{Z}_+} \mathbb{R}$. It is an easy exercise to show that Θ satisfies all the desired conditions. \square

As a consequence of Proposition 7.3 we can use the notation $\mathbb{E}(R, P)$ for $\mathbb{E}(R, \mathfrak{P})$ where \mathfrak{P} is some doubling spectrum starting with P . In the sequel we will assume that for a polytope P we have fixed a doubling spectrum \mathfrak{P} .

Remark 7.4. One can define the group $\mathbb{E}(R, P)$ using sequences of polytopes $\mathfrak{P}' = (P = P'_0 \subset P'_1 \subset \dots)$ that are more general than doubling spectra. In particular, suppose that $\mathfrak{P} = (P = P_0 \subset P_1 \subset \dots)$ is a doubling spectrum for P and $\mathfrak{P}' = (P'_0 \subset P'_1 \subset \dots)$ is a sequence of polytopes for which there exist isomorphisms $\varphi_i : P_i \rightarrow P'_i$ of polytopes that commute with the embeddings $P_i \subset P_{i+1}$ and $P'_i \rightarrow P'_{i+1}$. Then \mathfrak{P}' need not be a doubling spectrum in the strict sense since condition (ii) is not invariant under isomorphisms as just described. However, there evidently exists a natural isomorphism $\mathbb{E}(R, \mathfrak{P}) \approx \mathbb{E}(R, \mathfrak{P}')$.

For instance, if we start from the unimodular simplex $\Delta_n, n \in \mathbb{N}$, and consider the sequence $\mathfrak{P}' = (\Delta_n = P'_0 \subset P'_1 \subset \dots)$, in which $P'_{i+1} = P'_i \perp^v, i \in \mathbb{N}$, for the same column vector $v \in \text{Col}(\Delta_n)$, then the resulting sequence of unstable groups is naturally identified with the familiar sequence of groups of elementary matrices

$$E_{n+1}(R) \subset E_{n+2}(R) \subset \dots, \quad * \mapsto \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, $\mathbb{E}(R, \Delta_n) = \mathbb{E}(R)$ for all $n \in \mathbb{N}$.

Let $u, v \in \text{Col}(\mathfrak{P})$. It is easily observed that the $\langle (P_i)_v, u \rangle$ are the same for all P_i from \mathfrak{P} such that $u, v \in \text{Col}(P_i)$. We will use the notation $\langle \mathfrak{P}_v, u \rangle$ for this common value.

Proposition 7.5. *Let P be a polytope, $u, v \in \text{Col}(\mathfrak{P})$ and $\lambda, \mu \in R$. Then:*

- (a) $e_u^\lambda \circ e_u^\mu = e_u^{\lambda+\mu}$.
- (b) *The exact analogues of the equations in Theorem 6.1 hold once $\langle P_v, u \rangle$ is changed to $\langle \mathfrak{P}_v, u \rangle$.*
- (c) $\mathbb{E}(R, P)$ is perfect.

(a) and (b) are proved by an easy reduction to $R[P_i]$ with i big enough and (c) follows from (b) by the equation (4₂). In particular, the perfectness does not depend on whether or not P is balanced.

The following remark shows that naive analogies between the stable group of elementary matrices $\mathbb{E}(R)$ and $\mathbb{E}(R, P)$ may fail.

Remark 7.6. (a) In the special case of unit simplices we get the stable group of elementary matrices over R : $\mathbb{E}(R, \Delta_n) = \mathbb{E}(R)$ for all $n \in \mathbb{N}$.

(b) In general the groups $\mathbb{E}_R(P)$ are not perfect. However, after finitely many steps in the doubling spectrum one arrives at a polytope P'' for which this group is perfect.

In fact, after finitely many steps each base facet in P has been used for a doubling, and in the polytope P' then constructed each base facet has an invertible column vector. This property is preserved under further doublings. Therefore all the vectors δ^+ and δ^- that come up in doublings of P' are automatically decomposed, and after finitely many doublings starting from P' one arrives at a polytope P'' in which all the column vectors of P' , and thus all of P'' , are decomposed.

(c) The group $\mathbb{E}(R, P)$ is in general *not* the filtered union of the unstable groups $\mathbb{E}_R(P_i)$. Consider the simple example of the segment $2\Delta_1$. Then the second term

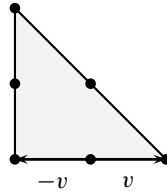


Fig. 7. The polytope $2\Delta_2$

in \mathfrak{P} can be identified with the triangle $2\Delta_2 = \text{conv}((0, 0), (2, 0), (0, 2)) \subset \mathbb{R}^2$ so that $2\Delta_1$ is the lower edge (see Figure 7). Consider the vectors $v = (1, 0)$ and $-v = (-1, 0)$ from $\text{Col}(2\Delta_1)$. Assume $2 \neq 0$ in R . Then the element $\varepsilon = (e_v^1 \circ e_{-v}^{-1} \circ e_v^1)^2 \in \mathbb{E}(R, \Delta_2)$ is not the identity automorphism of $R[\mathfrak{P}]$ (it switches signs on the second layer of $2\Delta_2$) whereas the restriction of ε to $R[2\Delta_1]$ is the identity automorphism. In particular there is no natural group embedding $\mathbb{E}_R(2\Delta_1) \rightarrow \mathbb{E}(R, 2\Delta_1)$.

(d) On the other hand, every element $e \in \mathbb{E}_R(P_i)$, $i \in \mathbb{Z}_+$ is a restriction to $R[P_i]$ of some element of $\mathbb{E}(R, P)$ and every element $\varepsilon \in \mathbb{E}(R, P)$ restricts to an element of $E_R(P_i)$ whenever i is big enough. Clearly, we have the following approximation principle: if two elements $\varepsilon, \varepsilon' \in \mathbb{E}(R, P)$ restrict to the same elements of $\mathbb{E}_R(P_i)$ for all sufficiently large i then $\varepsilon = \varepsilon'$.

(e) Unlike the group $\mathbb{E}(R, P)$, the Steinberg group $\text{St}(R, P)$, introduced in Section 8, is the direct limit of the corresponding unstable groups (see Remark 8.1 below).

The next results is just a standard fact in the classical situation of elementary matrices. However, the proof is no longer straightforward in the polytopal case, not even for balanced polytopes.

Theorem 7.7. *Let R be a ring and P a polytope (not necessarily balanced). Then the center $Z(\mathbb{E}(R, P))$ is trivial.*

Proof. Choose $\varepsilon \in Z(\mathbb{E}(R, P))$. There is no loss of generality in assuming $\varepsilon = e_{v_1}^{\lambda_1} \circ \dots \circ e_{v_k}^{\lambda_k}$ for some $k \geq 0$, $\lambda_1, \dots, \lambda_k \in R$ and $v_1, \dots, v_k \in \text{Col}(P)$. In particular, ε restricts to an automorphism of each of the $R[P_i]$, $i \in \mathbb{Z}_+$.

Step 1. We claim that for every $x \in L_P$ there exists $a_x \in R^*$ such that $\varepsilon(x) = a_x x$. (R^* denotes to the multiplicative group of invertible elements.)

Let $F \subset P$ be a facet. It is enough to show that every element $x \in L_P$ satisfies the condition $\varepsilon(x) = a_1 x_1 + \dots + a_s x_s$ for some $a_i \in R$ and $x_i \in L_P$ with $\text{ht}_F(x_i) \geq \text{ht}_F(x)$. In fact, by running F through $\mathbb{F}(P)$ we prove the claim.

In the argument below we consider the polytopes $P^{\perp F}$ for all facets $F \subset P$. Clearly, if F is not a base facet of an element of $\text{Col}(P)$, then $P^{\perp F}$ does not show up in a doubling spectrum starting with P – we use this doubled polytope only as an auxiliary object.

If $F = P_v$ for some $v \in \text{Col}(P)$, then the two automorphisms $\varepsilon, e_{\delta_{-(F,P)}}^1 \in \mathbb{E}(R, P)$ commute. If F is not the base facet of any of the column vectors, then

none of the products $\delta_{F,P}^- v$ and $v \delta_{F,P}^-$ exists for any $v \in \text{Col}(P)$. In fact, we have $\langle P_v^{-\perp F}, \delta_{F,P}^- \rangle = 0$ (by equation (4₁)) and $\langle P_{\delta_{F,P}^-}, v \rangle = \langle P^-, v \rangle = 0$. Since $v + \delta_{F,P}^- \neq 0$, Theorem 6.1 implies that ε and $e_{\delta_{F,P}^-}^1$ again commute. For simplicity put $\delta^- = \delta_{F,P}^-$.

Consider the point $x^\perp \in P^\perp$ that corresponds to x . We have

$$\varepsilon(x^\perp) \in R[P^{-\perp F}] = b_1 y_1 + \dots + b_t y_t$$

for some $b_j \in R$ and $y_j \in L_{P^{-\perp F}}$ with $\text{ht}_{P^-}(y_j) = \text{ht}_F(x)$, where $\text{ht}_F(x)$ is considered in P . It is easily seen that

$$(e_{\delta^-}^1 \circ \varepsilon)(x^\perp) = b_1 z_1 + \dots + b_t z_t + (R\text{-linear combination of points } u \text{ with } \text{ht}_{P^-}(u) > 0),$$

where $z_j = y_j + \text{ht}_F(x) \cdot \delta^- \in L_{P^-}$. Clearly, $\text{ht}_F(z_j) \geq \text{ht}_F(x)$.

On the other hand

$$(\varepsilon \circ e_{\delta^-}^1)(x^\perp) = \varepsilon(x) + (R\text{-linear combination of points } u \text{ with } \text{ht}_{P^-}(u) > 0)$$

Since ε and δ^- commute, we get what we want by comparing these two expressions.

Step 2. Let Q be a polytope, $w \in \text{Col}(Q)$ and α be an R -automorphism of $R[Q]$ of type $y \mapsto a_y y$, $a_y \in R^*$ for all $y \in L_Q$. Put

$$\alpha(w) = \frac{\alpha(z \cdot w)}{\alpha(z)}$$

where $z \in L_Q \setminus L_{Q_w}$ and the operation in \mathbb{Z}^n is written multiplicatively. The ratio is in fact independent of the choice of z because $\alpha(z' w) \alpha(z) = \alpha(z w) \alpha(z')$. Then

$$\alpha \circ e_w^1 \circ \alpha^{-1} = e_w^{\alpha(w)}. \tag{*}$$

(See also Lemma 4.4(b) in [BG2].)

Returning to our central automorphism ε we see that $a_x = a_{x'}$ whenever the points $x, x' \in L_P$ differ by a vector from $\text{Col}(P)$. Since we can replace P by some polytope obtained by successive doublings at base facets of column vectors, it is enough to apply the following lemma: clearly $a_y = 1$ for the lattice point y constructed there, and thus $a_x = 1$ since x and y differ by a sum of column vectors. \square

Lemma 7.8. *Let P be a polytope, $x \in L_P$, and v_1, \dots, v_k be column vectors of P . Set $P_0 = P$, $P_1 = P^{-v_1}$, $P_2 = (P_1)^{-v_2}, \dots$. Then there exists $y \in L_Q$, $Q = P_k$, such that x and y differ by a sum of column vectors of Q , and $y \in Q_{v_i}$ for $i = 1, \dots, k$.*

Proof. We construct a chain $x = x_0, \dots, x_k = y$ of lattice points $x_i \in P_i$ such that $x_i - x_{i-1}$ is a multiple of $\delta_{v_i}^+$ and $x_i \in (P_i)_{v_i}$ for $i = 0, \dots, k - 1$.

Let $i \geq 1$. If $x_{i-1} \in (P_{i-1})_{v_i}$, then we choose $x_i = x_{i-1}$. Otherwise $(P_{i-1})_{v_i} \neq (P_{i-1})_{v_j}$ for $j = 1, \dots, i - 1$, and we set $x_i = x_{i-1} + \text{ht}_{v_i}(x_{i-1}) \delta_{v_i}^+$. Since $\delta_{v_i}^+$ is parallel to the extension of $(P_{i-1})_{v_j}$ to P_i for $j = 1, \dots, i - 1$ (by (4₁)) and $\text{ht}_{v_i}(x_{i-1} + \text{ht}_{v_i}(x_{i-1}) \delta_{v_i}^+) = 0$, we have reached a lattice point with the desired properties. \square

8. Schur multiplier of $\mathbb{E}(R, P)$

Next we proceed in analogy with the ordinary algebraic K -theory and define the *polytopal Steinberg groups*. In the proofs below we use a number of modifications of Milnor’s arguments [M, §5]. Many of the difficulties that show up for general balanced polytopes are invisible in the special case of unit simplices.

Fix a doubling spectrum \mathfrak{P} starting with P . The *unstable* Steinberg groups $\text{St}_R(P_i)$ are defined as the groups generated by the x_v^λ , $v \in \text{Col}(P_i)$, $\lambda \in R$, which are subject to the relations

$$x_v^\lambda x_v^\mu = x_v^{\lambda+\mu}$$

and

$$[x_u^\lambda, x_v^\mu] = \begin{cases} \prod_{i=1}^n x_{u+iv}^{-\binom{n}{i}\lambda\mu^i} & \text{if } uv \text{ exists, } n = \langle P_v, u \rangle, \\ \prod_{i=1}^n x_{v+iu}^{\binom{n}{i}\mu\lambda^i} & \text{if } vu \text{ exists, } n = \langle P_u, v \rangle, \\ 1 & \text{if } u + v \notin \text{Col}(P) \cup \{0\}. \end{cases}$$

The *stable Steinberg group* $\text{St}(R, P)$ is defined through the obvious analogues of the equations above, where $\langle P_v, u \rangle$ and $\langle P_u, v \rangle$ are correspondingly changed to $\langle \mathfrak{P}_v, u \rangle$ and $\langle \mathfrak{P}_u, v \rangle$. Clearly, arguments similar to those in Proposition 7.3 show that $\text{St}(R, P)$ does not depend on the choice of the doubling spectrum \mathfrak{P} .

Remark 8.1. (a) There are natural surjective homomorphisms $\text{St}_R(P_i) \rightarrow \mathbb{E}_R(P_i)$ and $\text{St}(R, P) \rightarrow \mathbb{E}(R, P)$. Moreover, contrary to the groups of elementary automorphisms (see Remark 7.6), one has natural group homomorphisms between successive unstable Steinberg groups, and

$$\text{St}(R, P) = \lim_{\rightarrow} \text{St}_R(P_i).$$

However, these homomorphisms may be non-injective, even in the classical situation $P = \Delta_n$ – here we enter the topic of injective K_2 -stabilization.

(b) The group $\text{St}(R, P)$ is always perfect, like $\mathbb{E}(R, P)$.

(c) Remark 7.4 applies here as well: $\text{St}(R, P)$ can be computed from any ascending sequence of lattice polytopes that is isomorphic to a doubling spectrum.

Proposition 8.2. *Let P be a balanced polytope and R be a ring. Then the center $Z(\text{St}(R, P))$ is the kernel of the natural homomorphism $\text{St}(R, P) \rightarrow \mathbb{E}(R, P)$.*

Later on $\text{Ker}(\text{St}(R, P) \rightarrow \mathbb{E}(R, P))$ for P balanced will be called the *polytopal Milnor group* of R corresponding to P . We denote it by $K_2(R, P)$.

Proof of Proposition 8.2. As remarked, we have fixed a doubling spectrum $\mathfrak{P} = (P \subset P_1 \subset \dots)$. For every $i \in \mathbb{N}$ consider the subsets

$$U^{i+1} = \{u \in \text{Col}(P_{i+1}) \mid \langle P_i, u \rangle = 1\},$$

$$V^{i+1} = \{v \in \text{Col}(P_{i+1}) \mid \langle P_i, v \rangle = -1\}$$

of $\text{Col}(P_{i+1})$. (Unlike the classical situation the two sets U^{i+1} and V^{j+1} are not completely similar.)

We have the subgroups $\mathfrak{U}^{i+1}, \mathfrak{V}^{i+1} \subset \text{St}_R(P_{i+1})$ generated correspondingly by the $x_u^\lambda, v \in U^{i+1}$ and $x_v^\mu, v \in V^{i+1}$ ($\lambda, \mu \in R$). Assume $U^{i+1} = \{u_1, \dots, u_r\}$, $r = \#U^{i+1}$ and $V^{i+1} = \{v_1, \dots, v_s\}$, $s = \#V^{i+1}$.

Claim. The mappings

$$\begin{aligned} R^r &\rightarrow \mathfrak{U}^{i+1}, & (\lambda_1, \dots, \lambda_r) &\mapsto x_{u_1}^{\lambda_1} \cdots x_{u_r}^{\lambda_r}, \\ R^s &\rightarrow \mathfrak{V}^{i+1}, & (\mu_1, \dots, \mu_s) &\mapsto x_{v_1}^{\mu_1} \cdots x_{v_s}^{\mu_s} \end{aligned}$$

are group isomorphisms, where the free R -modules R^r and R^s are viewed as additive abelian groups. Moreover, the canonical surjection $\pi : \text{St}_R(P_{i+1}) \rightarrow \mathbb{E}_R(P_{i+1})$ is injective both on \mathfrak{U}^{i+1} and \mathfrak{V}^{i+1} .

First one observes that ww' does not exist for any pair

$$(w, w') \in (U^{i+1} \times U^{i+1}) \cup (V^{i+1} \times V^{i+1}).$$

Lemma 3.1 implies this for the pairs from $V^{i+1} \times V^{i+1}$ without the condition that P is balanced, whereas this condition is needed for the pairs from $U^{i+1} \times U^{i+1}$: if ww' existed then for some $(w, w') \in U^{i+1} \times U^{i+1}$ then $\langle P_i, ww' \rangle = \langle (P_{i+1})_{\delta^-}, ww' \rangle = 2$, contradicting the condition that P is balanced.

As a result of this observation all elements $\sigma \in \mathfrak{U}^{i+1}$ and $\rho \in \mathfrak{V}^{i+1}$ have presentations

$$(1) \quad \sigma = x_{u_1}^{\lambda_1} \cdots x_{u_r}^{\lambda_r} \quad \text{and} \quad (2) \quad \rho = x_{v_1}^{\mu_1} \cdots x_{v_s}^{\mu_s}.$$

For all points $x \in L_{P_i}$ and $y \in L_{P_{i+1}}$ with $\text{ht}_{P_i}(y) = 1$ we have

$$\begin{aligned} (\pi(\sigma))(x) &= x + \tau_1 \lambda_1(xu_1) + \cdots + \tau_r \lambda_r(xu_r) + f_{\geq 2} \\ (\pi(\rho))(y) &= y + \mu_1(yv_1) + \cdots + \mu_s(yv_s) \end{aligned}$$

where $f_{\geq 2}$ is an R -linear combination of the points from $L_{P_{i+1}}$ with height above P_i at least 2 and $\tau_i \in \{0, 1\}$ for $i \in [1, r]$. (The semigroup operation is written multiplicatively). It is immediate from the second equation that the presentation (2) is uniquely determined by $\pi(\rho)$. Observe that for each $i \in [1, r]$ there exists $x \in L_{P_i}$ such that the corresponding τ_i is 1. Therefore, by running x through L_{P_i} we see that likewise $\pi(\sigma)$ uniquely determines the presentation (1). Now the claim follows.

Since P is balanced, for all vectors $u \in U^{i+1}$, $v \in V^{i+1}$ and $w \in \text{Col}(P_i)$ and elements $\lambda, \mu, \nu \in F$ we have:

$$x_w^\nu x_u^\lambda x_w^{-\nu} = \begin{cases} x_{wu}^{-\nu\lambda} x_u^\lambda & \text{if } wu \text{ exists,} \\ x_{uw}^{\lambda\nu} x_u^\lambda & \text{if } uw \text{ exists,} \\ x_u^\lambda & \text{if } wu \notin \text{Col}(P), \end{cases}$$

and

$$x_w^\nu x_v^\mu x_w^{-\nu} = \begin{cases} x_{vw}^{\mu\nu} x_v^\mu & \text{if } vw \text{ exists,} \\ x_v^\mu & \text{if } vw \text{ does not exist.} \end{cases}$$

The listed cases exhaust all possibilities since $u + w \neq 0$, $v + w \neq 0$ and the product wv does not exist – we have $\langle (P_{i+1})_v, w \rangle = \langle P_i, w \rangle = 0$ and Proposition 3.3(a) applies. (It is not difficult to find examples for which uw exists.)

Since $wu \in U^{i+1}$, $uw \in U^{i+1}$ and $vw \in V^{i+1}$ whenever the corresponding product exists, we arrive at the inclusions:

$$z\mathfrak{U}^{i+1}z^{-1} \subset \mathfrak{U}^{i+1} \quad \text{and} \quad z\mathfrak{V}^{i+1}z^{-1} \subset \mathfrak{V}^{i+1}.$$

for all $z \in \text{St}_R(P_i)$. Changing z by z^{-1} we get

$$z\mathfrak{U}^{i+1}z^{-1} = \mathfrak{U}^{i+1} \quad \text{and} \quad z\mathfrak{V}^{i+1}z^{-1} = \mathfrak{V}^{i+1}. \tag{3}$$

for all $z \in \text{St}_R(P_i)$.

Now we are ready to prove that

$$\text{Ker}(\pi : \text{St}(R, P) \rightarrow \mathbb{E}(R, P)) = Z(\text{St}(R, P)).$$

Choose an element $z \in \text{St}(R, P)$ is such that $\pi(z) = 1$. We want to show that $zx_v^\lambda z^{-1} = x_v^\lambda$ for all $v \in \text{Col}(\mathfrak{P})$ and $\lambda \in R$. Since $\text{St}(R, P)$ is the inductive limit of the groups $\text{St}_R(P_i)$ and $\pi(z)$ restricts to an automorphism of $R[P_i]$ for $i \gg 0$, there exist $z_i \in \text{Ker}(\text{St}_R(P_i) \rightarrow \mathbb{E}_R(P_i))$ mapping to z provided $i \gg 0$ and such that z_{i+1} is the image of z_i in $\text{St}_R(P_{i+1})$. We can also assume $v \in \text{Col}(P_i)$ and that v gets decomposed in \mathfrak{P} at P_i . We will show that z_i maps to the center $Z(\text{Im}(\text{St}_R(P_i) \rightarrow \text{St}_R(P_{i+1})))$. Clearly, this implies that z is central in $\text{St}(R, P)$.

For the elements of $\text{St}_R(P_{i+1})$ we have

$$x_v^\lambda = [x_{\delta+(v, P_i)}^1, x_v^\lambda], \quad x_{\delta+(v, P_i)}^1 \in \mathfrak{U}^{i+1}, \quad x_v^\lambda \in \mathfrak{V}^{i+1},$$

where $v^1 \in \text{Col}(P_{i+1})$ is the column vector corresponding to v . The claim above and (3) imply that z_i commutes both with elements of \mathfrak{U}^{i+1} and \mathfrak{V}^{i+1} . Hence it also commutes with x_v^λ .

We have shown the inclusion $\text{Ker}(\pi : \text{St}(R, P) \rightarrow \mathbb{E}(R, P)) \subset Z(\text{St}(R, P))$. The other inclusion follows from Theorem 7.7. \square

Before the discussion of balanced polytopes in general let us treat a polytope that represents the case in which there are three column vectors with exactly one product.

Lemma 8.3. *Let $P = \text{conv}((0, 0), (3, 0), (1, 2), (0, 1))$ and R be a ring. Then the group $\text{St}(R, P)$ is a universal central extension of $\mathbb{E}(R, P)$.*

Proof. We have $\text{Col}(P) = \{u, v, w\}$ for $u = (0, -1)$, $v = (1, 0)$ and $w = (1, -1)$. The product table for $\text{Col}(P)$ consists of only the equation $uv = w$ (see Figure 8). We now describe the group $\text{St}(R, P)$ explicitly.

First we make the following observation: if Q is a polytope and $v, -v \in \text{Col}(Q)$ then Q^{-v} is naturally isomorphic to Q^{-v} so that the points of Q are mapped to themselves under this isomorphism. It follows from this observation and Lemma 5.1 that for producing a sequence of polytopes isomorphic to a doubling spectrum (starting with Q) one only needs to decompose at each step one of the original

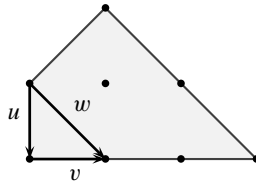


Fig. 8. Three column vectors with one product

column vectors of Q or one of the vectors of type either y^1 or δ^+ . (See Remarks 7.4 and 8.1.)

Fix a doubling spectrum $\mathfrak{P} = (P \subset P_1 \subset \dots)$. Consider the points $a = (0, 1)$, $b = (0, 0)$ and $c = (1, 0)$ from L_P . We have $a + u = b$ and $b + v = c$. Based on the aforementioned general observation one easily sees that for every i the set $\text{Col}(P_i)$ looks as follows. There is certain system of points in L_{P_i}

$$a = x_1, x_2, \dots, x_{j_0} = b, x_{j_0+1}, \dots, x_m = c, \quad 1 < j_0 < m$$

such that

$$x_{j'} - x_{j''} \in \text{Col}(P_i) \iff (j', j'') \in (Z_1 \cup Z_2 \cup Z_3 \cup Z_4) \setminus \{(1, 1), \dots, (m, m)\},$$

where

$$\begin{aligned} Z_1 &= [1, j_0 - 1] \times [1, j_0 - 1] & Z_2 &= [j_0, m] \times [1, j_0 - 1], \\ Z_3 &= [j_0, m - 1] \times [j_0, m - 1] & Z_4 &= \{m\} \times [1, m - 1]. \end{aligned}$$

Moreover, the numbers m , j_0 and $m - j_0$ can be arbitrarily big if i is big enough.

Now $\text{St}(R, P)$ admits the following description. Let A and B denote two disjoint copies of \mathbb{N} and let 0 be an ‘‘origin’’, $0 \notin A \cup B$. Then $\text{St}(R, P)$ is generated by symbols x_{ij}^λ where $\lambda \in R$, $i \neq j$ and

$$(i, j) \in (A \times A) \cup (B \times A) \cup (B \times B) \cup (\{0\} \times A) \cup (\{0\} \times B),$$

and these symbols are subject to the standard Steinberg relations:

$$x_{ij}^\lambda x_{ij}^\mu = x_{ij}^{\lambda+\mu} \quad \text{and} \quad [x_{ij}^\lambda, x_{kl}^\mu] = \begin{cases} x_{il}^{\lambda\mu} & \text{if } j = k \text{ and } i \neq l, \\ 1 & \text{if } j \neq k \text{ and } i \neq l. \end{cases}$$

In Figure 9 we have tried to visualize these data.

It only remains to notice that the proof of Theorem 5.10 in [M] goes through for $\text{St}(R, P)$ without any change. It is of course important that for a pair (i, k) from our index set we can always find j such that (i, j) and (j, k) are also in the index set. \square

For general balanced polytopes P one cannot visualize $\text{St}(R, P)$ in a similar way; see Example 10.3 below.

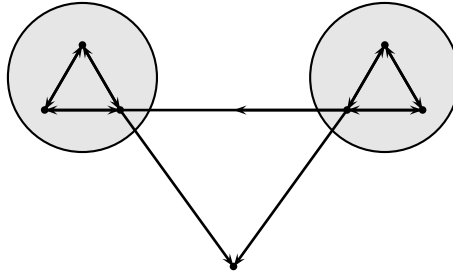


Fig. 9. The generators of $\text{St}(R, P)$

Theorem 8.4. *Let P be a balanced polytope and R be a ring. Then $\text{St}(R, P)$ is a universal central extension of $\mathbb{E}(R, P)$ and*

$$\text{Ker}(\text{St}(R, P) \rightarrow \mathbb{E}(R, P)) = Z(\text{St}(R, P)).$$

Proof. The second assertion is Proposition 8.2.

Suppose

$$1 \longrightarrow C \longrightarrow Y \xrightarrow{\psi} \text{St}(R, P) \longrightarrow 1$$

is a central extension, i. e. $C \subset Z(Y)$. We want to construct a splitting homomorphism $\xi : \text{St}(R, P) \rightarrow Y$. According to [M, §5] this will show the universality of the central extension $\pi : \text{St}(R, P) \rightarrow \mathbb{E}(R, P)$.

Pick $x, x' \in \text{St}(R, P)$. Since the extension ψ is central, the commutators of type $[y, y']$ with $y \in \psi^{-1}(x)$ and $y' \in \psi^{-1}(x')$ coincide. Therefore we can use the notation $[\psi^{-1}(x), \psi^{-1}(x')]$ for this common value.

For every generator x_w^λ of $\text{St}(R, P)$ with $w \in \text{Col}(P_i)$ and $\lambda \in R$ we choose an index $j \geq i$ such that the column vector w is decomposed at P_j and put

$$\xi(x_w^\lambda) = [\psi^{-1}(x_{\delta^+(w, P_j)}^1), \psi^{-1}(x_{w|}^\lambda)]. \tag{4}$$

(Here $w^1 \in \text{Col}(P_{j+1})$ is the corresponding vector.) All we have to show is that

- (i) the $\xi(x_w^\lambda)$ are independent of the choices of j ,
- (ii) the $\xi(x_w^\lambda)$ are subject to the same relations as the x_w^λ .

To this end we make the following observation. Let Q be the quadrangle considered in Lemma 8.3, and $u_0, v_0, w_0 \in \text{Col}(Q)$ be its column vectors. For some $k \geq 0$ Consider $u, v \in \text{Col}(P_k)$ such that uv exists. Then the assignments $x_{u_0}^* \mapsto x_u^*, x_{v_0}^* \mapsto x_v^*, x_{w_0}^* \mapsto x_{uv}^*$ ($* \in R$) give rise to a group homomorphism $\text{St}(R, Q) \rightarrow \text{St}(R, P)$. This observation follows from the definition of polytopal Steinberg groups and Lemma 5.1: after the identifications $u \leftrightarrow u_0, v \leftrightarrow v_0$ and $uv \leftrightarrow w_0$ the higher members of the doubling spectrum \mathcal{Q} of Q only admit column vectors that also show up in \mathfrak{P} . It is also crucial that the defining relations for $\text{St}(R, Q)$ are preserved by the corresponding elements of $\text{St}(R, P)$ because both P and Q are balanced.

By Lemma 8.3 every central extension of $\text{St}(R, Q)$ splits. On the other hand the homomorphism Ψ in the pull back diagram

$$\begin{array}{ccc}
 Y \times_{\text{St}(R, P)} \text{St}(R, Q) & \xrightarrow{\Psi} & \text{St}(R, Q) \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\psi} & \text{St}(R, P)
 \end{array}$$

is central since ψ is central. A splitting of Ψ yields the commutative triangle

$$\begin{array}{ccc}
 & \text{St}(R, Q) & \\
 & \swarrow & \searrow \\
 Y & \xrightarrow{\psi} & \text{St}(R, P)
 \end{array}$$

f

which simultaneously shows several things:

- (a) the definition of $\xi(x_w^\lambda)$ is in fact independent of the choice of j ,
- (b) $\xi(x_w^\lambda)\xi(x_w^\mu) = \xi(x_w^{\lambda+\mu})$ for all $\lambda, \mu \in R$,
- (c) for all $\lambda, \mu \in R$ and all $w_1, w_2 \in \text{Col}(\mathfrak{P})$ for which $w_1 w_2$ exists one has $[\xi(x_{w_1}^\lambda), \xi(x_{w_2}^\mu)] = \xi(x_{w_1 w_2}^{\lambda\mu})$.

In fact, by lifting δ_{uv, P_k}^+ and $(uv)^\perp$ to the appropriate vectors in $\text{Col}(\mathfrak{P}_0)$ one observes that

$$f(x_{w_0}^\lambda) = [\psi^{-1}(x_u^1), \psi^{-1}(x_v^\lambda)] = [\psi^{-1}(x_{\delta^+(uv, P_k)}^1), \psi^{-1}(x_{(uv)^\perp}^\lambda)] \tag{5}$$

for all $\lambda \in R$. Applying this formula to the column vectors $w, \delta^+(w, P_j)$, and w^\perp (playing the roles of uv, u and v) we prove the independence of $\xi(x_w^\lambda)$ of the choice of j . The other two properties of the $\xi(x_v^\lambda)$ follow similarly by using appropriate mappings $\text{St}(R, Q) \rightarrow \text{St}(R, P)$: for (b) we use the same mapping $x_{u_0}^* \mapsto x_{\delta^+}^*$, $x_{v_0} \mapsto x_w^*$, $x_{w_0} \mapsto x_w^*$ and for (c) we use the mapping $x_{u_0}^* \mapsto x_{w_1}^*$, $x_{v_0}^* \mapsto x_{w_2}^*$, $x_{w_0}^* \mapsto x_{w_1 w_2}^*$, $* \in R$.

Only one Steinberg relation remains to be checked. Consider a pair of column vectors $v, w \in \text{Col}(P_i)$ such that $v + w \notin \text{Col}(P_i) \cup \{0\}$. We want to show that $[\psi^{-1}(x_v^\lambda), \psi^{-1}(x_w^\mu)] = 1$ for all $\lambda, \mu \in R$. Like in the proof of Theorem 6.1 one must consider relations of this type separately.

Let j be as in (4). Then

$$[\psi^{-1}(x_{\delta^+(v, P_j)}^1), \psi^{-1}(x_{v^\perp}^\lambda)] \in \psi^{-1}(x_v^\lambda).$$

Assume we have shown that none of the products

$$\delta_{v, P_j}^+, \quad w\delta_{v, P_j}^+, \quad wv^\perp, \quad \text{and} \quad v^\perp w \tag{5}$$

exists. Then, since the corresponding sums are non-zero vectors, we get

$$[\psi^{-1}(x_{\delta^+(v, P_j)}^1), \psi^{-1}(x_w^\mu)], [\psi^{-1}(x_v^\lambda), \psi^{-1}(x_w^\mu)] \in C \subset Z(Y).$$

It follows that

$$1 = \left[[\psi^{-1}(x_{\delta^+(v, P_j)}^1), \psi^{-1}(x_v^\lambda)], \psi^{-1}(x_w^\mu) \right] = [\psi^{-1}(x_v^\lambda), \psi^{-1}(x_w^\mu)].$$

It remains to show that the products (5) do not exist. (Notice that this difficulty is absent in the case $P = \Delta_n$.)

For simplicity put $\delta^+ = \delta_{v, P_j}^+$. Thus δ^+ and v have the same base facet in P_{j+1} , namely P_j^\perp . (The arguments below use Proposition 3.3(a) several times.)

First observe that the inequality $\langle P_j^\perp, w \rangle > 0$ is impossible because otherwise wv would exist. This already excludes the existence of $w\delta^+$. Also, the product wv^\perp does not exist because $\langle (P_{j+1})_{v^\perp}, w \rangle = \langle P_j^-, w \rangle = 0$.

By the equations (4₁) and (4₂) we have $\langle F, \delta^+ \rangle \leq 0$ for every facet $F \subset P_{j+1}$ different from P_j^- . Since $w \in P_j^-$ the base facet $(P_{j+1})_w \subset P_{j+1}$ is different from P_j^- . Therefore $\langle (P_{j+1})_w, \delta^+ \rangle \leq 0$, and δ^+w does not exist.

Finally we have to exclude the existence of $v^\perp w$. There are two cases – either $\langle P_j^\perp, w \rangle < 0$ or $\langle P_j^\perp, w \rangle = 0$. If $\langle P_j^\perp, w \rangle < 0$ we have $\langle (P_{j+1})_w, v^\perp \rangle = \langle P_j^\perp, v^\perp \rangle = 0$ by Lemma 3.1, and we are done. If $\langle P_j^\perp, w \rangle = 0$ and $v^\perp w$ existed, then its image in P_j^- under the 90° -rotation would be vw – a contradiction because we have assumed that vw does not exist. \square

9. Functorial properties

The essential data that determines the stable elementary group $\mathbb{E}(P)$ for a balanced polytope P are

- (i) the matrix $CB(P) = (\langle F, v \rangle)$ whose rows are indexed by the column vectors v of P and whose columns are indexed by the *base facets* F of column vectors of P , and
- (ii) the partial product structure on the set of column vectors of P .

The following proposition shows that we are justified in saying that polytopes P and Q are *E-equivalent* if they coincide in these data.

Proposition 9.1. *Let P and Q be balanced polytopes, and R a ring. Suppose there exists a map $\mu : \text{Col}(P) \rightarrow \text{Col}(Q)$ such that the following conditions hold for all $v, w \in \text{Col}(P)$:*

- (i) $\langle P_w, v \rangle = \langle Q_{\mu(w)}, \mu(v) \rangle$ and
- (ii) $\mu(vw) = \mu(v)\mu(w)$ if vw exists.

(a) *Then the assignment $x_v^\lambda \mapsto x_{\mu(v)}^\lambda$ induces a homomorphism*

$$\text{St}(R, \mu) : \text{St}(R, P) \rightarrow \text{St}(R, Q).$$

(b) Suppose that for all $v \in \text{Col}(P)$ and all $w \in \text{Col}(Q) \setminus \mu(\text{Col}(P))$ the following holds: $-\mu(v) \neq w$, and none of the products $\mu(v)w$ and $w\mu(v)$ exists. (This condition obviously holds if μ is surjective.) Then one has induced homomorphisms $\mathbb{E}(R, \mu) : \mathbb{E}(R, P) \rightarrow \mathbb{E}(R, Q)$, and $K_2(R, \mu) : K_2(R, P) \rightarrow K_2(R, Q)$. $\mathbb{E}(R, \mu)$ is surjective if μ is so.

(c) If μ is bijective, then

$$\text{St}(R, P) \approx \text{St}(R, Q), \quad \mathbb{E}(R, P) \approx \mathbb{E}(R, Q), \quad K_2(R, P) \approx K_2(R, Q).$$

Proof. (a) The mapping μ extends to doubling spectra of P and Q and induces a map $\mu : \text{Col}(\mathfrak{P}) \rightarrow \text{Col}(\mathfrak{Q})$ as follows.

If P is doubled with respect to P_v , then we also double Q along $Q_{\mu(v)}$ and extend μ by setting $\mu(v^-) = \mu(v)^-, \mu(v^1) = \mu(v)^1, \mu(\delta^+) = \delta^+, \mu(\delta^-) = \delta^-$.

If Q is doubled with respect to a facet G that is not of type $Q_{\mu(v)}$ for some $v \in \text{Col}(P)$, then we put $\mu(v) = \mu(v)^-$.

Using Remark 5.2(b) one checks easily that the extension of μ again satisfies the conditions (i) and (ii).

Condition (i) implies that the product vw exists if and only if $\mu(v)\mu(w)$ exists. Furthermore it follows from Proposition 3.3(d) that $v = -w$ if and only if $\mu(v) = -\mu(w)$. In conjunction with (ii) this guarantees the compatibility of the assignment $x_v^\lambda \mapsto x_{\mu(v)}^\lambda$ with the Steinberg relations.

(b) We must check that $\text{St}(R, \mu)$ maps the center of $\text{St}(R, P)$ to the center of $\text{St}(R, Q)$. This certainly holds if x_v^λ for $v \in \text{Col}(\mathfrak{P})$ commutes with every x_w^λ for all $w \in \text{Col}(\mathfrak{Q}) \setminus \mu(\text{Col}(\mathfrak{P}))$. This is somewhat tedious, but, in view of Remark 5.2(b), straightforward to check. Hence the claim on induced homomorphisms follows from Theorem 8.4, and that on surjectivity is then trivial.

(c) is obvious in view of Proposition 8.2. \square

Remark 9.2. (a) If the matrix $\mathcal{CB}(P)$ has pairwise different rows, then it determines the partial product structure completely, since we can identify the product vw from $\mathcal{CB}(P)$ if it exists. Therefore, every other polytope Q such that $\mathcal{CB}(P) = \mathcal{CB}(Q)$ (up to a suitable bijection $\mu : \text{Col}(P) \rightarrow \text{Col}(Q)$) has the same stable Steinberg group. We will use this in Section 10.

Since we evaluate column vectors only against base facets (and *not* against *all* facets) to form $\mathcal{CB}(P)$, it is not clear whether the product vw can always be identified from $\mathcal{CB}(P)$.

(b) Though we cannot prove K_2 -functoriality for all maps μ discussed in Proposition 9.1, it is useful to note the St -functoriality, since it implies K_i -functoriality for $i \geq 3$; see [BrG5].

(c) Proposition 9.1 allows one to study polyhedral K -theory as a functor also in the polytopal argument. The map μ should be considered as a K -theoretic morphism from P to Q .

As an easy application of Proposition 9.1 one obtains

$$K_2(R, P \times Q) = K_2(R, P) \oplus K_2(R, Q).$$

for a ring R and every pair of balanced polytopes P and Q . The analogous equations hold for St and \mathbb{E} . In fact, one observes that $\text{Col}(P \times Q)$ is a disjoint union

of $\text{Col}(P)$ and $\text{Col}(Q)$ (where P and Q are considered as lattice subpolytopes of $P \times Q$), and that the column vectors coming from P are parallel to the base facets of those coming from Q , and vice versa. These properties extend to doubling spectra.

Finally we want to point out that the K -theoretic groups only depend on the projective toric variety associated with a polytope P . The *normal fan* $\mathcal{N}(P)$ of a finite convex (not necessarily lattice) polytope $P \subset \mathbb{R}^n$ is defined as the complete fan in the dual space $(\mathbb{R}^n)^* = \text{Hom}(\mathbb{R}^n, \mathbb{R})$ given by the system of cones

$$\{(\varphi \in (\mathbb{R}^n)^* \mid \max_P(\varphi) = F), F \text{ a face of } P\}.$$

Two polytopes $P, Q \subset \mathbb{R}^n$ are called *projectively equivalent* if $\mathcal{N}(P) = \mathcal{N}(Q)$. In other words, P and Q are projectively equivalent if and only if they have the same dimension, the same combinatorial type, and the faces of P are parallel translates of the corresponding ones of Q .

Remark 9.3. Let P and Q be *very ample* polytopes in the sense of [BrG1, §5]. This means that for any vertex $v \in P$ the affine semigroup $-v + (C_v \cap \mathbb{Z}^n) \subset \mathbb{Z}^n$ is generated by $-v + L_P$, where C_v is the cone in \mathbb{R}^n spanned by P at v (we assume that $P \subset \mathbb{R}^n$ and $\text{gp}(S_P) = \mathbb{Z}^{n+1}$), and similarly for Q . Then $\mathcal{N}(P) = \mathcal{N}(Q)$ if and only if the projective toric varieties $\text{Proj}(k[P])$ and $\text{Proj}(k[Q])$ are naturally isomorphic for some field k . These varieties are normal, but not necessarily projectively normal [BrG1, Example 5.5]

Projectively equivalent polytopes P and Q have the same set of column vectors: $\text{Col}(P) = \text{Col}(Q)$ (see [BrG1]), and the identity map on this set satisfies the condition of Proposition 9.1.

Proposition 9.4. *If P and Q are projectively equivalent balanced polytopes then $\text{St}(R, P) \approx \text{St}(R, Q)$ and $\mathbb{E}(R, P) \approx \mathbb{E}(R, Q)$.*

Remark 9.5. Proposition 9.4 does not generalize to unstable groups of elementary automorphisms. In fact, it is shown in [BrG1, Remark 5.3] that $\Gamma_{\mathbb{C}}(\Delta_1) \neq \Gamma_{\mathbb{C}}(2\Delta_1)$, the difference occurring exactly between $\mathbb{E}_{\mathbb{C}}(\Delta_1)$ and $\mathbb{E}_{\mathbb{C}}(2\Delta_1)$.

10. Stable groups for balanced polygons

First we classify all balanced polygons up to E -equivalence (and in some cases up to projective equivalence), introducing representatives for each class. Then we use this classification to describe the corresponding stable groups of elementary automorphisms in very explicit terms.

Theorem 10.1. *For a balanced polygon P there are only the following possibilities (up to affine-integral equivalence), each of which can in fact be realized and constitutes a E -equivalence class:*

- (a) P is a multiple of the unimodular triangle $P_a = \Delta_2$. Hence $\text{Col}(P) = \{\pm u, \pm v, \pm w\}$ and the column vectors are subject to the obvious relations,

- (b) P is projectively equivalent to the trapezoid $P_b = \text{conv}((0, 0), (0, 2), (1, 1), (0, 1))$, hence $\text{Col}(P) = \{u, \pm v, w\}$ and the relations in $\text{Col}(P)$ are $uv = w$ and $w(-v) = u$,
- (c) $\text{Col}(P) = \{u, v, w\}$ and $uv = w$ is the only relation,
- (d) $\text{Col}(P)$ has any prescribed number of column vectors, they all have the same base edge (clearly, there are no relations between them),
- (e) P is projectively equivalent to the unit lattice square P_e , hence $\text{Col}(P) = \{\pm u, \pm v\}$ with no relations between the column vectors,
- (f) $\text{Col}(P) = \{u, v\}$ so that $P_u \neq P_v$ with no relations in $\text{Col}(P)$.

Proof. It follows immediately from Proposition 9.1 that the polytopes in each class are E -equivalent, and that polytopes from different classes are not E -equivalent. Now let P be a balanced polygon.

Case 1. There are column vectors $u, v \in \text{Col}(P)$ such that $w = uv$ exists.

We claim that in this case $\text{Col}(P) \subset \{\pm u, \pm v, \pm w\}$. There exists a facet F of P such that $\langle F, v \rangle = \delta > 0$. Thus we have the following table whose entries are given by the “scalar products” of the vectors u, v, w with the facets P_u, P_v, F :

$$\begin{pmatrix} -1 & 1 & \gamma \\ 0 & -1 & \delta \\ -1 & 0 & \gamma + \delta \end{pmatrix}$$

Note that u, v, w have non-negative heights with respect to all other facets.

Evidently u and v form a basis of the lattice \mathbb{Z}^2 , and every column vector c is a linear combination of u and $v, c = \alpha u + \beta v$. Since $\langle P_u, c \rangle, \langle P_v, c \rangle \in \{0, \pm 1\}$, and at most one of these numbers can be negative, one easily checks that $\alpha \in \{-1, 0, 1\}$ and the following implications hold:

$$\alpha = 0 \implies \beta \in \{\pm 1\}, \alpha = 1 \implies \beta \in \{0, 1\}, \alpha = -1 \implies \beta \in \{0, -1, -2\}.$$

But $\alpha = -1, \beta = -2$ is impossible, since $\langle F, c \rangle \leq -2$ in this case. We have now shown that indeed $\text{Col}(P) \subset \{\pm u, \pm v, \pm w\}$.

If u, v, w are the only column vectors, then we are in case (c) and can stop the discussion, since this case is possible: see the proof of Lemma 8.3, where we have considered the polygon

$$P_c = \text{conv}((0, 0), (3, 0), (1, 2), (0, 1)).$$

So suppose at least one of $-u, -v, -w$ is a further column vector.

Assume that $-u \in \text{Col}(P)$. Then clearly $\gamma = 0$, and F is parallel to u . After a change of coordinates we can assume $u = (0, -1), v = (1, 0)$, so that $w = (1, -1)$. Since P_u is parallel to v , a parallel translation moves P into a position where the line segment $\text{conv}((0, 0), (1, 0))$ is contained in ∂P and $(1, 0)$ is a vertex. The point $(0, 1)$ must also lie in P_v , which is parallel to w . Since F is parallel to u , the lower end point of F must belong to P_u , and we can sketch part of P as in Figure 10. The “upper” end point of F must lie in $P_{-u} = P_v$ and so P has exactly the three facets

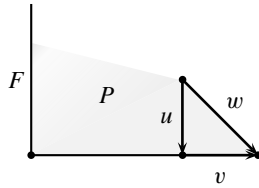


Fig. 10. Part of the polygone P

P_u, P_v, F . It follows that P is a multiple of the lattice unit triangle, and therefore $\text{Col}(P) = \{\pm u, \pm v, \pm w\}$. This is case (a).

If $-w$ is a column vector, then $-(\gamma + \delta) = -1$, and consequently $\gamma = 0, \delta = 1$. Moreover, $-w$ and, hence, u, v, w have height 0 with respect to all the other facets. Then $-u$ is a column vector, and we are again in case (a).

Therefore, in addition to u, v, w only $-v$ can be a column vector (unless we are in case (a)). Then F is the base facet of $-v$, and this implies $\delta = 1, \gamma = 0$ (since P is balanced), so that F is again parallel to u . In this case P is projectively equivalent to the polytope P_b , since all the facets except P_v and $P_{-v} = F$ must be parallel to v . (See Figure 2 where we have sketched an affine-integrally equivalent polytope).

Case 2. None of the products uv ($u, v \in \text{Col}(P)$) exists.

There are two possibilities: either all the column vectors share the same base edge or there are $u, v \in \text{Col}(P)$ with $P_u \neq P_v$. In the first situation it is clear that P can have an arbitrary number of column vectors: just consider the balanced quadrangles

$$P_{d,t} = \text{conv}((-1, 0), (3t, 0), (2t, 1), (0, 2)), \quad t \in \mathbb{N}.$$

In this situation $\text{Col}(P_{d,t}) = \{(s, -1) \mid s \in [0, t]\}$.

Now assume $P_u \neq P_v$ for some $u, v \in \text{Col}(P)$.

By Lemma 3.1 and Proposition 3.3(a) we have $\langle P_v, u \rangle = \langle P_u, v \rangle = 0$. We claim that there are only two possibilities: either $\text{Col}(P) = \{u, v\}$ or $\text{Col}(P) = \{\pm u, \pm v\}$.

First we show that any element of $\text{Col}(P)$ is parallel to either P_u or to P_v . In fact, if there is $w \in \text{Col}(P)$ which is not parallel to one of these edges then either $\langle P_v, w \rangle > 0$ or $\langle P_u, w \rangle > 0$ (for otherwise w would share the base edges with both u and v which is impossible by Lemma 3.1). But then Proposition 3.3(a) implies that one of the products wu or wv exists – a contradiction.

In particular we see that $\text{Col}(P) \subset \{\pm u, \pm v\}$. One only needs to show that if $-u \in \text{Col}(P)$ then $-v \in \text{Col}(P)$ as well. In fact, if $-u \in \text{Col}(P)$, then all the facets of P different from P_u and P_{-u} are parallel to u , and P_u and P_{-u} are parallel



Fig. 11. The polytopes $P_{d,t}$ and P_f

to v . So P is a parallelogram. Since u and v span the lattice \mathbb{Z}^2 , the parallelogram P is projectively equivalent to the unit square (up to affine-integral equivalence). In this case $\text{Col}(P) = \{\pm u, \pm v\}$.

That the case $\text{Col}(P) = \{u, v\}$ is also possible is shown by the balanced pentagon

$$P_f = \text{conv}((0, 0), (2, 0), (2, 1), (1, 2), (0, 2))$$

whose only column vectors are $(-1, 0)$ and $(0, -1)$. \square

After the classification of balanced polygons up to E -equivalence it remains to compute the stable elementary groups. In the following theorem we use block matrix notation in a self-explanatory way.

Theorem 10.2. *For a ring R we have the group isomorphisms:*

$$\mathbb{E}(R, P_a) = E(R), \tag{a}$$

$$\mathbb{E}(R, P_b) = \begin{pmatrix} E(R) & \text{End}_R(\oplus_{\mathbb{N}} R) \\ 0 & E(R) \end{pmatrix}, \tag{b}$$

$$\mathbb{E}(R, P_c) = \begin{pmatrix} E(R) & \text{End}_R(\oplus_{\mathbb{N}} R) & \text{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & E(R) & \text{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 0 & 1 \end{pmatrix}, \tag{c}$$

$$\mathbb{E}(R, P_{d,t}) = \begin{pmatrix} E(R) & \text{Hom}_R(\oplus_{\mathbb{N}} R, R^t) \\ 0 & \mathbf{Id}_t \end{pmatrix}, \quad t \in \mathbb{N}, \tag{d}$$

$$\mathbb{E}(R, P_e) = E(R) \times E(R), \tag{e}$$

$$\mathbb{E}(R, P_f) = \begin{pmatrix} E(R) & \text{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} E(R) & \text{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 1 \end{pmatrix}. \tag{f}$$

Proof. The first equation is just the definition of $E(R)$.

It is enough to prove the second, third and fourth equations – the last two follow from them since the unit square is the direct product of two line segments, and since P_f is E -equivalent to $P_{d,1} \times P_{d,1}$.

Let E_b, E_c and $E_{d,t}$ denote the groups on the right hand side of the three equations. By $A = \{1', 2', \dots\}$ and $B = \{1'', 2'', \dots\}$ we denote two mutually

disjoint copies of \mathbb{N} , which are also disjoint from the original \mathbb{Z}_+ , and introduce the following index sets:

$$\begin{aligned} I_b &= (A \times A) \cup (B \times A) \cup (B \times B) \\ I_c &= (A \times A) \cup (B \times A) \cup (B \times B) \cup (\{0\} \times A) \cup (\{0\} \times B), \\ I_{d,t} &= (A \times A) \cup (\{0, 1, \dots, t - 1\} \times A), \quad t \in \mathbb{N}. \end{aligned}$$

By identifying A and B with free R -bases of two copies of $\bigoplus_{\mathbb{N}} R$, say \mathbb{F}_A and \mathbb{F}_B , and $\{0, 1, \dots, t - 1\}$ with that of R^t , we can view the groups E_b, E_c , and $E_{d,t}$ as the corresponding subgroups of $\text{Aut}(\mathbb{F}_A \oplus \mathbb{F}_B)$, $\text{Aut}_R(\mathbb{F}_A \oplus \mathbb{F}_B \oplus R)$ and $\text{Aut}_R(\mathbb{F}_A \oplus R^t)$ in an obvious way. In particular, these groups consist of elementary automorphisms of free R -modules with respect to their distinguished bases.

For each of the three cases we denote by $e_{ij}^\lambda, \lambda \in R$ the associated elementary automorphism, (i, j) belonging to the corresponding index set. They are subject to the standard Steinberg relations.

Similar arguments as in the proof of Lemma 8.3 (and Proposition 9.1) show that $\text{St}(R, P_b)$ and $\text{St}(R, P_{d,t})$ can be assumed to be generated by symbols x_{ij}^λ , where $\lambda \in R, i \neq j$ and, respectively, $(i, j) \in I_b$ or $(i, j) \in I_{d,t}$. (Lemma 8.3 itself is about $\text{St}(R, P_c)$.) Moreover, these symbols are subject to the same Steinberg relations as the e_{ij}^λ . Therefore we have natural surjective group homomorphisms

$$\varphi_b : \text{St}(R, P_b) \rightarrow E_b, \quad \varphi_c : \text{St}(R, P_c) \rightarrow E_c, \quad \varphi_{d,t} : \text{St}(R, P_{d,t}) \rightarrow E_{d,t}.$$

By Proposition 8.2 we only need to show the equations

$$\text{Ker}(\varphi_b) = K_2(R, P_b), \quad \text{Ker}(\varphi_c) = K_2(R, P_c), \quad \text{Ker}(\varphi_{d,t}) = K_2(R, P_{d,t}).$$

First we show that $Z(E_b) = Z(E_c) = Z(E_{d,t}) = 1$ (Z is for the center). Since $Z(\mathbb{E}(R)) = 1$, the centers $Z(E_b), Z(E_c)$ and $Z(E_{d,t})$ are in the subgroups generated by $e_{ij}^\lambda, \lambda \in R$, where the (i, j) are respectively from $B \times A, (B \times A) \cup (\{0\} \times A) \cup (\{0\} \times B)$ and $\{0, 1, \dots, t - 1\} \times A$.

We have the group epimorphism $E_c \rightarrow E_b$ given by deleting the last column and row. Therefore, if $Z(E_b) = 1$ then $Z(E_c)$ is actually concentrated in the smaller subgroup, generated by e_{ij}^λ with $(i, j) \in (\{0\} \times A) \cup (\{0\} \times B)$.

Now we are done by the observation that for arbitrary homomorphisms $B_0 \in \text{End}(\bigoplus_{\mathbb{N}} R), B_1, B_2 \in \text{Hom}_R(\bigoplus_{\mathbb{N}} R, R)$ and $B \in \text{Hom}_R(\bigoplus_{\mathbb{N}} R, R^t)$ the equations

$$\begin{aligned} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \times \begin{pmatrix} 1 & B_0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & B_0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \\ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & B_1 \\ 0 & 1 & B_2 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & B_1 \\ 0 & 1 & B_2 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} A & 0 \\ 0 & \text{Id}_t \end{pmatrix} \times \begin{pmatrix} 1 & B \\ 0 & \text{Id}_t \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & \text{Id}_t \end{pmatrix} \times \begin{pmatrix} A & 0 \\ 0 & \text{Id}_t \end{pmatrix}$$

simultaneously for all $A_1, A_2, A \in \mathbb{E}(R)$ imply that B_0, B_1, B_2 and B are actually zero homomorphisms.

It follows that $K_2(R, P_b) \subset \text{Ker}(\varphi_b), K_2(R, P_c) \subset \text{Ker}(\varphi_c)$ and $K_2(R, P_{d,t}) \subset \text{Ker}(\varphi_{d,t})$.

To derive the opposite inclusions we apply the same arguments as in Proposition 8.2. They go through once we show that the analogue of the second part of the Claim in that proof remains true when we change π there by the unstable versions of each of the homomorphisms φ_b, φ_c and $\varphi_{d,t}$.

More precisely, for each natural number j we introduce the sets $I_b^{(j)}, I_c^{(j)}, I_{d,t}^{(j)}$ which are finite versions of I_b, I_c and $I_{d,t}$, defined via the sets $A^{(j)} = \{1', \dots, j'\}$ and $B^{(j)} = \{1'', \dots, j''\}$. They give rise to the corresponding unstable subgroups $E_b^{(j)} \subset E_b, E_c^{(j)} \subset E_c, E_{d,t}^{(j)} \subset E_{d,t}$. We have surjective group homomorphisms

$$\varphi_b^{(j)} : \text{St}(R, P_b)_j \rightarrow E_b^{(j)}, \quad \varphi_c : \text{St}(R, P_c)_j \rightarrow E_c^{(j)}, \quad \varphi_{d,t} : \text{St}(R, P_{d,t})_j \rightarrow E_{d,t}^{(j)},$$

whose sources are the appropriate unstable Steinberg groups. One should notice that for successive indices j these Steinberg groups may *not* correspond to successive members in the fixed doubling spectra – there may be big intervals of intermediate members.

Assume $j > 1$. We let

$$\mathfrak{U}_b, \mathfrak{V}_b \subset \text{St}(R, P_b)_j, \quad \mathfrak{U}_c, \mathfrak{V}_c \subset \text{St}(R, P_c)_j, \quad \mathfrak{U}_{d,t}, \mathfrak{V}_{d,t} \subset \text{St}(R, P_{d,t})_j$$

denote the subgroups which are the same for these unstable Steinberg groups as \mathfrak{U}^{i+1} and \mathfrak{V}^{i+1} for $\text{St}_R(P_{i+1})$ in the proof of Proposition 8.2.

What we want to show is that the restrictions

$$\varphi_b^{(j)}|_{\mathfrak{U}_b}, \quad \varphi_b^{(j)}|_{\mathfrak{V}_b}, \quad \varphi_c^{(j)}|_{\mathfrak{U}_c}, \quad \varphi_c^{(j)}|_{\mathfrak{V}_c}, \quad \varphi_{d,t}^{(j)}|_{\mathfrak{U}_{d,t}}, \quad \varphi_{d,t}^{(j)}|_{\mathfrak{V}_{d,t}}$$

are all injective group homomorphisms. We have the natural embeddings

$$E_b^{(j)} \subset E_{2j}(R), \quad E_c^{(c)} \subset E_{2j+1}(R), \quad E_{d,t}^{(j)} \subset E_{j+t}(R).$$

Under these embeddings the generators of $\mathfrak{U}_b (\mathfrak{U}_c, \mathfrak{U}_{d,t})$ are sent to standard elementary matrices e_{st}^λ with the same t . Similarly, the generators of $\mathfrak{V}_b (\mathfrak{V}_c, \mathfrak{V}_{d,t})$ are sent to standard elementary matrices e_{st}^λ with the same s . This, clearly, gives the result. \square

As the reader may guess Theorem 10.2 implies that the K_2 for balanced polygons is always either the usual K_2 or twice K_2 . We postpone the proof of this fact to [BrG5] where we treat all higher polyhedral K -groups simultaneously, the polyhedral Milnor K -group being a basic ingredient in the higher theory. Speaking loosely, all higher syzygies between elementary automorphisms come from unit simplices – a stable higher version of Theorem 2.2 in the polygonal case.

However, such a nice matrix theoretical interpretation of the stable groups of elementary automorphisms as in Theorem 10.2 is no longer possible for higher dimensional polytopes as explained in the example below. This makes the computation of polyhedral K -groups challenging.

Example 10.3. In the proof of Corollary 3.4 we have discussed the pyramid P over the unit square (see Figure 3). This polytope has 8 column vectors and 5 facets, each of which is the base facet for at least one column vector. In the table ($\langle F, v \rangle$) the rows corresponding to the column vectors whose base facet is the unit square have *two* entries 1. Moreover, every edge represents a column vector. All of this makes it impossible to construct a matrix representation in which elementary automorphisms are mapped to standard elementary matrices. (Though, we have the canonical embedding $\mathbb{E}_R(P) \rightarrow E_5(R)$, $5 = \#L_P$.)

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