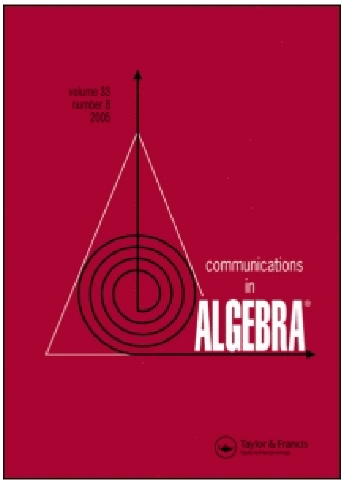

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Stanley-reisner rings with pure resolutions

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STANLEY–REISNER RINGS WITH PURE RESOLUTIONS

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Introduction

In this paper we continue our study, started in [3], of simplicial complexes Δ whose Stanley–Reisner rings have pure minimal free resolutions. (A complex of free modules over a polynomial ring is called *pure* if its maps can be represented by matrices φ_i all of whose non-zero entries have the same degree, which may depend on i). While it might be too ambitious to strive for their complete combinatorial classification, it seems to be a reasonable project to find the numerical invariants of all pure free resolutions that arise from simplicial complexes, at least in the presence of the Cohen–Macaulay property.

A very satisfactory case is that in which Δ is the maximal complex supported by its 1-skeleton Δ_1, or, equivalently, in which the defining ideal I_Δ of the Stanley–Reisner ring $K[\Delta]$ is generated by monomials of degree 2 (K is an arbitrary field). Fröberg [5] has shown that $K[\Delta]$ has a 2-linear resolution if and only if Δ_1 is a chordal graph, and we complement his result by proving that in all the other cases with a pure resolution Δ is a multi-cone over a 1-dimensional cycle. (A graph Γ on the vertex set V is chordal if $\#W = 3$ for all $W \subset V$ such that Γ_W is a cycle.) In particular the 1-dimensional simplicial complexes with pure free resolutions are completely classified, as well as those arising from a partially ordered set.

We next address the case of dimension 2. The main difficult case is that in which I_Δ is generated by degree 3 monomials and Δ is doubly Cohen–Macaulay. For each number n of vertices there exists at most one numerical type of resolution, and we succeed in constructing corresponding simplicial complexes for $n \equiv 0 \bmod 2$ and $n = 7$. A computer-aided exhaustive search for $n = 9$ has shown that no such complex exists, and we conjecture that the non-existence persists for all $n \equiv 1 \bmod 2$, $n \geq 9$.

Taylor [11] constructed a free resolution for an arbitrary ideal generated by monomials. It is not very difficult to find all the ideals I_Δ whose Taylor resolution is pure. (Then I_Δ has also a pure minimal free resolution.) Unfortunately one obtains only few Cohen–Macaulay rings this way.

In the final section of the paper we determine all ideals I_Δ that are generated by monomials of degree m and have a minimal pure free resolution in which each non-zero entry of the syzygy matrix of I_Δ has degree $m-1$. For example, if $m = 3$, then the monomials generating I_Δ can be represented by a subconfiguration of the Fano plane.

1 Simplicial complexes and free resolutions

We assume that the reader is familiar with the standard combinatorial and algebraic terminology developed in, e.g., Bruns and Herzog [2], Hibi [7], or Stanley [10]. Therefore we only introduce some notation and recall a few basic facts about simplicial complexes and free resolutions. Throughout this paper Δ denotes a simplicial complex on the vertex set V; its number of vertices is n, and it has dimension $d-1$. For subsets U, W of V the simplicial complex Δ_W consists of all those faces of Δ that are contained in W, whereas $\Delta \setminus U$ is formed by all faces that are disjoint to U. If $W = \{w\}$, then we write Δ_w and $\Delta \setminus w$ for $\Delta_{\{w\}}$ and $\Delta \setminus \{w\}$ respectively.

Let K be a field, and $S = K[X_1, \dots, X_n]$ be a polynomial ring over K whose indeterminates correspond bijectively to the vertices of Δ. The Stanley–Reisner ring $K[\Delta]$ of Δ with coefficients in K is the residue class ring S/I_Δ whose defining ideal is generated by those squarefree monomials in $X_1, \dots, X_n$ that represent non-faces of Δ. Since I_Δ is generated by homogeneous polynomials, $K[\Delta]$ carries a natural grading, and, like every finitely generated graded S-module, it has a minimal graded free resolution

$$(1)\ F_\bullet\colon 0 \to \bigoplus_j S(-j)^{\beta_{pj}} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_2} \bigoplus_j S(-j)^{\beta_{1j}} \xrightarrow{\varphi_1} S \to K[\Delta] \to 0;$$

here $S(-j)$ is a free S-module of rank 1 whose base element has degree j, and the maps φ_i are degree-preserving. In particular each entry of (a matrix representing) φ_i is a homogeneous polynomial; since the resolution is minimal, a non-zero entry has positive degree. (Because I_Δ is generated by monomials, $K[\Delta]$ has even a multi-graded minimal resolution, but we will not use the multi-graded structure directly.) We consider the image of φ_i to be generated by its rows.

The number p is the *projective dimension* of $K[\Delta]$. One always has $n - d \leq p \leq n$, and if $n - d = p$, then $K[\Delta]$ is *Cohen–Macaulay*. This holds true, if and only the dual complex $\mathrm{Hom}_S(F_\bullet, S(-n))$ is also acyclic, and if so, then it is the minimal graded free resolution of the graded canonical module of $K[\Delta]$.

The β_{ij} are called the *graded Betti numbers* of $K[\Delta]$, and their sums

$$\beta_i = \sum_{j \in \mathbb{Z}} \beta_{ij}$$

are the *Betti numbers* of $K[\Delta]$. Hochster [8, Theorem (5.1)] gave a combinatorial formula for the (multi-)graded Betti numbers of Δ:

$$(2) \qquad \beta_{ij} = \sum_{W \subset V,\ \#W = j} \dim_K \widetilde{H}_{j-i-1}(\Delta_W; K).$$

A *pure* free resolution is distinguished by the fact that for each i all the non-zero entries of φ_i have the same degree. If it is pure, then the resolution (1) can be written in the form

$$0 \longrightarrow S(-c_p)^{\beta_p} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_2} S(-c_1)^{\beta_1} \xrightarrow{\varphi_1} S \longrightarrow K[\Delta] \longrightarrow 0.$$

Its *shift type* is the sequence $c_p, \ldots, c_1$. Often it is however more instructive to use its *degree type* formed by the degrees $c_p - c_{p-1}, \ldots, c_2 - c_1, c_1$ of the entries of the maps φ_i. We say that the resolution is *m-pure* if $c_1 = m$, and that it is *m-linear* if its degree type is $1 \ \ldots \ 1\ m$. It is an obvious consequence of Hochster's formula that $K[\Delta_W]$ has a pure resolution for every subset $W \subset V$ along with $K[\Delta]$; its shift type is simply a truncation of the shift type of $K[\Delta]$.

Suppose $K[\Delta]$ is Cohen–Macaulay with a pure resolution of shift type $c_p, \ldots, c_1$, then the Betti numbers β_i and the multiplicity $e(K[\Delta])$ of $K[\Delta]$ can be expressed in terms of these shifts:

$$(3) \quad \text{(i)} \quad \beta_i = (-1)^{i+1} \prod_{j \neq i} \frac{c_j}{c_j - c_i} \qquad \text{and} \qquad \text{(ii)} \quad e(K[\Delta]) = \frac{1}{p!} \prod_{i=1}^{p} c_i;$$

see [2, 4.1.15] or the articles of Herzog and Kühl [6] and Huneke and Miller [9]. Note that $e(K[\Delta])$ is the number of facets of Δ. Since the β_i and $e(K[\Delta])$ are integers, the equations (3) impose restrictions on the shifts c_i.

The Gorenstein rings among the $K[\Delta]$ with pure resolution are easy to describe: $K[\Delta]$ is Gorenstein if and only if it is Cohen–Macaulay and its degree type is a palindrome. In fact, if $K[\Delta]$ is Gorenstein, then it has a self-dual resolution, and therefore a palindromic degree type; conversely, if the degree type is a palindrome, then $\beta_{n-d} = 1$, and a Cohen–Macaulay ring of type 1 is Gorenstein.

In the following we will use some easy observations about the shifts in graded resolutions. For the resolution (1) we set

$$m_i = \min\{j : \beta_{ij} \neq 0\} \qquad \text{and} \qquad M_i = \max\{j : \beta_{ij} \neq 0\}$$

and call these numbers the *minimal* and *maximal* shifts of $K[\Delta]$. Clearly, the resolution is pure if and only if $m_i = M_i$ for all i. Hochster's formula implies that $M_i \leq n = \#V$ for all i.

Proposition 1.1. *One has $m_1 < \cdots < m_p$, and if $K[\Delta]$ is Cohen–Macaulay, then also $M_1 < \cdots < M_p$.*

PROOF. Set $k = m_{i+1}$. Then the row of φ_{i+1} corresponding to any of the base elements of $S(-k)^{\beta_{i+1,k}}$ has a non-zero entry. If the corresponding column has shift s, then $m_{i+1} = k > s \geq m_i$. In the Cohen–Macaulay case we apply this argument to the dual complex, which is a minimal graded free resolution of the canonical module of $K[\Delta]$ and whose minimal shifts m_i^* are given by $n - M_{p-i}$. □

Suppose that I_Δ is generated by monomials of the same degree g. If these are pairwise coprime, then they form a regular sequence; the Koszul complex resolves $K[\Delta]$, which therefore has a pure resolution of degree type $g \ldots g$. Combinatorially this means that Δ is a multi-join of copies of the boundary complex of a $(g-1)$-simplex. Below we will often have to identify this case by a somewhat weaker information:

Proposition 1.2. *Let I_Δ be generated by monomials $M_1, \ldots, M_N$ of the same degree g. Then the following are equivalent:*
(a) *$M_1, \ldots, M_N$ are pairwise coprime;*
(b) *the non-zero entries of the map φ_2 have degree g.*

In fact, if M_i and M_j are not coprime, then the relation

$$(\mathrm{lcm}(M_i, M_j)/M_i)M_i - (\mathrm{lcm}(M_i, M_i)/M_j)M_j = 0$$

forces φ_2 to have entries of degree $< g$.

For a Cohen–Macaulay complex Δ the linearity of the resolution of $K[\Delta]$ can be recognized by the h-vector of Δ.

Proposition 1.3. *Suppose that $K[\Delta]$ is Cohen–Macaulay. Then it has an m-linear resolution if and only if*

$$h(\Delta) = \left(1, n-d, \binom{n-d+1}{2}, \ldots, \binom{n-d+m-2}{m-1}\right).$$

The numerator of the Hilbert function of $K[\Delta]$ (over the denominator $(1-t)^d$) is given by $h_0 + h_1 t + \cdots + h_s t^s$, where the h_i denote the components of $h(\Delta)$. Therefore the proposition is (almost) identical with Exercise 4.1.17 of [2], and since it is an exercise, we omit the proof.

In the following the m-Cohen–Macaulay property introduced by Baclawski [1] will play a certain role. One says that Δ is *m-Cohen–Macaulay* (over K) if for all $W \subset V$, $0 \leq \#W \leq m-1$, the restricted complex $\Delta \setminus W$ is Cohen–Macaulay and has the same dimension as Δ. This condition can be slightly weakened:

Proposition 1.4. *Suppose that $m \geq 2$.*

(a) *The following properties are equivalent:*

(i) *Δ is m-Cohen–Macaulay over K;*

(ii) *for each $W \subset V$, $\#W = m-1$, the restricted complex $\Delta \setminus W$ is Cohen–Macaulay over K and has the same dimension as Δ;*

(iii) *$K[\Delta]$ is Cohen–Macaulay, and the minimal and maximal shifts in its resolution satisfy the equations $m_i = M_i = d + i$ for $i = n-d-m+2, \ldots, i = n-d$.*

(b) *Under the equivalent conditions of* (a), *$K[\Delta]$ has a pure resolution if (and only if) the Stanley–Reisner rings $K[\Delta \setminus W]$ have pure resolutions of the same shift type for all $W \subset V$, $\#W = m-1$.*

PROOF. (a) We only need to show the implication (ii) $\Rightarrow$ (i) for $m = 2$; the general case follows by induction. If $\beta_{n-p,j} \neq 0$ for some $j \neq n$, then also $\beta_{n-p,j}(K[\Delta \setminus v]) \neq 0$ for some vertex v of V: choose v outside a proper subset W of V that contributes to $\beta_{n-p,j}$. However, this is impossible since, by our assumption, $K[\Delta \setminus v]$ has projective dimension $\#(V \setminus \{v\}) - d = n - 1 - d$.

Since β_{n-d} is non-zero and $\beta_{n-d} = \beta_{n-p,n}$, we have $M_{n-d} = m_{n-d} = n$. If the projective dimension of $K[\Delta]$ would exceed $n-d$, then $M_{n-d+1} > m_{n-d} = n$, in contradiction to the inequality noticed above.

The equivalence of (i) and (iii) follows from similar arguments. It is due to Baclawski [1], and can also be found in [2, Section 5.6].

(b) Again it is enough to treat the case $m = 2$. It follows from the preceding argument that the homology of the whole complex Δ only contributes to β_{n-p}. If there were proper subsets $W, W' \subset V$ of different cardinalities such that the homologies of Δ_W and $\Delta_{W'}$ contribute to the same Betti number of $K[\Delta]$, then for vertices $w \notin W$ and $w' \notin W'$ the Stanley–Reisner rings $K[\Delta \setminus w]$ and $K[\Delta \setminus w']$ cannot have pure resolutions of the same shift type. □

2 Ideals generated in degree 2

In this section we will exploit Hochster's formula in order to give a complete classification of those simplicial complexes Δ for which $K[\Delta]$ has a 2-pure resolution. In particular I_Δ is generated by elements of degree 2, what can be described combinatorially by saying that Δ is the (necessarily) unique maximal member of the family of simplicial complexes with the same 1-skeleton as Δ.

Fröberg [5] classified all Δ for which $K[\Delta]$ has a 2-linear resolution: these are exactly those simplicial complexes for which I_Δ is generated by monomials of degree 2 and whose 1-skeleton is a chordal graph. Therefore it is enough to find the simplicial complexes with a 2-pure, but not 2-linear resolution.

Theorem 2.1. *The Stanley–Reisner ring of a simplicial complex Δ has a 2-pure, but not 2-linear resolution exactly in the following two cases:*
(i) *Δ is a multi-cone over a 1-dimensional cycle; in particular the free resolution of $K[\Delta]$ is of degree type 2 1 ... 1 2;*
(ii) *Δ is the multi-join of copies of • •; the free resolution of $K[\Delta]$ is the Koszul complex of I_Δ, and thus of degree type 2 2 ... 2 2.*

Proof. That the Stanley–Reisner ring of a (multi-cone over a) 1-dimensional cycle has a free resolution of the type given follows easily from Hochster's formula or from the fact that a sphere is Gorenstein, in conjunction with our observations about the shifts in minimal free resolutions of Cohen–Macaulay rings.

If Δ is the multi-join of copies of • •, then I_Δ is generated by pairwise coprime monomials of degree 2. These monomials form a regular sequence, and therefore the Koszul complex of I_Δ resolves $K[\Delta]$.

Conversely suppose that $K[\Delta]$ has a 2-pure, but not 2-linear resolution. Then, by virtue of Fröberg's theorem, the 1-skeleton of Δ has a cycle Γ of length at least 4 without a chord. Note that $\Gamma = \Delta_{\mathrm{vert}(\Gamma)}$. If $V = \mathrm{vert}(\Gamma)$, then $\Gamma = \Delta$, and we are done. Otherwise there exists a vertex x outside Γ. We have $\widetilde{H}_1(\Gamma, K) \neq 0$, and because of our hypothesis Hochster's formula implies that $\widetilde{H}_0(\Delta_W, K) = 0$ for all subsets W of V with $\#W = g - 1$ where g is the length of Γ.

Let v be a vertex of Γ with neighbouring vertices u and w on Γ. We choose $W = (\mathrm{vert}(\Gamma) \setminus \{u, w\}) \cup \{x\}$. Then Δ_W is connected, and this is only possible if $\{v, x\}$ is a face of Δ. If x is the only vertex of Δ outside Γ, then we are again done: the 1-skeleton of Δ is the 1-skeleton of a cone over Γ, and therefore Δ is such a cone.

Finally assume that there exists a vertex $y \neq x$ of Δ, $y \notin \mathrm{vert}(\Gamma)$. We choose non-neighbouring vertices u and w of Γ, and set $W = \{u, w, x, y\}$. If $\{x, y\} \notin \Delta$, then, according to our previous observation, Δ_W is a 4-cycle. By Hochster's formula this implies that the free resolution of $K[\Delta]$ is of degree type $a_p\, a_{p-1} \ldots a_2$ 2 2. As noticed in Section 1, this is only possible if I_Δ is generated by pairwise coprime monomials of degree 2, and thus case (ii) of the theorem applies.

The only remaining possibility is that $\{x, y\} \in \Delta$. Since x, y are arbitrary vertices outside Γ, the complex Δ must indeed be a multi-cone over Γ. ❑

The following corollaries were proved in the authors' previous article [3].

Corollary 2.2. *Let Δ be a simplicial complex of dimension one. Then its Stanley–Reisner ring $K[\Delta]$ has a pure resolution if and only if Δ is one of the the following:* (i) *complete graph;* (ii) *forest;* (iii) *cycle.*

In fact, I_Δ contains all squarefree monomials of degree 3. So, if I_Δ is generated by monomials of degree 3, then Δ is a complete graph. Otherwise I_Δ is generated by degree 2 monomials, and we can apply the theorem. If $K[\Delta]$ has a 2-linear resolution, then, as an immediate consequence of Hochster's formula, Δ must be a forest. If Δ is of the type (i) of the theorem, then it is a cycle, and if it is of type (ii), then it is necessarily a 4-cycle.

Corollary 2.3. *Suppose that P is a Cohen–Macaulay partially ordered set of rank $d-1$ (with $d \geq 2$) which possesses the rank decomposition $P = P_0 \cup P_1 \cup \cdots \cup P_{d-1}$ with each $\# P_i \geq 2$. Then the Stanley–Reisner ring $K[\Delta(P)]$ of P has a pure resolution if and only if we have one of the following:*
(i) *$d = 2$ and P is a cycle;*
(ii) *$d \geq 3$, each P_i contains exactly 2 elements, and elements $x \neq y$ are incomparable if and only if they are of the same rank;*
(iii) *the number of maximal chains of P is $\# P - d + 1$.*

Note that $I_{\Delta(P)}$ is generated by the monomials representing pairs of incomparable elements. The cases (i) and (ii) correspond to the cases (i) and (ii) of the theorem, and in presence of the Cohen–Macaulay condition, (iii) is equivalent to the 2-linearity of the resolution. In fact, the number of maximal chains of P equals the sum over all entries of the h-vector. These are non-negative, and $h_0 = 1$, $h_1 = \# P - d$. Now one applies Proposition 1.3.

3 Simplicial complexes of dimension 2

Let Δ be a simplicial complex of dimension 2. Then I_Δ contains all squarefree monomials of degree 4. Thus, if it is generated by such monomials, then Δ is the 2-skeleton of a simplex. If I_Δ is generated by degree 2 monomials and $K[\Delta]$ has a pure free resolution, then the structure of Δ is completely described by Fröberg's theorem or Theorem 2.1. It remains to consider the case in which I_Δ has a system of generators consisting of monomials of degree 3.

It is impossible that the second syzygy matrix of $K[\Delta]$ has entries of degree 3, since in this case I_Δ would be generated by pairwise coprime monomials. But then Δ is the multi-join of copies of , and has dimension at least 3. There are only finitely many cases in which the second syzygy matrix has degree 2; these will be enumerated in Proposition 5.1 below. Therefore the condition in Theorem

3.1 that the second syzygy matrix of $K[\Delta]$ is generated by linear elements is harmless.

Of course $K[\Delta]$ may have a 3-linear resolution, and already at this point a severe complication comes up: whether $K[\Delta]$ has a 3-linear resolution, depends in general on the field of coefficients – in contrast to the results in Section 1, where the choice of the field K is irrelevant. The standard example of such behaviour is a triangulation of the real projective plane for which $K[\Delta]$ has a 3-linear resolution if and only if char $K \neq 2$; see, e.g., [2, p. 228].

The potential numerical types of 3-pure, but not 3-linear resolutions of Stanley–Reisner rings of 2-dimensional simplicial complexes are essentially given by the next theorem. The question that remains open in (b) will be discussed in Remark 3.4.

Theorem 3.1. *Let Δ be a simplicial complex of dimension 2 for which $K[\Delta]$ has a 3-pure, but not 3-linear resolution. Assume that the second syzygy matrix of $K[\Delta]$ is generated by linear elements.*

(a) *Then Δ is Cohen–Macaulay over K, and one of the following conditions hold:*

(i) *$\#V \geq 6$ is even, Δ is 2-Cohen–Macaulay over K, and the resolution degrees of $K[\Delta]$ are* 2 1 ... 1 3*; one has*

$$f(\Delta) = (n, n(n-1)/2, n(n-2)/2),$$
$$h(\Delta) = (1, n-3, (n-2)(n-3)/2, (n-2)/2);$$

(ii) *$\#V \geq 7$ is odd, Δ is 3-Cohen–Macaulay over K, and the resolution degrees of $K[\Delta]$ are* 1 2 1 ... 1 3*; one has*

$$f(\Delta) = (n, n(n-1)/2, n(n-1)/2),$$
$$h(\Delta) = (1, n-3, (n-2)(n-3)/2, n-1).$$

(b) *Simplicial complexes of type* (a)(i) *exist for all even numbers $\#V \geq 6$, and a simplicial complex of type* (a)(ii) *exists for $\#V = 7$.*

PROOF. Part (b) is covered by Propositions 3.2 and 3.3 below.

For part (a) we set $n = \#V$ and $p = \operatorname{proj\,dim} K[\Delta]$ as usual. Then the shifts $c_i, i \geq 1$, in the minimal free resolution of $K[\Delta]$ satisfy the inequalities $c_{i-1} < c_i$ and $i + 2 \leq c_i \leq n$. In particular we have $p \leq n-2$, and if $p = n-2$, then $c_i = i + 2$ for $i = 1, \dots, p$. This however is impossible, since we assume that $K[\Delta]$ does not have a 3-linear resolution. It follows that $p \leq n-3$, and therefore $p = n-3$: note that $p \geq n - \dim K[\Delta] = n - 3$. Furthermore $c_p = n$, and there exists exactly one index i with $c_i = c_{i-1} + 2$. In other words, the degree type of the minimal resolution is

$$1 \dots 1 2 1 \dots 1 3.$$

Assume that the '2' occurs at place m, and that it is followed on the left by at least two degrees 1. We choose a subset W of V of cardinality $m+3$, and replace Δ by Δ_W. Then Δ_W fulfills all our assumptions, whence we may assume that $m = p - 2$. An evaluation of the Herzog–Kühl formula (3)(i) yields

$$\beta_1 = m(m+4)(m+5)(m+6)/(6(m+3)),$$

a number which for $m \in \mathbb{Z}$ is never an integer. Hence at most a single '1' may follow the '2'.

Now we replace Δ by $\Delta' = \Delta_W$ where $W \subset V$ has cardinality $n-1$. Evaluating the formula (3)(i) for Δ' gives $m(m+4)/6$ for its first Betti number, and so m must be even. Therefore, if n is odd, exactly one '1' follows the '2', and if n is even, the degree 2 matrix is the last one in the resolution.

The statement concerning the 2-Cohen–Macaulay property in (i) and the 3-Cohen–Macaulay property in (ii) follows immediately from 1.4.

In order to compute the f-vector and the h-vector, one determines the multiplicity $e(K[\Delta])$ from the Huneke–Miller formula (3)(ii), and uses that $e(K[\Delta]) = f_2(\Delta)$. □

For even $n \geq 6$ we now construct the simplicial complexes whose existence has been claimed in part (b) of the theorem.

Proposition 3.2. *Let $n \geq 6$ be an even integer. Then the simplicial complex Δ whose facets are*

$$\{i, i+1, i+2\}, \{i, i+1, i+4\}, \ldots, \{i, i+1, i+(n-2)\}, \quad i = 1, \ldots, n,$$

is 2*-Cohen–Macaulay and has a pure resolution of degree type* 2 1 … 1 3 *over every field K. (The numbers giving the vertices are to be read modulo n).*

Proof. In view of Proposition 1.4(b) we must show that for all vertices v the restriction $\Delta' = \Delta \setminus v$ is Cohen–Macaulay with 3-linear resolution. By its definition, Δ is invariant under the action of the cyclic group whose generator sends vertex i to vertex $i+1$, $i = 1, \ldots, n$. Therefore we may assume that $v = 1$.

We order the faces of Δ' in blocks B_j, $j = 4, \ldots, n$. The blocks B_j are constructed by descending induction, starting with $j = n$: B_j contains all facets F with $j \in F$ and $F \notin B_n \cup \cdots \cup B_{j+1}$. If j is even, then the facets in B_j are ordered in the sequence

$$\{2,3,j\}, \{4,5,j\}, \ldots, \{j-4, j-3, j\},$$
$$\{3, j-1, j\}, \{5, j-1, j\}, \ldots, \{j-3, j-1, j\}, \{j-2, j-1, j\},$$

and if j is odd, then they are arranged in the sequence

$\{3, 4, j\}, \{5, 6, j\}, \dots, \{j-4, j-3, j\},$
$\{2, j-1, j\}, \{4, j-1, j\}, \dots,, \{j-3, j-1, j\}\{j-2, j-1, j\}.$

For a shelling we remove the facets as follows: first those of B_n in the order specified (i.e. starting with $\{2, 3, n\}$), then those of B_{n-1} etc., the last facet being $\{2, 3, 4\}$. It is not hard to see that at each step the intersection of the removed facet with the subcomplex generated by the 'later' facets is generated by 1-faces. Therefore Δ' is shellable.

The number of facets of Δ' is $(n-3)(n-2)/2$, and since it has the maximum number of vertices and 1-faces, its h-vector is $(1, n-4, (n-3)(n-4)/2)$. It follows from 1.3 that Δ' has a 3-linear resolution. □

Before we write down the simplicial complex Δ with 7 vertices whose existence has been claimed in 3.1(b), let us observe that each 1-face of Δ must be contained in exactly three facets; this claim will be justified in Remark 3.4(a) below. Each 1-face of the simplicial complex with 6 vertices given in 3.2 is contained in at least two and at most three facets, and therefore we simply try to add a seventh vertex and all the 'missing' facets. This idea works, but it is useful to specify Δ with a different enumeration of its vertices:

Proposition 3.3. *The simplicial complex Δ with the facets*

$$\{i, i+1, i+2\}, \{i, i+1, i+4\}, \{i, i+2, i+4\}, \quad i = 1, \dots, 7,$$

has a pure resolution of degree type 1 2 1 3.

PROOF. It is enough to show that $\Delta \setminus v$ has a pure resolution of degree type 2 1 3 for all vertices v. By the cyclic symmetry of Δ we may assume $v = 7$. The substitution $4 \mapsto 1, 6 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 1 \mapsto 5, 5 \mapsto 6$ transforms $\Delta \setminus 7$ into the complex with 6 vertices specified in Proposition 3.2. □

In the following remark we discuss the problem of finding simplicial complexes of resolution type 3.1(a)(ii) with at least 9 vertices. Moreover we list some examples with other resolution types.

Remark 3.4. (a) Let Δ be a simplicial complex with an odd number n of faces and resolution type 3.1(a)(ii), and choose a vertex v of Δ. Then the link of v in Δ is a 3-connected graph in which each vertex is adjacent to exactly 3 edges. In fact, Δ has the h-vector $(1, n-3, (n-2)(n-3)/2, (n-2)/2)$, and $\Delta \setminus x$ has the resolution type 3.1(a)(i), and therefore the h-vector $(1, n-4, (n-3)(n-4)/2, (n-3)/2)$, whence $\text{link}_\Delta\{x\}$ has the h-vector $(1, n-3, (n+1)/2)$, and thus the f-vector $(n-1, 3(n-1)/2)$. Note that $\text{link}_\Delta\{v\}$ is 3-Cohen–Macaulay along with Δ; this follows easily from the exact sequence connecting the Stanley–Reisner rings of $\text{star}_\Delta\{v\}$, Δ and $\Delta \setminus v$. Hence $\text{link}_\Delta\{v\}$ must be 3-connected, and in particular each vertex must be adjacent to at least, and thus exactly three edges. This implies that each 1-face of Δ is contained in exactly 3 facets.

(b) One sees easily that for each vertex v of one of the 'even' simplicial complexes constructed in Proposition 3.2 there exist two vertices u, w such that the 1-faces $\{v, u\}$ and $\{v, w\}$ are contained in exactly $n/2$ facets, whereas all other 1-faces $\{v, x\}$ are contained in exactly 2 facets.

This phenomenon is further illustrated by the following assertion whose verification we leave to the reader. Let Δ be a simplicial complex of type 3.1(a)(i), and $v \neq w$ vertices of Δ; then the following are equivalent: (i) $\Delta \setminus \{v, w\}$ is Cohen–Macaulay; (ii) the link of w in $\Delta \setminus v$ is a tree; (iii) $\{v, w\}$ is contained in exactly $n/2$ facets of Δ.

(c) It follows from (a) and (b) that it is impossible for $n \geq 8$ to supplement such an 'even' complex by one more vertex in order to obtain an 'odd' complex of resolution type 3.1(a)(ii).

The example constructed in 3.2 for $n = 6$ is uniquely determined by its resolution type, and therefore the same holds for the example with $n = 7$ in 3.3. However, for $n = 8$ there exists a second isomorphism type which also carries a cyclic symmetry and has the property observed in (b), and there are many more isomorphism types without these extra features; see (e).

(d) An exhaustive computer search has shown that there exist no simplicial complexes of type 3.1(a)(ii) with 9 vertices (at least not over a field of characteristic 13). This leads us to conjecture that for $n \geq 9$ there exist no simplicial complexes of this type.

The following criterion could be used to detect them. Let Δ be a 2-dimensional simplicial complex with an odd number $n \geq 7$ of vertices such that I_Δ is generated by degree 3 monomials; then the following are equivalent: (i) $K[\Delta]$ has a resolution of type 3.1(a)(ii); (ii) $f_2(\Delta \setminus \{v, w\}) = (n-3)(n-4)/2$ and $\widetilde{H}_1(\Delta \setminus \{v, w\}; K) = 0$ for all vertices $v \neq w$.

In fact, if condition (i) is fulfilled, then the shift $n - 2$ does not appear in the resolution, so that $\widetilde{H}_i(\Delta \setminus \{v, w\}; K) = 0$ for all i. That $\Delta \setminus \{v, w\}$ has exactly $(n-3)(n-4)/2$ facets, follows from the formula (3)(ii). Conversely, if (ii) is satisfied, then $\widetilde{H}_2(\Delta \setminus \{v, w\}; K) \cong \widetilde{H}_1(\Delta \setminus \{v, w\}; K) = 0$, and since $\widetilde{H}_0(\Delta \setminus \{v, w\}; K) = 0$ by hypothesis, the shift 7 does not appear in the resolution. Using Proposition 1.1 the reader may check, that this is the case if and only if the resolution has the desired type.

(e) In the search we have found 85 isomorphism classes of simplicial complexes with 8 vertices which (over a field of characteristic 13) have resolution type 3.1(a)(i) and the following property: each 1-face is contained in at least two and at most three facets. Several of these complexes can be supplemented by a ninth vertex such that the resulting complex is 2-Cohen–Macaulay and has the property stated in (a). In particular this property does not exclude the existence of a simplicial complex of type 3.1(a)(ii) for $n \geq 9$.

(f) We have searched for further 2-Cohen–Macaulay complexes Δ with cyclic symmetry and pure resolutions, and found the following types.

n	d	degree type of resolution
7	4	1 2 4
8	4	2 1 1 4
	5	2 1 5
9	4	2 1 1 1 4
		1 2 1 1 4
	5	1 2 1 5
	6	2 1 6
10	5	2 1 1 1 5
	6	2 1 1 6

4 Pure Taylor resolutions

In this section we shall entirely work in an algebraic setup; therefore we use the language of ideals generated by monomials instead of that of simplicial complexes. Let I be an ideal in the polynomial ring $R = K[X_1, \dots, X_n]$ that is generated by (not necessarily squarefree) monomials $M_1, \dots, M_N$. Taylor [11] (see also Eisenbud [4]) constructed a free resolution of I given by the complex

$$T_\bullet : 0 \longrightarrow F_N \xrightarrow{\varphi_N} F_{N-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

with $F_k = \bigwedge^k R^N$, $k = 1, \dots, N$ and

$$\varphi_k(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} \frac{\operatorname{lcm}(M_{i_1}, \dots, M_{i_k})}{\operatorname{lcm}(M_{i_1}, \dots, \widehat{M}_{i_j}, \dots, M_{i_k})} e_{i_1} \wedge \dots \wedge \widehat{e}_{i_j} \wedge \dots \wedge e_{i_k}.$$

Here $e_1, \dots, e_N$ is the canonical basis of R^N, and $\widehat{\ }$ indicates that M_{i_j} and e_{i_j} are to be omitted. In order to make the maps in this complex homogeneous of degree zero one has to assign a grading to $\bigwedge^k R^N$ in such a way that the degree of the basis element $e_{i_1} \wedge \cdots \wedge e_{i_k}$ equals the degree of the monomial $\operatorname{lcm}(M_{i_1}, \dots M_{i_k})$. It is clear that $T_\bullet$ is a pure resolution with shifts $c_N, \dots, c_1$ if and only if $\deg(\operatorname{lcm}(M_{i_1}, \dots, M_{i_k}))$ depends only on k, and not on the monomials M_{i_j}.

The resolution above is already defined over the ring $\mathbb{Z}$ of integers, and we could replace the field K by $\mathbb{Z}$. In general the Taylor resolution is not a minimal one. If it is pure, then R/I also has a pure minimal free resolution over $\mathbb{Z}$, and therefore over an arbitrary field. In this case the projective dimension p of R/I is the minimum index for which $c_p = c_{p+1}$ (with $d_{N+1} = d_N$).

We say that I is of *pure Taylor type* $c_N, \ldots, c_1$ if it is generated by squarefree monomials and the Taylor resolution of R/I is a pure free resolution with shifts $c_N, \ldots, c_1$. (The restriction to squarefree monomials is not essential since every monomial ideal can be 'deformed' into a squarefree one with the same numerical data.) In the following we want to determine all the shifts $c_N, \ldots, c_1$ which can occur in a pure Taylor resolution. Furthermore we will see that the ideal I whose resolution has the shifts $c_N, \ldots, c_1$ is uniqely determined.

It is best to translate the problem of finding all ideals of pure Taylor type into one of Boolean algebra. Let B be a boolean algebra (with join $\sqcup$, meet $\sqcap$, complement $^-$, and maximal element $\hat{1}$). A function $V\colon B \to \mathbb{Z}$ is called a *valuation* if it satisfies the identity $V(b_1 \sqcap b_2) + V(b_1 \sqcup b_2) = V(b_1) + V(b_2)$. We say that V in *non-negative*, if $V(b) \geq 0$ for all b. The valuations on B form a free abelian group. If B is finite, then every choice of values $v(\alpha)$ for the atoms α of B uniquely extends to a valuation of B; therefore the group of valuations of V is a finitely generated free abelian group of rank equal to the number of atoms of B.

Proposition 4.1. *Let $\mathbf{B}_N$ be the free boolean algebra on N elements $\mu_1, \ldots, \mu_N$. Then the following are equivalent:*
(a) *there exists a squarefree monomial ideal I of pure Taylor type $c_N, \ldots, c_1$;*
(b) *there exists a non-negative valuation V on $\mathbf{B}_N$ with $V(\mu_{i_1} \sqcup \ldots \sqcup \mu_{i_k}) = c_k$ for all $k = 1, \ldots, N$ and $1 \leq i_1 < \cdots < i_k \leq N$.*

PROOF. Suppose that $I \subset K[X]$ is of pure Taylor type $c_N, \ldots, c_1$. We identify each squarefree monomial M with the set of variables dividing M. Then the assignment $\mu_i \mapsto M_i$ extends to a unique homomorphism Φ of Boolean algebras from $\mathbf{B}_N$ into the set of subsets of X. We define V by $V(\beta) = \#(\Phi(\beta))$ for all elements $\beta \in \mathbf{B}_N$. It is obvious that V has the properties required for (b).

Conversely, let V be a non-negative valuation on $\mathbf{B}_N$. For each atom $\alpha \in \mathbf{B}_N$ we choose indeterminates $X_{\alpha,1}, \ldots, X_{\alpha,V(\alpha)}$, and define the monomial M_i, $i = 1 \ldots, N$, by

$$M_i = \prod_{\alpha \leq \mu_i} X_{\alpha,1} \cdots X_{\alpha,V(\alpha)}.$$

It is clear that the ideal generated by $M_1, \ldots, M_N$ has pure Taylor type $c_N, \ldots, c_1$. □

The proof of Proposition 4.1 describes how to construct all ideals of pure Taylor type. Furthermore it shows that such an ideal is uniquely determined by its type. In order to test whether there exists an ideal of type $c_N, \ldots, c_1$, we define a valuation on $\mathbf{B}_N$ by setting

$$V(\mu_{i_1} \sqcup \ldots \sqcup \mu_{i_k}) = c_k$$

for all $k = 1, \ldots, N$ and $1 \le i_1 < \cdots < i_k \le N$, and $V(\hat{1}) = c_N$. It is an easy exercise in boolean algebra to show that this is indeed possible, and to compute the values of V on the atoms of $\mathbf{B}_N$.

Each atom α has the form $\alpha = \bar{\mu}_{i_1} \sqcap \cdots \sqcap \bar{\mu}_{i_{N-k}} \sqcap \mu_{j_1} \sqcap \cdots \sqcap \mu_{j_k}$. By the symmetry in the definition of V, $V(\alpha)$ depends only on k, and we set $a_k = V(\alpha)$. Using the principle of inclusion-exclusion one obtains

$$a_k = v(\alpha) = \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} c_{N-k+i}$$

for $k = 1, \ldots, n$, and $a_0 = 0$. The numbers a_k, $k = 1, \ldots, n$, are called the *atomic weights* of I.

Proposition 4.2. *There exists an ideal I of pure Taylor type $c_N, \ldots, c_1$ if and only if*

$$\sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} c_{N-k+i} \ \ge\ 0$$

for $k = 1, \ldots, N$. Furthermore I is uniquely determined (up to isomorphism).

Using the atomic weights we can reformulate the construction in the proof of Proposition 4.1 as follows. Let Δ be an $(N-1)$-simplex. Then we associate with each $(k-1)$-face of Δ a family of a_k indeterminates, and define the monomials $M_1, \ldots, M_N$ by forming for each vertex v of Δ the product over all indeterminates associated with a face containing v.

It is not difficult to characterize those ideals I which yield Cohen-Macaulay residue class rings. Unfortunately their number is quite small.

Proposition 4.3. *Let $I \subset R = K[X_1, \ldots, X_n]$ be an ideal of pure Taylor type. Then the following are equivalent:*
(a) *R/I is Cohen-Macaulay;*
(b) *I is unmixed;*
(c) *the generators $M_1, \ldots, M_N$*
  (i) *form a regular sequence (equivalently, are pairwise coprime), or*
  (ii) *are of the form $M_i = L_1 \cdots \widehat{L_i} \cdots L_N$ where $L_1, \ldots, L_N \neq 1$ are pairwise coprime squarefree monomials. (In this case the residue class ring R/I has projective dimension 2.)*

PROOF. The implication (a) $\Rightarrow$ (b) is well-known.

For (b) $\Rightarrow$ (c) we choose an index k with $a_k \neq 0$. The set $\{1, \ldots, N\}$ can be covered by $m = \lceil N/k \rceil$ subsets $J_1, \ldots, J_m$ of cardinality k. Each J_k corresponds to the atom $\alpha_k = \bar{\mu}_{i_1} \sqcap \cdots \sqcap \bar{\mu}_{i_{N-k}} \sqcap \mu_{j_1} \sqcap \cdots \sqcap \mu_{j_k}$ with $\{j_1, \ldots, j_k\} = J_k$. For each v we choose an indeterminate X_{u_v} among those indeterminates which correspond to the atom α_v in the sense of the proof of Proposition 4.1. Then each monomial M_i contains one of the X_{u_v} as a factor. Therefore the ideal $(X_{u_1}, \ldots, X_{u_m})$ is a prime ideal containing I, and a fact a minimal such ideal: only the monomials M_i with $i \in J_v$ contain X_{u_v}.

On the other hand we can also find a minimal covering of $\{1, \ldots, N\}$ by $N - k + 1$ subsets of cardinality k, and therefore a minimal prime ideal of I which has height $N - k + 1$. The equation $\lceil N/k \rceil = N - k + 1$ has only the solutions $k = 1$ and $k = N - 1$.

Furthermore these arguments show that there is exactly one index k with $a_k \neq 0$ if I is unmixed. Thus an unmixed ideal I has the atomic weights (i) $a_1 > 0$ and $a_i = 0$ for $i \neq 1$ or (ii) $a_{N-1} > 0$ and $a_i = 0$ for $i \neq N - 1$. In case (i) the monomials $M_1, \ldots, M_N$ are pairwise coprime, and therefore form a regular sequence. If $N \geq 2$, then in case (ii) the monomials M_i are given by $M_i = L_1 \cdots \widehat{L_i} \cdots L_N$ where $L_1, \ldots, L_N$ are pairwise coprime squarefree monomials of the same degree.

The implication (c) $\Rightarrow$ (a) follows immediately from the fact that in both cases (i) and (ii) the codimension of I equals the projective dimension of R/I. □

The reader may check the following more general assertion: let I be an ideal of pure Taylor type $c_N, \ldots, c_1$ with atomic weights $a_1, \ldots, a_N$, and set $u = \min\{k \colon a_k \neq 0\}$ and $w = \max\{k \colon a_k \neq 0\}$; then

$$\begin{aligned} \operatorname{codim} I &= \lceil N/w \rceil, \\ \operatorname{proj\,dim} R/I &= N - u + 1 \\ &= \max\{\operatorname{codim} P \colon P \text{ is a minimal prime overideal of } I\}. \end{aligned}$$

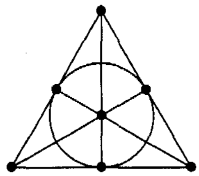
5 Special cases

In this section we want to determine all squarefree monomial ideals with a pure minimal free resolution of degree type $g_p \ \ldots \ g_2 \, m{-}1 \ m$. Since the case $m = 2$ is completely covered by Fröberg's theorem and Theorem 2.1 above, it is sufficient to treat the case $m \geq 3$. Since it is not necessary to specify the polynomial ring in which the ideals under consideration live, we simply denote it by R.

There is a trivial class of ideals I whose free resolution has the required type: choose a regular sequence $L_1, \ldots, L_N$ of monomials of degree $m - 1$ and multiply each of them by a new indeterminate Z. Then R/I is resolved by the Koszul complex associated with the sequence $L_1, \ldots, L_N$ except that the rows of the first matrix are formed by $ZL_1, \ldots, ZL_N$. We call the sequence $ZL_1, \ldots, ZL_N$ a *spider* (with body Z and legs $L_1, \ldots, L_N$).

We start with the case in which $m = 3$; it is slightly more complicated than that of $m \geq 4$. The following propsition supplements Theorem 3.1 since it covers a case excluded in that theorem.

Proposition 5.1. *Let $M_1, \ldots, M_N$ be squarefree monomials generating an ideal with a minimal pure resolution of degree type $g_p \ \ldots \ g_2 \ 2 \ 3$. Suppose that $M_1, \ldots, M_N$ is not a spider. Then $M_1, \ldots, M_N$ are represented by lines of the Fano plane*



where the vertices are the indeterminates and each line stands for the product of its vertices. In particular we have $3 \leq N \leq 7$. The degree type of the resolution is 1 2 3 *for $N = 3, 4$, and* 1 1 2 3 *for $N = 5, 6, 7$. (There are* 2 *non-isomorphic cases for $N = 4$.)*

Conversely, each set of lines of the Fano plane represents a squarefree monomial ideal with a minimal pure resolution of degree type $g_p \ \ldots \ g_2 \ 2 \ 3$.

PROOF. The rows of the second matrix in the minimal free resolution of R/I can always be chosen among the rows of the second matrix in the Taylor resolution. Since their entries have degree 2, the greatest common divisor of two of $M_1, \ldots, M_N$ is either 1 or an indeterminate.

We next observe that it is impossible to have two coprime monomials, say UVW and XYZ. In fact, if another one of $M_1, \ldots, M_N$ is a monomial in these 6 variables, then it has exactly two variables in common with UVW or XYZ, which is impossible. But if there is no such monomial, then we must have a row with degree 3 entries in the second matrix of the free resolution. This follows from Hochster' formula (2) in Section 1 or by elementary arguments.

Therefore, if $i \neq j$, then the greatest common divisor of M_i and M_j is an indeterminate. Elementary combinatorial arguments yield that the monomials $M_1, \ldots, M_N$ are indeed represented by lines in the Fano plane.

Conversely, given a subset of lines of the Fano plane, one can easily compute the minimal free resolution of the residue class ring it define. □

For $m \geq 4$ one gets the following classification; again we may leave the details of the proof to the reader. (For $m = 3$ the ideals listed in the proposition below leave out that one which corresponds to all 7 lines of the Fano plane.)

Proposition 5.2. *Let $m \geq 4$, and let X be an symmetric $(m+1) \times (m+1)$ matrix whose entries above the main diagonal are pairwise different indeterminates, whereas those on the main diagonal are* 1*. Let $L_1, \ldots, L_{m+1}$ be the products of the entries of the rows of X.*

Then, for a sequence $M_1, \ldots, M_N$ of squarefree monomials that is not a spider, the residue class ring R/I, $I = (M_1, \ldots, M_N)$, has a pure minimal free resolution of degree type $g_p \ \ldots \ g_2 \ m{-}1 \ m$ if and only if $M_1, \ldots, M_N$ is given by a subset of $L_1, \ldots, L_{m+1}$.

The degree type is $1 \ 2 \ \ldots \ m{-}1 \ m$ if $N = m + 1$, and $u \ u{+}1 \ \ldots \ m{-}1 \ m$ if $N = m + 1 - u$, $u = 1, \ldots, m{-}2$.

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