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Unimodular triangulations of simplicial cones by short vectors



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ABSTRACT

We establish a bound for the length of vectors involved in a unimodular triangulation of simplicial cones. The bound is exponential in the square of the logarithm of the multiplicity, and improves previous bounds significantly. The proof is based on a successive reduction of the highest prime divisor of the multiplicity and uses the prime number theorem to control the length of the subdividing vectors.

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1. Introduction

In this paper we discuss the triangulation of simplicial cones C into unimodular subcones by “short” vectors. Length is measured by the basic simplex Δ_C of C that is spanned by the origin and the extreme integral generators of C : we want to find an upper bound for the dilatation factor c for which all subdividing vectors are contained in $c\Delta_C$. Roughly speaking, the larger the multiplicity μ of C (given by the lattice normalized volume of Δ_C), the more subdivision steps are to be expected, and they inevitably increase the length of the subdividing vectors. Therefore μ is the natural parameter on which estimates for c must be based, at least for fixed dimension d .

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A prominent case in which bounds for c come up is the desingularization of toric varieties. The standard argument applied in this situation leads to rather bad bounds. A slight improvement was reached by Bruns and Gubeladze [1, Theorem 4.1] who gave a bound that is better, but still exponential in μ . The main result of this paper is a bound essentially of order $\mu^{\log \mu}$ for fixed dimension d (Corollary 4.6). The next goal would be a bound that is polynomial in μ , but we do not know if such exists.

It seems that the only general technique for triangulating a (simplicial) cone into unimodular subcones is successive stellar subdivision: one chooses an integral vector x in C and replaces C by the collection of subcones that are spanned by x and the facets of C that are visible from C . This simple procedure allows successive refinement of triangulations if simultaneously applied to all cones that contain x .

We start from the basic and rather easy observation that a unimodular triangulation by iterated subdivision can be reached very quickly by short vectors if μ is a power of 2. In order to exploit this observation for arbitrary μ , two crucial new ideas are used: (i) not to diminish the multiplicity μ in every subdivision step (as usually), but to allow it to grow towards a power of 2, and (ii) to control this process by the prime number theorem. While we formulate our results only for simplicial cones, they can easily be generalized (see Remark 4.7.)

The bound in [1, Theorem 4.1] was established in order to prove that multiples cP of lattice polytopes P can be covered by unimodular simplices as soon as c exceeds a threshold that depends only on the dimension d (and not on P or μ). If only unimodular covering is aimed at (for polytopes or cones), one can do much better than for triangulations: the threshold for c has at most the order of d^6 . In particular, it is independent of μ . See Bruns and Gubeladze [2, Theorem 3.23]. A polynomial bound of similar magnitude for the unimodular covering of cones is given in [2, Theorem 3.24]. These polynomial bounds are based on the first part of von Thaden's PhD thesis [5], whereas the results of this paper cover the second part of [5].

The most challenging problem in the area of this paper is to show that the multiples cP of a lattice polytope P have unimodular triangulations for *all* $c \gg 0$. The best known result in arbitrary dimension is the Knudsen–Mumford–Waterman theorem that guarantees the existence of such c . We refer the reader to [2, Chapter 3] and to [3] for an up-to-date survey. The paper [3] contains an explicit upper bound for c .

For unexplained terminology and notation we refer the reader to [2].

2. Auxiliary results

Our first theorem will show that there is a sublinear bound in $\mu(C)$ on the length of the subdividing vectors in a unimodular triangulation if the multiplicity $\mu(C)$ of the cone C is a power of 2. We always assume that a simplicial cone C is generated by its extreme integral generators, i.e., the primitive integral vectors contained in the extreme rays. The multiplicity then is the lattice normalized volume of the simplex Δ_C spanned by them and the origin.

Note that for unimodular cones D the Hilbert basis $\text{Hilb}(D)$ that appears in the theorem consists only of the extreme integral generators.

Theorem 2.1. *Let $d \geq 3$ and let $C = \mathbb{R}_+v_1 + \dots + \mathbb{R}_+v_d \subset \mathbb{R}^d$ be a simplicial d -cone with $\mu(C) = 2^l$ ($l \in \mathbb{N}$). Then there exists a unimodular triangulation $C = C_1 \cup \dots \cup C_k$ such that*

$$\text{Hilb}(C_j) \subset \left(\frac{d}{2} \left(\frac{3}{2} \right)^l \right) \Delta_C, \quad 1 \leq j \leq k.$$

Proof. The proof of this theorem is similar to the proof of Theorem 4.1 in [1]. We consider the following sequence:

$$h_k = 1, \quad k \leq 0, \quad h_k = \frac{1}{2}(h_{k-1} + \dots + h_{k-d}), \quad k \geq 1.$$

Because

$$h_k - h_{k-1} = \frac{1}{2}h_{k-1} - \frac{1}{2}h_{k-d-1}$$

for $k \geq 2$ and $h_1 > h_l$ for $l \leq 0$, it follows by induction that this sequence is increasing. Since for $k \geq 2$

$$h_k = \frac{1}{2}h_{k-1} + \frac{1}{2}(h_{k-2} + \dots + h_{k-d-1}) - \frac{1}{2}h_{k-d-1} = \frac{3}{2}h_{k-1} - \frac{1}{2}h_{k-d-1} < \frac{3}{2}h_{k-1},$$

and because $h_1 = d/2$, $h_2 < 3d/4$, we arrive at

$$h_k \leq \frac{d}{2} \left(\frac{3}{2} \right)^{k-1}$$

for $k \geq 1$. This inequality will be needed in the following.

So, let $\mu(C) = 2^l$ ($l \in \mathbb{N}$). If C is already unimodular (i.e. $l = 0$), we are done. If C is not unimodular (i.e., $l \geq 1$), then choose $i_1, \dots, i_m \in \{1, \dots, d\}$ with $1 \leq m \leq d$ and $i_j < i_k$ for $j < k$ such that the vectors v_{i_1}, \dots, v_{i_m} generate a minimal non-unimodular subcone. Then we set

$$u = \frac{1}{2}(v_{i_1} + \dots + v_{i_m}).$$

That u is an integral vector follows from a more general fact; see equation (1) below.

Now we apply stellar subdivision to the cone C by the vector u , which gives us the cones

$$C_{i_s} = \mathbb{R}_+v_1 + \dots + \mathbb{R}_+v_{i_s-1} + \mathbb{R}_+u + \mathbb{R}_+v_{i_s+1} + \dots + \mathbb{R}_+v_d, \quad 1 \leq s \leq m \leq d.$$

For these cones of the first generation (we regard the initial cone C as the cone belonging to the 0-th generation) we have

$$\mu(C_{i_s}) = \left| \det \left(v_1, \dots, v_{i_s-1}, \frac{1}{2}(v_{i_1} + \dots + v_{i_m}), v_{i_s+1}, \dots, v_d \right) \right| = \frac{1}{2} \mu(C) = 2^{l-1}.$$

If $\mu(C_{i_s}) = 1$, then the procedure stops. Otherwise it is continued until we end with a triangulation of the initial cone C by unimodular cones of the l -th generation.

For the vectors w_k which have been used for the stellar subdivisions of the cones of the $(k - 1)$ -th generation we get

$$w_k \in h_k \Delta_C.$$

We will prove this claim by induction on k . For $k = 1$ it is obvious because $(v_{i_1} + \dots + v_{i_m})/2 \in (d/2)\Delta_C$. For $k > 1$, all generators u_1, \dots, u_d of a certain cone $C' = \mathbb{R}_+ u_1 + \dots + \mathbb{R}_+ u_d$ of the $(k - 1)$ -th generation either belong to the initial vectors v_1, \dots, v_d or are vectors which have been used for stellar subdivisions of cones of *different* generations. So by induction it follows

$$u_i \in h_{n_i} \Delta_C, \quad n_i \leq k - 1,$$

where the n_i are pairwise different. The equality

$$w_k = \frac{1}{2}(u_{j_1} + \dots + u_{j_v}), \quad 1 \leq v \leq d,$$

immediately leads us to

$$w_k \in \frac{1}{2}(h_{k-1} + \dots + h_{k-d})\Delta_C = h_k \Delta_C,$$

because the h_i are increasing. Hence, we are done. \square

The theorem motivated us to come up with a triangulation algorithm which first triangulates the underlying cone into subcones D with $\mu(D) = 2^l$ ($l \in \mathbb{N}$). Such triangulations play a central role in this paper and therefore we give them a special name.

Definition 2.2. A *2-triangulation* is a triangulation Σ whose simplicial cones have multiplicities equal to powers of 2.

Since multiplicities of simplicial cones can be interpreted as orders of groups, our terminology is a close analogy to the notion of 2-group.

The next, purely number theoretic lemma will be essential in the process of finding a 2-triangulation of a given cone C . (By ld we denote the base 2 logarithm.)

Lemma 2.3. *Let m and p be two odd integers with $p/2 < m < p$. Then there exist natural numbers $s \leq \text{ld}(p)$ and $t < p/2$ such that*

$$2^s t = (2^{s-1} - 1)p + m.$$

Proof. Because both m and p are odd, there exist a natural number $s > 1$ and another odd number q such that $p - m = 2^{s-1}q$. Now, let $t = (p - q)/2$. Then, t is a natural number, since p and q are both odd. Hence,

$$2^s t = 2^{s-1}(p - q) = (2^{s-1} - 1)p + m,$$

which proves the lemma. \square

Remark 2.4. Improving Lemma 2.3 in the sense that we could find natural numbers $s \ll \text{ld}(p)$, $t < p/2$ and x such that $2^s t = xp + m$ for given odd numbers m and p with $p/2 < m < p$ would critically affect the numerical quality of the bound we are going to give later on.

In the following we will use an upper bound for the prime number counting function that J. Rosser and L. Schoenfeld provided in [4].

Theorem 2.5. *For $x > 0$ let $\pi(x)$ denote the number of prime numbers p with $p < x$. Then for all $x > 1$ we have*

$$\pi(x) < 1.25506 \cdot \frac{x}{\log(x)}.$$

As pointed out above, we want to subdivide a given simplicial cone C into a 2-triangulation by successive stellar subdivision. Therefore it is useful to replace a large prime number p that divides $\mu(C)$ by smaller prime numbers in the passage from C to the subcones resulting from a subdivision step.

Let the primitive vectors $v_1, \dots, v_d \in \mathbb{Z}^d$ generate a simplicial cone C of dimension d , and let U be the sublattice of \mathbb{Z}^d spanned by these vectors. Then $\mu(C)$ is the index of U in \mathbb{Z}^d , and each residue class has a representative in

$$\text{par}(v_1, \dots, v_d) = \{q_1 v_1 + \dots + q_d v_d : 0 \leq q_i < 1\}.$$

If p divides $\mu(C)$, then there is an element of order p in \mathbb{Z}^d/U , and consequently there exists a vector

$$x = \frac{1}{p} \sum_{i=1}^d z_i v_i \in \mathbb{Z}^d, \quad z_i \in \mathbb{Z}, \quad 0 \leq z_i < p. \tag{1}$$

The next lemma shows that we can find an element x such that the coefficients z_i avoid prime numbers between 3 and p for a subset of the indices i whose size can be bounded by a function of p .

Lemma 2.6. *With the notation introduced, let $M \subset \{1, \dots, d\}$ such that*

$$|M| \leq \frac{\log(p)}{\tau}, \quad \tau = 1.25506.$$

Then there exists an element x of order p modulo U such that none of the coefficients z_i , $i \in M$, is an odd prime $< p$.

Proof. Let $b \bmod a$ denote the remainder of b modulo $a \neq 0$, chosen between 0 and $|a| - 1$.

Let x be an element of order p modulo U given as above. Then all the elements

$$x_j = \frac{1}{p} \sum_{i=1}^d (jz_i \bmod p)v_i, \quad j = 1, \dots, p - 1,$$

lie in $\text{par}(v_1, \dots, v_d)$ and have order p modulo U since p is prime.

We consider the maps $a \mapsto ja \bmod p$, $0 < j < p$. We must find a factor j such that $jz_i \bmod p$ is not an odd prime $< p$ for $i \in M$. If $z_i = 0$, then this condition does not exclude any factor j . Otherwise it excludes q factors where q is the number of odd primes $< p$. In total we must exclude at most $|M|q$ factors j . But in view of [Theorem 2.5](#) we have

$$|M|q = |M|(\pi(p) - 1) < |M| \left(\tau \cdot \frac{p}{\log(p)} - 1 \right) \leq p - |M| \leq p - 1$$

for $|M| > 0$. Furthermore, the lemma is obviously true for $|M| = 0$. \square

3. The algorithm

Before we describe the triangulation procedure precisely, we give an informal outline. The aim of this procedure is a 2-triangulation Σ of the original cone C so that the generators of the cones $D \in \Sigma$ are relatively short with respect to the simplex Δ_C .

For this purpose, we successively apply stellar subdivision with carefully selected vectors $x \in C$ to the cone $C = \mathbb{R}_+v_1 + \dots + \mathbb{R}_+v_d \subset \mathbb{R}^d$. If p is a prime divisor of $\mu(C)$, then there exists a vector

$$x = \frac{1}{p} \left(\sum_{j=1}^d z_j v_j \right) \in \text{par}(v_1, \dots, v_d) \setminus \{0\}.$$

In order to end up with a 2-triangulation of C , we want z_j to be either a composite number or small, namely $z_j \leq p/2$. In general, z_j cannot be expected to have this

property. Therefore we add a certain multiple kv_j ($k \in \mathbb{N}$) of v_j to the vector x if z_j is a prime number and $z_j > p/2$. This results in a vector $x' \in C$ with

$$x' = \frac{1}{p} \left(\sum_{j=1}^d z'_j v_j \right)$$

such that all z'_j are of the form $z'_j = 2^{g_j} t_j$ where $t_j \leq p/2$ or t_j is a composite number. By [Lemma 2.3](#) we can achieve this goal.

Of course, we wish the vectors x' to be as short as possible. We must avoid the situation that both z'_j is big and v_j is a long vector because then x' would be long. Here the upper bound for the prime number counting function comes into play via [Lemma 2.6](#) (see [Lemma 3.1](#) for a more detailed explanation), which guarantees that certain vectors are being multiplied by numbers z'_j/p with $z'_j < p$.

Power 2 triangulation – P2T

Input: The initial cone C

Output: The 2-triangulation $\hat{T}(C)$ of C

- 1: $\hat{T}(C) := \{C\}$
- 2: $\hat{A}(C) := \{C\}$
- 3: $\xi_C(-i) := v_i$ for $i = 1, \dots, d$
- 4: $\tau := 1.25506$
- 5: $\xi_C(i) := 0$ for $i \in \mathbb{N}_0$
- 6: **while** $\hat{T}(C)$ contains a cone $D = \mathbb{R}_+ \xi_D(i_1) + \dots + \mathbb{R}_+ \xi_D(i_d)$ (where $i_1 > i_2 > \dots > i_d \geq -d$) such that $\mu(D)$ is not a power of 2 **do**
- 7: $p := \max\{p \in \mathbb{P} : p \mid \mu(D)\}$
- 8: **FIND** $x = 1/p \left(\sum_{j=1}^d z_j \xi_D(i_j) \right) \in \text{par}(\xi_D(i_1), \dots, \xi_D(i_d)) \setminus \{0\}$ (which exists due to [Lemma 2.6](#)) for which
 - (1) $z_j \notin \mathbb{P}$ or
 - (2) $z_j \leq p/2$ or
 - (3) $z_j = 2$ and $p = 3$
 for all $j \leq \log(p)/\tau$
- 9: **for all** $j = \lfloor \log(p)/\tau \rfloor + 1, \dots, d$ **do**
- 10: **if** $z_j \notin \mathbb{P}$ or $z_j \leq p/2$ or $z_j = 2$ **then**
- 11: $z'_j := z_j$
- 12: **else**
- 13: $z'_j := z_j + kp$ with $k \in \mathbb{N}$ such that $z_j + kp = 2^s t$ where $s \leq \text{ld}(p)$ and $t < p/2$ (apply [Lemma 2.3](#))
- 14: **end if**
- 15: **end for**
- 16: $x' := 1/p \left(\sum_{j=1}^d z'_j w_j \right)$

```

17: for all  $E \in \hat{T}(C)$  with  $x' \in E$  do
18:   Apply stellar subdivision to  $E$  by  $x'$  (let  $E_j$  ( $j = 1, \dots, m$ ) be the resulting
      cones)
19:    $\hat{T}(C) := (\hat{T}(C) \setminus \{E\}) \cup \{E_j : j = 1, \dots, m\}$ 
20:    $\hat{A}(C) := \hat{A}(C) \cup \{E_j : j = 1, \dots, m\}$ 
21: end for
22:  $\nu := \max\{i : \xi_E(i) \neq 0\}$ 
23: for all  $j = 1, \dots, m$  do
24:   for all  $k \leq \nu$  do
25:      $\xi_{E_j}(k) := \xi_E(k)$ 
26:   end for
27:    $\xi_{E_j}(\nu + 1) := x'$ 
28: end for
29: end while
30: Return  $\hat{T}(C)$ 

```

The set $\hat{A}(C)$ contains the original cone C and all cones being created in the course of the P2T algorithm. The set $\hat{T}(C)$ is a strict subset of $\hat{A}(C)$ unless $\mu(C)$ is a power of 2. $\hat{A}(C)$ has been introduced out of technical reasons; it will help us to analyze certain properties of the resulting triangulation.

The line 8 in the P2T algorithm could be easily elaborated in a way how to determine the vector x constructively, but because we are – in the context of this article – only interested in the existence of x , we leave it at this point.

The triangulation of the initial cone C resulting from P2T has the desired properties. We verify them in the following.

Lemma 3.1. *Let $D = \mathbb{R}_+ w_1 + \dots + \mathbb{R}_+ w_d$ be a cone to which we apply a stellar subdivision by a vector x' in the P2T algorithm. Then x' is of the form*

$$x' = \frac{1}{p_{\max}} \left(\sum_{j=1}^d z'_j w_j \right), \quad p_{\max} = \max\{p \in \mathbb{P} : p \mid \mu(D)\},$$

such that:

- (1) for all j we have $z'_j = 2^{g_j} m_j$ with $g_j \in \mathbb{N}$, $g_j \leq \text{ld}(p_{\max})$ and
 - (a) $m_j \leq 2p_{\max}/3$, or
 - (b) $m_j < p_{\max}$ is a composite number;
- (2) (a) $z'_j/p < 1$ for $j \leq \log(p)/\tau$,
 - (b) $z'_j/p \leq p/2$ for the remaining j .

This lemma has already been stated implicitly in lines 8 and 13 of the P2T algorithm.

The next definition will be helpful in showing that the multiplicities of the cones in the final set $\hat{T}(C)$ are relatively small and that the length of every chain of cones

$$E_0 = D \subset E_1 \subset E_2 \dots \subset E_L = C,$$

where E_i is generated from E_{i+1} by stellar subdivision and D belongs to the resulting 2-triangulation of C , is relatively short.

Definition 3.2. Let n be a natural number, $n = \prod_{i=1}^{\infty} p_i^{\alpha_i}$ be its prime decomposition. Then we define $\phi(n) = 2(\text{ld}(n) - \eta(n))$, where $\eta(n) = \sum_{i=1}^{\infty} \alpha_i$. (Hence $\phi(n) = \sum_{i=1}^{\infty} \alpha_i (2\text{ld}(p_i) - 2)$.)

The function ϕ has some obvious nice properties, which we will need in the following.

Lemma 3.3.

- (1) $\phi(ab) = \phi(a) + \phi(b)$ for $a, b \in \mathbb{N}$,
- (2) $\phi(a/b) = \phi(a) - \phi(b)$ for $a, b \in \mathbb{N}$, $b \mid a$,
- (3) $n = 2^s$ with $s \in \mathbb{N}$ if and only if $\lfloor \phi(n) \rfloor = 0$.

Lemma 3.4. Let $D, E \in \hat{A}(C)$ such that E results from D by stellar subdivision in the course of the P2T algorithm. Then

$$\phi(\mu(E)) \leq \phi(\mu(D)) - 1.$$

Proof. Due to lines 8 and 13 of the algorithm,

$$\mu(E) = \mu(D) \frac{f}{p_{\max}} \cdot 2^l,$$

where $p_{\max} = \max\{p \in \mathbb{P} : p \mid \mu(D)\}$ and $f, l \in \mathbb{N}$. Furthermore, f is

- (1) either composite – i.e., $f = u \cdot v < p_{\max}$ (with $u, v \in \mathbb{N}$) – or
- (2) $f \leq 2p_{\max}/3$.

By [Lemma 3.3](#) and because by definition $p_{\max} \mid \mu(D)$, we have

$$\phi(\mu(E)) = \phi(\mu(D)) - \phi(p_{\max}) + \phi(f) + \phi(2^l) = \phi(\mu(D)) + \phi(f) - \phi(p_{\max}).$$

In case (1)

$$\phi(f) - \phi(p_{\max}) = \phi(u) + \phi(v) - 2\text{ld}(p_{\max}) - 2 \leq -2,$$

which proves the lemma in this case. In case (2)

$$\phi(f) - \phi(p_{\max}) \leq 2 \cdot (\text{ld}(p_{\max}) + \text{ld}(2) - \text{ld}(3) - \text{ld}(p_{\max})) \leq -1,$$

which proves the lemma for the second case. \square

Theorem 3.5. *For a simplicial d -cone C the P2T algorithm computes a 2-triangulation of C .*

Proof. The algorithm applies successive stellar subdivisions to the initial cone C . It stops when all multiplicities are powers of 2, and that it stops after finitely many iterations follows from [Lemma 3.4](#). \square

4. Bounds

Lemma 4.1. *Let $D \in \hat{A}(C)$ be an arbitrary cone resulting from the P2T algorithm. Furthermore, we define*

$$\chi(D) = \max\{i : \xi_D(i) \neq 0\}.$$

Then

$$\chi(D) \leq \phi(\mu(C)) - 1.$$

Proof. Let $D \in \hat{A}(C)$. By the algorithm, there is chain of cones

$$E_0 = D \subset E_1 \subset E_2 \dots \subset E_L = C$$

such that E_i is generated from E_{i+1} by stellar subdivision. [Lemma 3.4](#) implies that $\phi(\mu(D)) \leq \phi(\mu(C)) - L$. On the other hand, by construction, $\chi(D) = \chi(C) + L$, where $\chi(C) = -1$. Therefore

$$\chi(D) = L - 1 \leq \phi(\mu(C)) - \phi(\mu(D)) - 1.$$

This proves the lemma. \square

Theorem 4.2. *For all $D \in \hat{A}(C)$ we have*

$$\mu(D) \leq 2^{1/2 \cdot \text{ld}(\mu(C)) \cdot (\text{ld}(\mu(C)) + 3)}$$

Proof. By the algorithm, there is a chain of cones

$$D = E_0 \subset E_1 \subset E_2 \dots \subset E_L = C, \quad L \in \mathbb{N}_0$$

such that E_i is generated from E_{i+1} by stellar subdivision. Furthermore, let $p_{\max}(n) = \max\{p \in \mathbb{P} : p \mid n\}$ for any natural number n . Then obviously $p_{\max}(\mu(E_{i+1})) \geq p_{\max}(\mu(E_i))$ (see lines 8 and 13 of the algorithm).

Now, choose s such that $p_{\max}(\mu(E_i)) \leq 3$ for all $i \leq s$. Due to lines 8 and 13 of the algorithm, we have

$$\mu(E_i) = \frac{z_i}{p} \cdot \mu(E_{i+1}) \leq \mu(E_{i+1}), \quad 0 \leq i < s$$

since $z_i \leq p$ and $p \in \{2, 3\}$.

On the other hand, again by lines 8 and 13, we have that

$$\mu(E_i) = \mu(E_{i+1}) \frac{f}{p_{\max}(\mu(E_{i+1}))} \cdot 2^l, \quad i > s,$$

where $f, l \in \mathbb{N}$. Furthermore,

- (1) f is either composite – i.e., $f = u \cdot v < p_{\max}$ (with $u, v \in \mathbb{N}$) – (see lines 8 and 13 of the algorithm) and $l = 0$, or
- (2) $f \leq p/2$ and $l \leq \text{ld}(p_{\max}(\mu(E_{i+1})))$.

In both cases it follows that

$$\frac{\mu(E_i)}{2^{\eta(\mu(E_i))}} \leq \frac{1}{2} \cdot \frac{\mu(E_{i+1})}{2^{\eta(\mu(E_{i+1}))}}, \quad i > s.$$

On the other hand, $p_{\max}(n) \leq 2n/2^{\eta(n)}$ for every natural number n . Therefore,

$$p_{\max}(\mu(E_i)) \leq \frac{2\mu(E_i)}{2^{\eta(\mu(E_i))}} \leq \frac{1}{2^{L-i}} \cdot \mu(C), \quad i > s$$

This implies that $L - s \leq \lfloor \text{ld}(\mu(C)) \rfloor$. Otherwise we would have that $p_{\max}(\mu(E_{s+1})) \leq 2$. Furthermore, it follows that

$$\mu(E_i) \leq p_{\max}(\mu(E_{i+1})) \cdot \mu(E_{i+1}) \leq \frac{1}{2^{L-i-1}} \cdot \mu(C) \cdot \mu(E_{i+1}), \quad i > s.$$

For $\text{ld}(\mu(C)) \geq 2$ we have that

$$\begin{aligned} \mu(D) &= \prod_{i=0}^s \frac{\mu(E_i)}{\mu(E_{i+1})} \cdot \prod_{i=s+1}^{L-1} \frac{\mu(E_i)}{\mu(E_{i+1})} \cdot \mu(E_L) \leq \mu(C) \prod_{i=s+1}^{\lfloor \text{ld}(\mu(C)) \rfloor + s - 1} \frac{\mu(C)}{2^{L-i-1}} \\ &\leq 2^{1/2 \cdot \text{ld}(\mu(C)) \cdot (\text{ld}(\mu(C)) + 3)}, \end{aligned}$$

For $\mu(C) = 3$ the algorithm stops after the first iteration, because there is a vector x as given in line 8 of the algorithm, where $z_j \in \{0, 1, 2\}$ for all j . Hence, the resulting cones do have multiplicities equal to 1 or 2. Therefore, for all cones D we have that

$$\mu(D) \leq 2 \leq 2^{1/2 \cdot \text{ld}(\mu(C)) \cdot (\text{ld}(\mu(C)) + 3)}.$$

Furthermore, for $\mu(C) \in \{1, 2\}$ the algorithm even stops before the first iteration (see lines 1 and 6), which implies that $C = D$. Hence, in this case

$$\mu(D) = \mu(C) \leq 2^{1/2 \cdot \text{ld}(\mu(C)) \cdot (\text{ld}(\mu(C))+3)}$$

which finishes the proof. \square

The next theorem is the central numerical consequence resulting from the P2T algorithm, namely a length bound on the vectors involved. In the theorem and its proof we use the notation of P2T.

Theorem 4.3. *Let $D \in \hat{T}(C)$. Then, for all $s \geq 0$:*

$$\xi_D(s) \in \left(\frac{d}{2} \cdot \mu(C) \cdot 4^s\right) \Delta_C.$$

Proof. To simplify notation we set $\mu = \mu(C)$. We prove the theorem via induction on s . So, let $s = 0$. If $\xi_D(0) = 0$, there is nothing to prove.

So suppose that $\xi_D(0) \neq 0$. By the construction of $\xi_D(0)$ it follows that this vector was used for the stellar subdivision of the initial cone C . Hence, $\xi_D(0)$ is of the form

$$\xi_D(0) = \frac{1}{p} \sum_{i=1}^d z'_i v_i \in \mathbb{Z}^d \setminus \{0\},$$

where $z'_i/p \leq p/2 \leq \mu/2$ for all i (Lemma 3.1). Therefore $x \in (d/2)\mu\Delta_C$, which finishes the case $s = 0$.

For the induction step assume the statement is true for s replaced by $s - 1 \geq 0$. Again there is nothing to prove if $\xi_D(s) = 0$. Otherwise $\xi_D(s) \neq 0$ is a vector used for stellar subdivision. With the same notation as above, it follows by construction of $\xi_D(s)$ that

$$\xi_D(s) = \frac{1}{p} \left(\sum_{i=1}^d z'_i \xi_D(j_i) \right) \in \mathbb{Z}^d \setminus \{0\}$$

such that $s > j_1 > j_2 > \dots > j_d$. Furthermore p is a prime number $\leq \mu$. Now, let l be chosen such that $j_l > -1 \geq j_{l+1}$ (or $l = d$), implying that $\xi_D(j_l + 1), \dots, \xi_D(j_d) \in \{v_1, \dots, v_d\}$. We set

$$q = \left\lfloor \frac{\log(p)}{\tau} \right\rfloor$$

and distinguish three cases.

(1) $q = 0$. This is equivalent to $p = 3$. It follows by induction that

$$\xi_D(s) \in \left(\sum_{i=1}^l \left(\frac{d}{2} \mu 4^{j_i}\right) \frac{z'_i}{3} + \sum_{i=l+1}^d \frac{z'_i}{3} \right) \Delta_C,$$

because $s > j_1 > j_2 > \dots > j_d$. But, for $p = 3$ we have that $z'_i < 3$ for all i (see line 8 of the algorithm). Hence,

$$\xi_D(s) \in \frac{d}{2}\mu \left(\sum_{i=1}^l 4^{j_i} + 1 \right) \Delta_C \subset \left(\frac{d}{2}\mu 4^s \right) \Delta_C,$$

since $\mu \geq p = 3$.

(2) $l \leq q$ (and $q \neq 0$). Then, again, it follows by induction that

$$\xi_D(s) \in \left(\sum_{i=1}^l \left(\frac{d}{2}\mu 4^{j_i} \right) \frac{z'_i}{p} + \sum_{i=l+1}^d \frac{p}{2} \right) \Delta_C.$$

In the algorithm $z'_i/p < 1$ for all $i \leq \log(p)/\tau$ and therefore $z'_i/p < 1$ for all $i \leq l$. Hence,

$$\xi_D(s) \in \frac{d}{2}\mu \left(\sum_{i=1}^l 4^{j_i} + 1 \right) \Delta_C \subset \left(\frac{d}{2}\mu 4^s \right) \Delta_C,$$

which finishes the argument in this case.

(3) $l > q > 0$. By induction it follows that

$$\xi_D(s) \in \left(\sum_{i=1}^q \left(\frac{d}{2}\mu 4^{j_i} \right) \frac{z'_i}{p} + \sum_{i=q+1}^l \left(\frac{d}{2}\mu 4^{j_i} \right) \frac{z'_i}{p} + \sum_{i=l+1}^d \frac{p}{2} \right) \Delta_C.$$

From the first two sums we can extract the factor $(d/2)\mu$ and bound the third summand by $(d/2)\mu$.

Because $s > j_1 > j_2 > \dots > j_d$ and $z'_i/p < 1$ for all $i \leq q$, as well as $z'_i/p < p/2$ for $i > q$, we have that

$$\sum_{i=1}^q 4^{j_i} \frac{z'_i}{p} + \sum_{i=q+1}^l 4^{j_i} \frac{z'_i}{p} + 1 \leq \sum_{i=1}^q 4^{j_i} + 1 + \sum_{i=q+1}^l 4^{j_i} \frac{p}{2}.$$

Furthermore, $\sum_{i=k}^l \lambda^i = \frac{\lambda^{l+1} - \lambda^k}{\lambda - 1}$ for each $\lambda \neq 1$ and $k, l \in \mathbb{N}$ ($k \leq l$). Hence,

$$\sum_{i=1}^q 4^{j_i} + 1 \leq \frac{1}{3} \cdot (4^{j_1+1} - 4^{j_q}) + 1 \leq \frac{1}{3} \cdot 4^{j_1+1},$$

because $l > q$ and $j_l \geq 0$ which implies that $j_q \geq 1$. Therefore,

$$\sum_{i=1}^q 4^{j_i} + 1 + \sum_{i=q+1}^l 4^{j_i} (p/2) \leq \frac{1}{3} \cdot 4^{j_1+1} + \frac{p}{6} \cdot 4^{j_{q+1}+1}$$

Note that $j_i + r \leq j_{i-\lfloor r \rfloor} + 1$ for $r \in \mathbb{R}_+$. It implies

$$j_{q+1} = j_{q+1} + q - q \leq j_1 - q,$$

hence

$$\frac{1}{3} \cdot 4^{j_1+1} + \frac{p}{6} \cdot 4^{j_{q+1}+1} \leq \frac{1}{3} \cdot 4^{j_1+1} + \frac{p}{6} \cdot 4^{j_1-q+1}$$

On the other hand,

$$4^q = 4^{\lfloor \log(p)/\tau \rfloor} \geq 4^{\log(p)/\tau-1} \geq \frac{p^{\log(4)/\tau}}{4} \geq \frac{p}{4}.$$

Finally,

$$\frac{1}{3} \cdot 4^{j_1+1} + \frac{p}{6} \cdot 4^{j_1-q+1} \leq \frac{1}{3} \cdot 4^{j_1+1} + \frac{4}{6} \cdot 4^{j_1+1} \leq 4^s,$$

which finishes the proof. \square

Theorem 4.4. *Let $D \in \hat{T}(C)$. Then, for all $s \in \mathbb{Z}$ and $d \geq 2$:*

$$\xi_D(s) \in \left(\frac{d}{2} \cdot \mu(C) \cdot 4^{\phi(\mu(C))} \right) \Delta_C.$$

Proof. The theorem follows from [Lemma 4.1](#) and [Theorem 4.3](#). Due to [Theorem 4.3](#)

$$\xi_D(s) \in \left(\frac{d}{2} \cdot \mu(C) \cdot 4^s \right) \Delta_C$$

for $s \geq 0$, but on the other hand $\max\{i : \xi_D(i) \neq 0\} \leq \phi(\mu(C)) - 1$ by [Lemma 4.1](#), which shows that the theorem is true for all $s \geq 0$. Furthermore, $\xi_D(s) \in \Delta_C$ for $s < 0$ by definition. Because $d \geq 2$ and $\phi(n) \geq 0$ for all natural numbers $n \geq 1$ one has

$$\xi_D(s) \in \Delta_C \subset \left(\frac{d}{2} \cdot \mu(C) \cdot 4^{\phi(\mu(C))} \right) \Delta_C$$

for all $s < 0$, which finishes the proof. \square

If we now collect the results from [Theorem 4.2](#), [Theorem 4.4](#) and [Lemma 2.1](#) and additionally keep in mind that the P2T algorithm produces a 2-triangulation of the cone C , then we arrive at the desired result.

Theorem 4.5. *Every simplicial d -cone $C = \mathbb{R}_+v_1 + \dots + \mathbb{R}_+v_d \subset \mathbb{R}^d$, $d \geq 2$, has a unimodular triangulation $C = D_1 \cup \dots \cup D_t$ such that for all i*

$$\text{Hilb}(D_i) \subset \left(\frac{d^2}{4} \cdot \mu(C) \cdot 4^{\phi(\mu(C))} \cdot \left(\frac{3}{2} \right)^{1/2 \cdot \text{ld}(\mu(C)) \cdot (\text{ld}(\mu(C))+3)} \right) \Delta_C.$$

Using an upper bound for the function ϕ , we can simplify the bound somewhat:

Corollary 4.6. *Let $\varepsilon = 5 + 3/2 \cdot \text{ld}(3/2)$ and $\varrho = 1/2 \cdot \text{ld}(3/2)$. So, $\varepsilon \approx 5.88$ and $\varrho \approx 0.29$. Then every simplicial d -cone $C = \mathbb{R}_+v_1 + \dots + \mathbb{R}_+v_d \subset \mathbb{R}^d$, $d \geq 2$, which is not already unimodular (i.e., $\mu(C) > 1$) has a unimodular triangulation $C = D_1 \cup \dots \cup D_t$ such that for all i*

$$\text{Hilb}(D_i) \subset \left(\frac{d^2}{64} \cdot \mu(C)^{\varrho \cdot \text{ld}(\mu(C)) + \varepsilon} \right) \Delta_C.$$

Proof. The corollary follows from [Theorem 4.5](#) and the fact that $\phi(n) \leq 2 \text{ld}(n) - 2$ for all natural numbers $n > 1$. This means that $4^{\phi(\mu(C))} \leq 1/16 \cdot \mu(C)^4$ and

$$\left(\frac{3}{2} \right)^{1/2 \cdot \text{ld}(\mu(C)) \cdot (\text{ld}(\mu(C)) + 3)} = \mu(C)^{1/2 \cdot \text{ld}(3/2) \cdot \text{ld}(\mu(C)) + 3/2 \cdot \text{ld}(3/2)}. \quad \square$$

Remark 4.7. In every iteration of the P2T algorithm it is guaranteed that the set $\hat{T}(C)$ constitutes a triangulation of the initial cone C . Even more so, after every iteration the set $\hat{T}(C)$ is a refined triangulation of the previous set $\hat{T}(C)$ via successive stellar subdivisions. In particular, the algorithm would also end up with a triangulation of a cone C in case we start it with a triangulation $\hat{T}(C) = \{D_1, \dots, D_N\}$. Furthermore, the resulting cones would also coincide on the boundary, because in every iteration of the P2T algorithm stellar subdivision with a vector x is applied to every cone, which contains x (see lines 17 and 18 of the algorithm). Now, every cone C can be triangulated into simplicial cones C' generated by extreme rays of C . So, if we start the algorithm with $\hat{T}(C) = \{C'_1, \dots, C'_N\}$, it would end up with a 2-triangulation of C which is a refinement of the start triangulation. Hence, the algorithm essentially works for an arbitrary cone. We can replace the basic simplex Δ_C by the convex hull Γ of the origin and the extreme integral generators of C , and $\mu(C)$ by the lattice normalized volume of Γ .

References

- [1] W. Bruns, J. Gubeladze, Unimodular covers of multiples of polytopes, *Doc. Math.* 7 (2002) 463–480.
- [2] W. Bruns, J. Gubeladze, *Polytopes, Rings and K-Theory*, Springer, 2009.
- [3] C. Haase, A. Paffenholz, L.C. Piechnik, F. Santos, Existence of unimodular triangulations – positive results, preprint, arXiv:1405.1687.
- [4] J. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* 6 (1962) 64–94.
- [5] M.v. Thaden, Unimodular covers and triangulations of lattice polytopes, PhD thesis, 2007, Osnabrück.