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A remark on regularity of powers and products of ideals

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ABSTRACT

We give a simple proof for the fact that the Castelnuovo–Mumford regularity and related invariants of products of powers of ideals are asymptotically linear in the exponents, provided that each ideal is generated by elements of constant degree. We provide examples showing that the asymptotic linearity is false in general. On the other hand, the regularity is always given by the maximum of finitely many linear functions whose coefficients belong to the set of the degrees of generators of the ideals.

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1. Introduction

Let I be an homogeneous ideal of a polynomial ring $R = K[x_1, \dots, x_n]$. Cutkosky, Herzog and Trung [7] and, independently, Kodiyalam [10] proved that the Castelnuovo–Mumford regularity $\text{reg}(I^a)$ of the powers of I is a linear function of $a \in \mathbb{N}$ for large a . In [7, Remark pg. 252] it is asserted that the same result holds for products of powers of ideals I_1, \dots, I_m , i.e., $\text{reg}(I_1^{a_1} \cdots I_m^{a_m})$ is a linear function in $a = (a_1, \dots, a_m) \in \mathbb{N}^m$ if $a_i \gg 0$ for every i . We show that this is actually the case when each ideal I_i is generated by polynomials of a given degree, but false in general. The method of proof allows us to easily generalize the results to ideals in arbitrary standard graded K -algebras.

We refer the reader to [5] for basic commutative algebra.

After the first version of this note had been uploaded to [arXiv.org](https://arxiv.org), M. Chardin informed us that the main result is a special case of a theorem proved by Bagheri, Chardin and Há [2, Theorem 4.6]. We hope that our simple approach to the problem and the accompanying examples are nevertheless welcome.

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2. Regularity for powers and products

For a finitely generated graded non-zero R -module M and a nonnegative integer $j \leq \text{pd}_R(M)$ we denote the largest degree of a minimal generator of the j -th syzygy module by $t_j^R(M)$, and, by convention, set $t_j^R(M) = -\infty$ if $j > \text{pd}_R(M)$. By definition one has $\text{reg}_R(M) = \max\{t_j^R(M) - j : j \geq 0\}$.

Let I_1, \dots, I_m be non-zero homogeneous ideals of R and let f_{i1}, \dots, f_{ig_i} be a minimal homogeneous generating system of I_i . Set $d_{ij} = \text{deg } f_{ij}$ and

$$B = R[z_{ij} : 1 \leq i \leq m, 1 \leq j \leq g_i] = K[x_1, \dots, x_n, z_{ij} : 1 \leq i \leq m, 1 \leq j \leq g_i].$$

To simplify the exposition we set

$$I^a = I_1^{a_1} \dots I_m^{a_m} \quad \text{for } a \in \mathbb{N}^m,$$

and we say that a formula (or a property) holds for $a \gg 0$ if there exist a $b \in \mathbb{N}^m$ such that it holds for every $a \in b + \mathbb{N}^m$.

The multigraded Rees algebra $R(I_1, \dots, I_m)$ can be seen as the subalgebra of $R[t_1, \dots, t_m]$ whose elements have the form $\sum_{a \in \mathbb{N}^m} F_a t^a$ with $F_a \in I^a$. The polynomial ring $R[t_1, \dots, t_m]$ is naturally $\mathbb{Z} \times \mathbb{Z}^m$ -graded if we extend the \mathbb{Z} -grading on R by setting $\text{deg}(x_i) = (1, 0)$ and $\text{deg } t_j = (0, e_j)$ where e_j denotes the j -th unit vector in \mathbb{Z}^m . Evidently $R(I_1, \dots, I_m)$ is a graded subalgebra. It has a structure of B -module via the R -algebra map sending z_{ij} to $f_{ij}t_i$. This map is $\mathbb{Z} \times \mathbb{Z}^m$ -graded if we equip B with the graded structure induced by the assignment $\text{deg}(x_i) = (1, 0)$ and $\text{deg}(z_{ij}) = (d_{ij}, e_i)$.

Let $A = K[z_{ij} : 1 \leq i \leq m, 1 \leq j \leq g_i] \subset B$ with the induced $\mathbb{Z} \times \mathbb{Z}^m$ -graded structure. Note that A has no elements of degree $(g, 0) \in \mathbb{Z} \times \mathbb{Z}^m$ such that $g \neq 0$. Therefore, given a finitely generated $\mathbb{Z} \times \mathbb{Z}^m$ -graded A -module M , the maximum \mathbb{Z} -degree in an a -graded component,

$$\rho_M(a) = \sup\{i \in \mathbb{Z} : M_{(i,a)} \neq 0\}, \quad a \in \mathbb{Z}^m,$$

is finite. The key to the result about the behavior of regularity is a description of $\rho_M(a)$: it is the supremum over the values of finitely many linear functions $L_k(a)$ where each L_k is defined on a certain subset of \mathbb{Z}^m . In order to describe these subsets we set $[m] = \{1, \dots, m\}$, and

$$\text{supp } L = \{i \in [m] : \alpha_i \neq 0\} \subset [m]$$

for a linear polynomial $L = \sum_{i=1}^m \alpha_i u_i + \alpha_0$. The nonzero coefficients of the relevant linear functions are given by \mathbb{Z} -degrees d_{ij} of the indeterminates of A :

Lemma 2.1. *Let M be a finitely generated $\mathbb{Z} \times \mathbb{Z}^m$ -graded A -module. Then there exist $w_1, \dots, w_c \in \mathbb{Z}^m$ and linear functions L_1, \dots, L_c on \mathbb{Z}^m ,*

$$L_k(u) = \sum_{i=1}^m \lambda_{ki} u_i + \lambda_{k0}, \quad \lambda_{k0} \in \mathbb{Z}, \quad \lambda_{ki} \in \{0\} \cup \{d_{i1}, \dots, d_{ig_i}\}, \quad i = 1, \dots, m$$

such that

$$\rho_M(a) = \sup\left\{L_k(a) : 1 \leq k \leq c \text{ and } a \in w_k + \sum_{i \in \text{supp } L_k} \mathbb{N}e_i\right\}$$

for every $a \in \mathbb{Z}^m$. In particular we have

$$\rho_M(a) = \sup\{L_k(a) : 1 \leq k \leq c \text{ and } \text{supp } L_k = [m]\} \quad \text{for } a \gg 0.$$

Proof. We represent M as a quotient F/U of the graded free module F by a graded submodule U . Then we replace U with its initial submodule $\text{in}(U)$ with respect to some term order. Next we filter $F/\text{in}(U)$ so that the successive quotients are shifted copies of quotients of A by monomial prime ideals of A . It follows that the (multigraded) Hilbert series

$$\text{HS}_M(x, s) = \sum_{(i,a) \in \mathbb{Z} \times \mathbb{Z}^m} \dim M_{(i,a)} x^i s^a \in \mathbb{Q}[[x, s_1, \dots, s_m]]$$

can be written as the sum of the Hilbert series of residue class rings $A/P_k(-v_k, -w_k)$ where P_k an ideal generated by a subset of the variables of A and $(-v_k, -w_k) \in \mathbb{Z} \times \mathbb{Z}^m$ and is a shift. That is,

$$\text{HS}_M(x, s) = \sum_{i=1}^c x^{v_k} s^{w_k} \text{HS}_{A/P_k}(x, s)$$

and therefore

$$\rho_M(a) = \sup\{\rho_{A/P_k}(a - w_k) + v_k : k = 1 \dots, c\}.$$

Set

$$S_k = \{i : \text{there exists } j \text{ such that } z_{ij} \notin P_k\}$$

and

$$\lambda_{ki} = \max\{d_{ij} : z_{ij} \notin P_k\}$$

if $i \in S_k$ and $\lambda_{ki} = 0$ otherwise. Since

$$\text{HS}_{A/P_k}(x, s) = \frac{1}{\prod_{z_{ij} \notin P_k} (1 - x^{d_{ij}} s_i)}$$

we have

$$\rho_{A/P_k}(a) = \begin{cases} \sum_{j=1}^m \lambda_{ki} a_i & \text{if } a \in \sum_{i \in S_k} \mathbb{N} e_i, \\ -\infty & \text{otherwise.} \end{cases}$$

Summing up, we obtain the desired description where L_k is defined as

$$L_k(u) = \sum_{i=1}^m \lambda_{ki}(u_i - w_{ki}) + v_k. \quad \square$$

Set $\mathfrak{m} = (x_1, \dots, x_n)$ and $\mathcal{R} = R(I_1, \dots, I_m)$. Consider the Koszul complex $K(\mathfrak{m}, \mathcal{R})$ and its homology $H(\mathfrak{m}, \mathcal{R})$. By construction $H_j = H_j(\mathfrak{m}, \mathcal{R})$ is a finitely generated $\mathbb{Z} \times \mathbb{Z}^m$ graded B -module annihilated by \mathfrak{m} and $\dim_K H_j(\mathfrak{m}, \mathcal{R})_{(u,a)} = \beta_{ju}(I^a)$. Hence H_j is a finitely generated $\mathbb{Z} \times \mathbb{Z}^m$ graded A -module and

$$\rho_{H_j}(a) = t_j^R(I^a).$$

Then Lemma 2.1 applies so that $t_j^R(I^a)$ is given as the supremum of finitely many linear functions each of which is evaluated on a translated (possibly lower dimensional) non-negative orthant. Indeed, one can

explicitly compute such a representation by following the steps described in the proof of Lemma 2.1 and applying them to the Koszul homology module H_i .

For simplicity we state only the main consequence about the asymptotic behavior of $t_i(I^a)$ and $\text{reg}(I^a)$. It is of course determined by the linear functions L_k with $\text{supp } L_k = [m]$.

Theorem 2.2. *Let I_1, \dots, I_m be given as above. Then, for every $0 \leq j \leq n - 1$, there exist $b_j \in \mathbb{N}$ and linear functions $L_k^{(j)}(u) = \sum_{i=1}^m \lambda_{ki}^{(j)} u_i + \lambda_{k0}^{(j)}$ with $1 \leq k \leq b_j$ such that $\lambda_{ki}^{(j)} \in \{d_{i1}, \dots, d_{ig_i}\}$ and $\lambda_{k0}^{(j)} \in \mathbb{Z}$ and such that*

$$t_j^R(I^a) = \sup\{L_k^{(j)}(a) : 1 \leq k \leq b_j\} \quad \text{for } a \gg 0.$$

In particular, $\text{pd}_R(I^a)$ is constant for $a \gg 0$ and

$$\text{reg}(I^a) = \sup\{L_k^{(j)}(a) - j : 0 \leq j \leq n - 1 \text{ and } 1 \leq k \leq b_j\} \quad \text{for } a \gg 0.$$

If I_i is generated in a single degree, say d_i , then there is only one choice for $\lambda_{ki}^{(j)}$, $i > 0$, namely d_i . If this holds for all ideals I_i , then the supremum is obviously given by a single linear polynomial, and we obtain

Corollary 2.3. *If each I_i is generated in a single degree, say d_i , then $\text{pd}_R(I^a)$ is constant for $a \gg 0$, say equal to p , and for each j , $0 \leq j \leq p$, there exists a linear polynomial $L^{(j)}(u) = \sum_{i=1}^m d_i u_i + \lambda_0^{(j)}$ with $\lambda_0^{(j)} \in \mathbb{Z}$ such that*

$$t_j^R(I^a) = L^{(j)}(a) \quad \text{for } a \gg 0.$$

In particular, there exists an linear polynomial $L(u) = \sum_{i=1}^m d_i u_i + \lambda_0$ such that

$$\text{reg}(I^a) = L(a) \quad \text{for } a \gg 0.$$

The conclusion of the corollary holds for a single ideal without any hypothesis on the degrees of the generators since the supremum of finitely many linear functions on \mathbb{Z} is always given by a single one for large values of the argument.

Remark 2.4. Let J_i be a homogeneous reduction of I_i for every i . Then $\mathcal{R} = R(I_1, \dots, I_m)$ is a finitely generated algebra over $R(J_1, \dots, J_m)$. It follows that the Koszul homology $H(x, \mathcal{R})$ is finitely generated over a polynomial ring whose variables have degree $(v, e_i) \in \mathbb{Z} \times \mathbb{Z}^m$ where v varies in the set of degrees of the generators of J_i . Hence the coefficients $\lambda_{ki}^{(j)}$ in Theorem 2.2 can be taken in the set of the degrees of the generators of J_i . Therefore Corollary 2.3 holds more generally if each I_i has a reduction generated in a single degree. Note that Kodiyalam proved in [10] that for a single ideal I_1 one has $\text{reg}(I_1^a) = ad + b$ for $a \in \mathbb{N}$ and $a \gg 0$ where d is the minimum of $t_0(J)$ where J varies in the set of the homogeneous reduction of I_1 .

Remark 2.5. Under the assumption that I_i is generated in degree d_i , in [4, Remark 3.2.] we have observed that $\text{reg}(I^a) \leq \sum_{i=1}^m d_i a_i + \text{reg}_0 \mathcal{R}$ for all $a \in \mathbb{N}^m$ where $\text{reg}_0 \mathcal{R}$ is the regularity computed according to the \mathbb{Z} -graded structure of $\mathcal{R} = R(I_1, \dots, I_m)$ induced by inclusion $R \rightarrow \mathcal{R}$. So the constant λ_0 in Corollary 2.3 is $\leq \text{reg}_0 \mathcal{R}$. But in general the inequality can be strict. For example, already for $m = 1$ if I_1 is an ideal such that $\{a \in \mathbb{N} : \text{reg } I_1^a > ad_1\}$ is finite and not empty, then $\lambda_0 = 0$ and $\text{reg}_0 \mathcal{R} > 0$. Examples of ideals of this kind are described in [3] and [6].

Remark 2.6. So far we have considered ideals in polynomial rings. If R is not a polynomial ring (but still standard graded), then the projective dimension and the regularity of R -modules computed via syzygies

can be infinite. Nevertheless, one has a bound for $t_j^R(I^a)$ for all j as in [Theorem 2.2](#). Instead of the Koszul complex, the resolution of K over the polynomial ring, one must use the free resolution of K over R (of infinite length if R is not a polynomial ring).

In order to generalize [Theorem 2.2](#) and [Corollary 2.3](#) to a general result for ideals in an arbitrary standard graded algebra R , we consider the regularity based on local cohomology. It is defined by

$$\text{Reg}_R(M) = \sup\{i + j : H_m^i(M)_j \neq 0\}$$

When R is a standard graded polynomial ring over a field, then $\text{reg}_R(M) = \text{Reg}_R(M)$ by a theorem of Eisenbud and Goto (for example, see [\[5, Theorem 4.3.1\]](#)).

Theorem 2.7. *The conclusions of [Theorem 2.2](#) and [Corollary 2.3](#) hold for ideals in arbitrary standard graded K -algebras, if reg is replaced by Reg .*

Proof. We write R as a residue class ring of a standard graded polynomial ring S over K . Then $\text{Reg}_R(I^a) = \text{Reg}_S(I^a) = \text{reg}_S(I^a)$. Therefore it can be computed from the Koszul homologies $H_j(\mathfrak{m}, I^a)$ where \mathfrak{m} is now the graded maximal ideal of S .

We set $\mathcal{R} = R(I_1, \dots, I_m)$ as above. Then $H_j(\mathfrak{m}, I^a)$ is the degree a component of $H_j(\mathfrak{m}, \mathcal{R})$ in the \mathbb{Z}^m -grading. After this observation the proof of [Theorem 2.2](#) can be copied. \square

Remark 2.8. In [\[8\]](#) the author provides a linear upper bound for $\text{Reg}(I^a)$ where I_1, \dots, I_m are ideals in a standard graded algebra R over an Artinian local ring.

Remark 2.9. The results above can be immediately generalized to the behavior of $\text{reg}(I^a M)$ (or $\text{Reg}(I^a M)$) where M is a finitely generated graded R -module. Instead of the multi-Rees \mathcal{R} alone, one considers the finitely generated \mathcal{R} -module

$$\mathcal{M} = M(I_1, \dots, I_m) = \bigoplus_a I^a M.$$

The Koszul homology $H_j(\mathfrak{m}, \mathcal{M})$ is a finitely generated A -module, and the remaining arguments need no change. The case $m = 1$ is due to Trung and Wang [\[11\]](#).

3. Examples

We now discuss an example showing that, in general, $t_j^R(I^a)$ and $\text{reg}(I^a)$ need not be linear functions of a for $a \gg 0$.

Example 3.1. Consider the ideals $I_1 = (x, y^2)$, $I_2 = (x^2, y)$ and in the polynomial ring $R = K[x, y]$. One has

$$t_i(I_1^{a_1} I_2^{a_2}) = \begin{cases} \max\{2a_1 + a_2, a_1 + 2a_2\} + i & \text{if } i = 0, 1 \text{ and } (a_1, a_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}, \\ 0 & \text{if } i = 0 \text{ and } (a_1, a_2) = (0, 0). \end{cases}$$

So that

$$\text{reg}(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2, a_1 + 2a_2\} \text{ for all } (a_1, a_2) \in \mathbb{N}^2.$$

To establish these formulas we propose two approaches. First we follow the strategy outlined above and deduce the result from the study of the multigraded Hilbert series of the Koszul homology modules $H_i(x, y, R(I_1, I_2))$. Secondly we write down the resolution of $I_1^{a_1} I_2^{a_2}$ directly.

The presentation of the Rees algebra $\mathcal{R} = R(I_1, I_2)$ is given by:

$$B/P \simeq \mathcal{R}, \quad B = R[z_{11}, z_{12}, z_{21}, z_{22}],$$

with $\mathbb{Z} \times \mathbb{Z}^2$ graded structure induced by $\deg(x) = \deg(y) = (1, 0, 0)$, $\deg(z_{11}) = (1, 1, 0)$, $\deg(z_{12}) = (2, 1, 0)$, $\deg(z_{21}) = (2, 0, 1)$, $\deg(z_{22}) = (1, 0, 1)$. Here

$$P = (y^2 z_{11} - x z_{12}, y z_{21} - x^2 z_{22}, x y z_{11} z_{22} - z_{12} z_{21}).$$

Set $A = K[z_{11}, z_{12}, z_{21}, z_{22}]$ so that $H_0 = H_0(x, y, \mathcal{R}) = \mathcal{R}/(x, y) = A/(z_{12} z_{21})$. Using a filtration or by direct computation one has:

$$HS_{H_0}(x, s_1, s_2) = \frac{1}{(1 - x s_1)(1 - x^2 s_1)(1 - x s_2)} + \frac{x^2 s_2}{(1 - x s_1)(1 - x s_2)(1 - x^2 s_2)}. \tag{3.1}$$

From this we deduce that

$$t_0(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2 : a \in \mathbb{N}^2, a_1 + 2a_2 : a \in e_2 + \mathbb{N}^2\},$$

that is,

$$t_0(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2, a_1 + 2a_2\} \quad \text{for all } a \in \mathbb{N}^2.$$

Similarly for $H_1 = H_1(x, y, \mathcal{R})$ one obtains:

$$HS_{H_1}(x, s) = \frac{x^3 s_1}{(1 - x s_1)(1 - x^2 s_1)(1 - x s_2)} + \frac{x^3 s_2}{(1 - x s_1)(1 - x^2 s_2)(1 - x s_2)}. \tag{3.2}$$

Then it follows that

$$t_1(I_1^{a_1} I_2^{a_2}) = \sup\{2a_1 + a_2 + 1 : (a_1, a_2) \in e_1 + \mathbb{N}^2, a_1 + 2a_2 + 1 : (a_1, a_2) \in e_2 + \mathbb{N}^2\}.$$

Hence

$$t_1(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2 + 1, a_1 + 2a_2 + 1\}$$

for all $a \in \mathbb{N}^2 \setminus \{(0, 0)\}$.

Let us now sketch the approach via free resolutions. First one checks that $I_1^{a_1} I_2^{a_2}$ is minimally generated by the $1 + a_1 + a_2$ elements

$$\begin{matrix} x^{a_1} y^{a_2}, & x^{a_1-1} y^{a_2+2}, & x^{a_1-2} y^{a_2+4}, & \dots, & y^{a_2+2a_1}, \\ \mid & & & & \\ x^{a_1} y^{a_2}, & x^{a_1+2} y^{a_2-1}, & x^{a_1+4} y^{a_2-2}, & \dots, & x^{a_1+2a_2}. \end{matrix}$$

This confirms our formula

$$t_0(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2, a_1 + 2a_2\} \quad \text{for all } a \in \mathbb{N}^2.$$

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