

CONIC DIVISOR CLASSES OVER A NORMAL MONOID ALGEBRA

WINFRIED BRUNS

In [2] J. Gubeladze and the author have studied the divisorial ideals of an algebra $R = K[M]$ where K is a field and M a normal affine monoid. The divisorial ideals which have a monomial K -basis cut out from the group $\text{gp}(M)$ by a translate of the cone \mathbb{R}_+M have been called *conic*. Up to isomorphism the conic ideals are exactly the direct summands of the extension $R^{1/n}$ of R where $R^{1/n}$ is the K -algebra over the monoid $(1/n)M = \{x/n : x \in M\} \subset \mathbb{Q} \otimes \text{gp}(M)$.

In this note we want to extend the discussion in [2]. In a rather straightforward manner we will represent the divisor classes by the full-dimensional open cells in a certain cell complex. The conic classes can then be identified with the full-dimensional open cells in a decomposition of the torus $(\mathbb{R} \otimes \text{gp}(M))/\text{gp}(M)$. Furthermore, these classes can be characterized by the relative compactness of a certain group associated with them. For torsion classes this group is finite.

By Hochster's theorem $R^{1/n}$ is a Cohen-Macaulay ring. Therefore conic classes are Cohen-Macaulay. Using a criterion of Stückrad and Vogel [7] for the Cohen-Macaulay property of Segre products, Baetiça [1] has given examples of Cohen-Macaulay divisorial ideals that are not conic. We will review his construction and streamline the arguments somewhat.

In the last part of the paper we investigate the multiplicities of the conic classes in the decomposition of $R^{1/n}$, as n varies. This multiplicity turns out to be a quasi-polynomial for all $n \geq 1$, given by the number of lattice points in the union of the interiors of certain cells of the cell complex mentioned above. As Watanabe [9] has already noticed, this argument can be used for the computation of the Hilbert-Kunz multiplicity of R in characteristic $p > 0$. In addition it yields some assertions about the Hilbert-Kunz function of R .

1. CONIC DIVISOR CLASSES

Let M be a normal affine monoid of rank d . The divisor class group of $K[M]$ has been discussed in [2]. We refer the reader to this article for details that may be missing in the following. We always assume that M is positive, i. e. 0 is the only invertible element in M .

Changing the embedding if necessary, we may assume that $\mathbb{Z}^d = \text{gp}(M)$. One has $M = \mathbb{Z}^d \cap \mathbb{R}_+M$ since M is normal, and the cone \mathbb{R}_+M has an irredundant representation

$$\mathbb{R}_+M = \{x \in \mathbb{R}^d : \sigma_i(x) \geq 0, i = 1, \dots, s\}$$

as an intersection of halfspaces defined by primitive integral forms σ_i , $i = 1, \dots, s$ on \mathbb{R}^d . (“Primitive integral” means that the elements of the canonical basis of \mathbb{R}^d have coprime integral values.)

Every (fractional) divisorial ideal of R is isomorphic to a (fractional) monomial ideal of R , and the latter are precisely the ideals

$$\mathcal{D}(u) = K \cdot \{x \in \mathbb{Z}^d : \sigma(x) \geq u\} \quad u \in \mathbb{R}^s.$$

Here $\sigma(x) = (\sigma_1(x), \dots, \sigma_s(x))$, and the inequality must be read componentwise. (We identify monomials with their exponent vectors.) Of course, $\mathcal{D}(u) = \mathcal{D}(\lceil u \rceil)$ with $\lceil u \rceil = (\lceil u_1 \rceil, \dots, \lceil u_s \rceil)$.

The ideals $\mathcal{D}(u)$ and $\mathcal{D}(v)$ are isomorphic as R -modules if and only if there exists a monomial $x \in \mathbb{Z}^d$ such that $\mathcal{D}(u) = x\mathcal{D}(v)$. For $u, v \in \mathbb{Z}^s$ this amounts to $u = v + \sigma(x)$ for some $x \in \mathbb{Z}^d$. Therefore the divisor class group $\text{Cl}(R)$ is given by $\mathbb{Z}^s / \sigma(\mathbb{Z}^d)$.

Some more information is contained in the topological group $\mathbb{R}^s / \sigma(\mathbb{Z}^d)$ and its cell decomposition $\bar{\Delta}$ defined as follows. First, let Δ be the cell decomposition of \mathbb{R}^n defined by all the hyperplanes through integral points that are parallel to the coordinate hyperplanes. Then the open s -cells δ of Δ are in 1-1-correspondence to \mathbb{Z}^s via their “upper right” corners

$$\lceil x \rceil \in \mathbb{Z}^s, \quad x \in \delta.$$

The “upper closures”

$$[\delta] = \{y \in \mathbb{R}^s : \lceil y \rceil = \lceil x \rceil\}, \quad x \in \delta,$$

are the fibers of the map $\mathbb{R}^s \rightarrow \mathbb{Z}^s$, $y \mapsto \lceil y \rceil$. Clearly $[\delta]$ is a semi-open cube and the union of open cells of Δ . Moreover, these semi-open cubes correspond bijectively to the divisorial ideals of M . The cell decomposition is invariant under the operation of $\sigma(\mathbb{Z}^d)$ on \mathbb{R}^s via translations, and so Δ induces a cell decomposition $\bar{\Delta}$ of $\mathbb{R}^s / \sigma(\mathbb{Z}^d)$. Collecting all our arguments, we obtain

Theorem 1.1. *The following sets are in a natural bijective correspondence:*

- (a) $\text{Cl}(R)$;
- (b) *the fibers of the map $\mathbb{R}^s / \sigma(\mathbb{Z}^d) \mapsto \text{Cl}(R)$ induced by the assignment $x \mapsto \lceil x \rceil$, $x \in \mathbb{R}^s$;*
- (c) *the open s -cells of $\bar{\Delta}$.*

As pointed out in the introduction, the *conic* divisorial ideals

$$\mathcal{C}(y) = K \cdot (\mathbb{Z}^d \cap (y + \mathbb{R}_+ M)), \quad y \in \mathbb{R}^d.$$

are of special interest. Clearly,

$$\mathcal{C}(y) = \mathcal{D}(\sigma(y))$$

so that the conic divisorial ideals are exactly those among the $\mathcal{D}(u)$ for which u can be chosen in $\sigma(\mathbb{R}^d)$.

If one restricts the cell decomposition of \mathbb{R}^s to $\mathbb{R}^d \cong \sigma(\mathbb{R}^d)$, then one obtains the cell decomposition Γ of \mathbb{R}^d by the hyperplanes

$$H_{i,z} = \{x \in \mathbb{R}^d : \sigma_i(x) = z\}, \quad i = 1, \dots, s, \quad z \in \mathbb{Z}.$$

It induces a cell decomposition $\bar{\Gamma}$ of the torus $\mathbb{R}^d/\mathbb{Z}^d$, which we can also define by restricting $\bar{\Delta}$ to the submanifold $\mathbb{R}^d/\mathbb{Z}^d$ (via σ). Figure 1 gives an example of such a decomposition. We have marked the cone \mathbb{R}_+M and the semiopen square $(-1, 0]^2$, a fundamental domain for the action of \mathbb{Z}^2 . Evidently $\bar{\Gamma}$ has exactly 3 open cells of dimension 2.

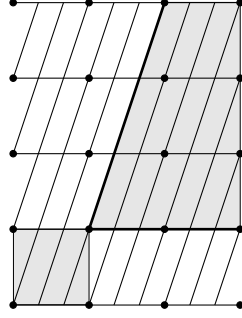


FIGURE 1.

Corollary 1.2. *The following sets are in a natural bijective correspondence:*

- (a) *the set of conic divisor classes;*
- (b) *the fibers of the map $\mathbb{R}^d/\mathbb{Z}^d \rightarrow \mathbb{R}^s/\sigma(\mathbb{Z}^d)$ induced by the assignment $x \mapsto \lceil \sigma(x) \rceil$;*
- (c) *the full dimensional cells of $\bar{\Gamma}$.*

Note that \mathbb{R}_+M contains a point x with $\sigma_i(x) > 0$ for all i . This implies that the full-dimensional open cells of $\bar{\Gamma}$ are contained in full-dimensional open cells of Δ .

We can now characterize the conic classes in terms of the sizes of certain subgroups of $\mathbb{R}^s/\sigma(\mathbb{Z}^d)$:

Corollary 1.3. *A divisor class is*

- (a) *a torsion element if and only if it is represented by an ideal $\mathcal{D}(u)$ for which $\mathbb{Z}\bar{u}$ is a finite subgroup of $\mathbb{R}^s/\sigma(\mathbb{R}^d)$;*
- (b) *conic if it is represented by an ideal $\mathcal{D}(u)$ for which $\mathbb{Z}\bar{u}$ is a relatively compact subgroup of $\mathbb{R}^s/\sigma(\mathbb{Z}^d)$ (i. e. the closure of $\mathbb{Z}\bar{u}$ is compact).*

The next proposition connects the property of being conic with a condition of Stanley [8, p. 41].

Proposition 1.4. *The divisorial ideal $\mathcal{D}(u)$ is conic if and only if the coset $\sigma(\mathbb{R}^d) - \lceil u \rceil$ in $\mathbb{R}^s/\sigma(\mathbb{R}^d)$ contains a point in the semi-open cube $Q = (-1, 0]^s$.*

Proof. We can replace u by $\lceil u \rceil \in \mathbb{Z}^s$, and therefore assume that $u \in \mathbb{Z}^s$.

Suppose first that $\mathcal{D}(u) = \mathcal{C}(x)$. Then $u = \lceil \sigma(x) \rceil$ and $\sigma(x) - u = \sigma(x) - \lceil \sigma(x) \rceil$ is in $Q \cap (\sigma(\mathbb{R}^d) - u)$.

Conversely, suppose that $v \in (Q \cap \sigma(\mathbb{R}^d)) - u$. Then $u + v \in \sigma(\mathbb{R}^d)$, and $\mathcal{D}(u) = \mathcal{D}(u + v)$. \square

Let T be the torsion subgroup of $\text{Cl}(R) \cong \mathbb{Z}^s/\sigma(\mathbb{Z}^d)$. Then $\text{Cl}(R)/T$ is a free abelian group of rank $m = s - d$. It can be identified with a rank m lattice L in the m -dimensional vector space $\mathbb{R}^s/\sigma(\mathbb{R}^d)$. Roughly speaking, the conic classes form the set of lattice points in a polytope in this vector space.

Corollary 1.5. *Let P be the polytope in $\mathbb{R}^s/\sigma(\mathbb{R}^d)$ spanned by the images of the conic classes in $\text{Cl}(R)/T$. Then*

- (a) *a class $c \in \text{Cl}(R)$ is conic if and only if its residue class modulo T belongs to P ;*
- (b) *P has dimension $m = s - d$, and is centrally symmetric with respect to $[\omega]/2$ modulo T where ω is the canonical module of R .*

Proof. Let P' be the image of the semi-open cube Q in $\mathbb{R}^s/\sigma(\mathbb{R}^d)$. By the proposition a class is conic if and only if its image modulo T belongs to P' . Therefore the convex hull P of the images of the conic classes has property (a). It is of dimension m since it contains the origin of L (given by the trivial class) and the classes of a system of generators of $\text{Cl}(R)$: the classes of the monomial divisorial prime ideals generate $\text{Cl}(R)/T$ and are conic (see [2, 3.4]). Therefore P must have dimension m . The central symmetry with respect to $[\omega]/2$ results from the fact that $[\omega] - c$ is conic if c is conic; see loc. cit. \square

2. CONIC AND COHEN-MACAULAY CLASSES

As pointed out above, the conic classes represent Cohen-Macaulay modules. In general however, there exist Cohen-Macaulay divisorial monomial ideals that are not conic, as shown by Baetica [1], using a theorem of Stückrad and Vogel on the Cohen-Macaulay property of Segre products. Baetica applies it to divisorial ideals over Segre products of 3 polynomial rings. However, he does not use the full strength of the results in [7], and his arguments can be improved.

Let R_i , $i = 1, 2$, be a positively graded, finitely generated K -algebra with irrelevant maximal ideal \mathfrak{m}_i . Furthermore, let M_i be a graded module over R_i , $i = 1, 2$ such that $\text{depth } M_i \geq 2$. Then the k -th local cohomology of the module $M = M_1 \# M_2$ over the Segre product $R = R_1 \# R_2$ (with respect to the irrelevant maximal ideal \mathfrak{m} of R) is the direct sum

$$(*) \quad \bigoplus_{i+j=k+1} (H_{\mathfrak{m}_1}^i(M_1) \# H_{\mathfrak{m}_2}^j(M_2)) \oplus (M_1 \# H_{\mathfrak{m}_2}^k(M_2)) \oplus (H_{\mathfrak{m}_1}^k(M_1) \# M_2).$$

This is just the Künneth formula for Serre cohomology [7, 0.2.10], rewritten in terms of local cohomology. (See [7, p. 38] for the correspondence between Serre and local cohomology.) Since $\text{depth } M_i \geq 2$, one has $H_{\mathfrak{m}_i}^k(M_i) = 0$ for $k = 0, 1$, and so $\text{depth } M \geq 2$ by [7, 0.2.12].

In order to control the vanishing of the local cohomology we introduce the following notation: for a graded maximal Cohen-Macaulay module N over a positively graded affine K -algebra S of dimension e , with irrelevant maximal ideal \mathfrak{n} , we set

$$b(N) = \inf\{k : N_k \neq 0\},$$

$$h(N) = \sup\{k : H_{\mathfrak{n}}^e(N)_k \neq 0\}.$$

Under certain conditions N and $H_n^e(N)$ are “gap free”: $N_k \neq 0$ for all $k \geq b(N)$ and $H_n^e(N)_k \neq 0$ for all $k \leq h(N)$. By graded local duality, $H_n^e(N)$ is the graded K -dual of the maximal Cohen-Macaulay module $\text{Hom}_S(N, \omega)$ where ω denotes the canonical module of S . Therefore it is enough to discuss gap freeness for N . If the graded component S_1 is not contained in a minimal prime ideal of S , then it is not contained in an associated prime ideal of N , and after the extension of K to an infinite field it follows that S_1 contains a nonzerodivisor of N , whence N is gap free.

It is now easy to generalize [7, Th. I.4.6] to more than two factors in the Segre product:

Proposition 2.1. *Suppose that R_1, \dots, R_n are positively graded Cohen-Macaulay K -algebras of dimensions $d_1, \dots, d_n \geq 2$, with irrelevant maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$, and let M_i be a graded maximal Cohen-Macaulay module over R_i , $i = 1, \dots, n$. If $(R_i)_1$ is not contained in a minimal prime ideal of R_i for $i = 1, \dots, n$, then the following are equivalent:*

- (a) $M_1 \# \dots \# M_n$ is Cohen-Macaulay;
- (b) for all nonempty, proper subsets $I \subset \{1, \dots, n\}$ one has

$$\min\{h(M_i) : i \in I\} < \max\{b(M_j) : j \notin I\};$$

- (c) there exists a permutation j_1, \dots, j_n of $1, \dots, n$ such that $M_{j_1} \# \dots \# M_{j_t}$ is Cohen-Macaulay over $R_{j_1} \# \dots \# R_{j_t}$, $t = 1, \dots, n$.

Proof. An iterative use of the formula (*) allows one to compute the local cohomology of the Segre product. The k -th local cohomology of $M = M_1 \# \dots \# M_n$ over $R = R_1 \# \dots \# R_n$ (with respect to the irrelevant maximal ideal \mathfrak{m} of R) is the direct sum of the R -modules

$$H_{\mathfrak{m}_{i_1}}^{d_{i_1}}(M_{i_1}) \# \dots \# H_{\mathfrak{m}_{i_t}}^{d_{i_t}}(M_{i_t}) \# M_{i_{t+1}} \# \dots \# M_{i_n}$$

where $0 \leq t \leq n$, $1 \leq i_1 < \dots < i_t \leq n$, $i_{t+1} < \dots < i_n$, $\{i_1, \dots, i_s\} = \{1, \dots, s\}$, and

$$k = d_{i_1} + \dots + d_{i_t} - (t - 1).$$

In addition to (*) this computation only uses that $H_{\mathfrak{m}_i}^j(M_i) = 0$ for $j < d_i$.

The equivalence of (a) and (b) follows immediately, and only the implication (b) \implies (c) needs to be proved. We have to find a permutation j_1, \dots, j_n such that

$$\begin{aligned} h(M_{j_1} \# \dots \# M_{j_t}) &= \min(h(M_{j_i} : 1 \leq i \leq t) < b(M_{j_{t+1}})) \\ b(M_{j_1} \# \dots \# M_{j_t}) &= \max(b(M_{j_i} : 1 \leq i \leq t) > h(M_{j_{t+1}})) \end{aligned}$$

for $t = 1, \dots, n - 1$.

Reordering $1, \dots, n$ if necessary, we can assume that $b(M_1) \leq \dots \leq b(M_n)$. We set $u_1 = 1$. Then we split the sequence $2, \dots, n$ into blocks

$$B_v = \{u_v + 1, \dots, u_{v+1}\}, \quad v = 1, \dots, w,$$

where $h(M_i) \geq b(M_{u_v})$, $i = u_v + 1, \dots, u_{v+1} - 1$, but $h(M_{u_{v+1}}) < b(M_{u_v})$. Such a decomposition exists by (b), applied successively to the subsets $\{1, \dots, u_v\}$. Each of the blocks B_1, \dots, B_w is then cyclically permuted to

$$B'_v = \{u_{v+1}, u_v + 1, \dots, u_{v+1} - 1\}$$

and the desired permutation is finally given by the concatenation $1, B'_1, \dots, B'_w$. It is not hard to check that this permutation indeed satisfies the desired inequalities. \square

Let R_1, \dots, R_n be polynomial rings over K of dimensions $d_1, \dots, d_n \geq 2$ with the standard grading by total degree. Then their Segre product $S = R_1 \# \dots \# R_n$ is a monoid ring over the Segre product

$$\mathbb{Z}_+^{d_1} \# \dots \# \mathbb{Z}_+^{d_n}$$

where the Segre product construction is simply applied to the monoids of monomials. It is not hard to see that $\text{Cl}(S) \cong \mathbb{Z}^{n-1}$, and that the divisorial ideals over S are represented by the Segre products of shifted copies of the R_i ,

$$D = R_1(-s_1) \# \dots \# R_n(-s_n), \quad (s_1, \dots, s_n) \in \mathbb{Z}^n.$$

The divisor class of D is completely determined by the differences $(s_2 - s_1, \dots, s_n - s_1) \in \mathbb{Z}^{n-1}$. One has $b(R_i(-s_i)) = s_i$ and $h(R_i(-s_i)) = s_i - d_i$.

Note that the proof of the implication (b) \implies (c) contains an algorithm by which one can check the Cohen-Macaulay property of the Segre product. It is especially simple if all the dimensions d_i are equal of constant value d : if $s_1 \leq \dots \leq s_n$, then D is Cohen-Macaulay if and only if $s_{i+1} < s_i + d$ for $i = 1, \dots, n-1$, and the other cases are obtained by permutation. For $n = 3$ and $d_1 = d_2 = d_3 = 3$ the Cohen-Macaulay classes are indicated in Figure 2. The conic classes correspond to the lattice points in the polytope given by the inequalities $-2 \leq x, y, y - x \leq 2$. In particular there exist more Cohen-Macaulay classes than conic ones.

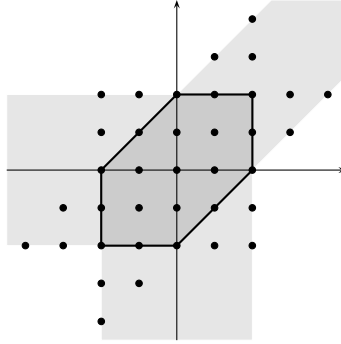


FIGURE 2.

It follows immediately from the computation of the local cohomology that in the case $d_1 = \dots = d_n = d$ only the depths $kd - (k - 1)$ are possible for $k = 1, \dots, n$. For $n = 3, d = 3$, the (unbounded) areas of the classes of depth $2d - 1$ have been shadowed.

Remark 2.2. Baetica [1] has also investigated Segre products $T = R^{(c)} \# S^{(d)}$ of Veronese subalgebras of polynomial rings R and S . If $\gcd(c, d) = 1$, then $\text{Cl}(T) \cong \mathbb{Z}$. For example, if $\dim R = \dim S = 2$ and $c = 3, d = 2$, then the set of Cohen-Macaulay classes is $\{-5, -3, \dots, 5, 7\}$ (with respect to a suitable choice of generator) and does not form an interval in $\text{Cl}(T)$.

3. THE DECOMPOSITION OF $R^{1/n}$

Let M be an affine monoid and $G = \text{gp}(M)$. The monoid $(1/n)M$ contains M , and so $R^{1/n}$ contains R as a subalgebra. Moreover, $\text{gp}((1/n)M) = (1/n)G$. For each residue class $c \in (1/n)G/G$ we set

$$I_c = R^{1/n} \cap K \cdot c.$$

Then I_c is an R -submodule of $R^{1/n}$, and $R^{1/n}$ decomposes into the direct sum of its R -submodules I_c , $c \in (1/n)G/G$.

Now suppose that M is normal. Then $I_c = \mathbb{R}_+ M \cap (M + c)$, and the parallel translation by $-c$ yields

$$I_c \cong \mathcal{C}(-c).$$

Therefore I_c is a conic divisorial ideal.

We consider the cell complex $\bar{\Gamma}$ on the torus $\mathbb{R}^d/\mathbb{Z}^d$ (after the identification of $\text{gp}(M)$ and \mathbb{Z}^d). For each open d -cell $\bar{\gamma}$ of $\bar{\Gamma}$ we choose an open d -cell γ of Γ representing it and set

$$[\gamma] = \{x \in \mathbb{R}^d : \lceil \sigma(x) \rceil = \sigma(y)\}, \quad y \in \gamma.$$

Then $[\gamma]$ is the union of finitely many open cells of Γ contained in the closure of γ . Moreover, all the conic ideals $\mathcal{C}(x)$, $x \in [\gamma]$, coincide. In the following we write \mathcal{C}_γ for $\mathcal{C}(x)$, $x \in [\gamma]$.

Theorem 3.1. *Let $v_\gamma(n)$ be the multiplicity with which the isomorphism class of \mathcal{C}_γ occurs in the decomposition of $R^{1/n}$ as an R -module. Then*

$$v_\gamma(n) = \# \left([\gamma] \cap \frac{1}{n} \mathbb{Z}^d \right).$$

In particular, there exists a quasi-polynomial $q_\gamma : \mathbb{Z} \rightarrow \mathbb{Z}$ with rational coefficients such that $v_\gamma(n) = q_\gamma(n)$ for all $n \geq 1$. One has

$$q_\gamma(n) = \text{vol}(\gamma)n^d + a_\gamma n^{d-1} + \tilde{q}_\gamma(n), \quad n \in \mathbb{Z},$$

where $a_\gamma \in \mathbb{Q}$ is constant and \tilde{q}_γ is a quasi-polynomial of degree $\leq d - 2$.

Proof. The direct summand I_c of $R^{1/n}$, $c \in (1/n)G/G$, is isomorphic to \mathcal{C}_γ if and only if $(-c + \mathbb{Z}^d) \cap [\gamma] \neq \emptyset$. Moreover, $(-c + \mathbb{Z}^d) \cap [\gamma]$ contains at most one point, since $\lceil \sigma(x) \rceil \neq \lceil \sigma(y) \rceil$ if $x - y \in \mathbb{Z}^d$, $x \neq y$. This proves the first assertion.

For the second we decompose $[\gamma]$ into the open cells σ of Γ of which it is the union. The generating function

$$H_\sigma(t) = \sum_{n=0}^{\infty} \# \left(\sigma \cap \frac{1}{n} \mathbb{Z}^d \right) t^n$$

is a rational function of degree 0. In fact, it is the Hilbert series of the canonical module of the Ehrhart ring of the rational polytope $\bar{\sigma}$, and therefore has degree 0. Summing over the cells σ we obtain that

$$H_\gamma(t) = \sum_{n=0}^{\infty} v_\gamma(n) t^n$$

is a rational function of degree (at most) 0. Therefore $v_\gamma(n)$ is given by a quasi-polynomial q_γ for all $n \geq 1$.

Only the d -dimensional cell γ itself contributes to the degree d term of q_γ . By Ehrhart reciprocity the number of points in $\gamma \cap (1/n)\mathbb{Z}$ is $(-1)^d \bar{q}(-n)$ where \bar{q} is the Ehrhart quasi-polynomial of the closure $\bar{\gamma}$ of γ . Since $\bar{\gamma}$ is a full-dimensional rational polytope, the leading coefficient of \bar{q}_γ is constant and equal to the volume of $\bar{\gamma}$.

Next we have to show that the coefficient of n^{d-1} in q_γ is constant. It gets contributions from $\bar{\gamma}$ and from the Ehrhart quasi-polynomials of the open $(d-1)$ -cells. The coefficient of n^{d-1} in \bar{q}_γ is constant since the affine hulls of the facets of $\bar{\gamma}$ contain lattice points (Stanley [8, p. 237]). In fact, the support forms of \mathbb{R}_+M map \mathbb{Z}^d surjectively onto \mathbb{Z} . For the same reason the coefficient of n^{d-1} is constant in the Ehrhart polynomials of the $(d-1)$ -cells since these are the interiors of rational polytopes in the affine hulls of the facets of $\bar{\gamma}$ (which, as stated already, contain lattice points). \square

In this proof we have used results of Ehrhart and Stanley on lattice points in rational polytopes and Hilbert functions; for example, see [8] or 4.4.5, 6.3.5 and 6.3.11 in [3].

As a corollary we obtain a description of the Hilbert-Kunz function of $K[M]$ for normal M and a nice formula for the Hilbert-Kunz multiplicity of a graded algebra with normalization $K[M]$. The argument has similarly been used by Watanabe [9]. Also see Conca [4] and Eto [5] for results on the Hilbert-Kunz multiplicity of monoid rings. By $\mu_R(N)$ we denote the minimal number of generators of an R -module N .

By definition, the *Hilbert-Kunz multiplicity* of a finitely generated graded module B over a positively graded algebra A over a field K of prime characteristic p is

$$e_{\text{HK},A}(B) = \lim_{e \rightarrow \infty} \frac{\dim_K B/\mathfrak{m}^{[p^e]}B}{p^{de}}$$

where \mathfrak{m} is the irrelevant maximal ideal of A generated by all elements of positive degree.

Corollary 3.2. *Let K be an algebraically closed field of characteristic $p > 0$ and M a normal affine monoid of rank d .*

- (a) *Set $R = K[M]$ and let \mathfrak{m} be the maximal ideal of R generated by the monomials different from 1. Then*

$$\dim_K R/\mathfrak{m}^{[p^e]} = \mu_R(R^{1/p^e}) = \sum_{\gamma} \mu_R(\mathcal{C}_\gamma) v_\gamma(p^e), \quad e \in \mathbb{Z}, e \geq 1,$$

is the value of the quasi-polynomial $q_{\text{HK}} = \sum_{\gamma} \mu_R(\mathcal{C}_\gamma) v_\gamma$ at p^e . It has constant leading coefficient

$$e_{\text{HK},R}(R) = \sum_{\gamma} \mu_R(\mathcal{C}_\gamma) \text{vol}(\gamma) \in \mathbb{Q},$$

and also the coefficient of its degree $d-1$ term is constant and rational.

- (b) *Suppose S is a graded subalgebra of R such that R is a finite S -module and the normalization of S . Then $e_{\text{HK},S}(S) = \sum_{\gamma} \text{vol}(\gamma) \mu_S(\mathcal{C}_{\gamma}) \in \mathbb{Q}$.*

Proof. One has the isomorphism $\bar{R}/\mathfrak{m}^{[p^e]} \cong \bar{R}^{1/p^e}/\mathfrak{m}\bar{R}^{1/p^e}$ of K -vector spaces, and so Nakayama's lemma yields

$$\dim_K R/\mathfrak{m}^{[p^e]} = \mu_R(\bar{R}^{1/p^e}) = \sum_{\gamma} v_{\gamma}(p^e) \mu_R(\mathcal{C}_{\gamma}).$$

Now all the assertions in (a) follow immediately from the theorem.

For (b) we note the exact sequence

$$0 \rightarrow S \rightarrow R \rightarrow T \rightarrow 0$$

in which T has Krull dimension $< d$. Therefore $e_{\text{HK},S}(S) = e_{\text{HK},S}(R)$. The decomposition of R^{1/p^e} over R is, a fortiori, a decomposition over S , and the formula for $e_{\text{HK},S}(R)$ follows again from the theorem. \square

That the coefficient of the degree $d - 1$ term is constant, as we have derived from combinatorial arguments, is in fact true in much more generality; see Huneke, McDermott, and Monsky [6].

Corollary 3.2 allows one to give non-trivial examples of algebras $K[M]$ for which the quasi-polynomial q_{HK} is a true polynomial. The Ehrhart quasi-polynomial of a polytope with integral vertices is a polynomial. Therefore it is sufficient that all cells of Γ have integral vertices, and this holds if the support forms $\sigma_1, \dots, \sigma_s$ of the cone $C = \mathbb{R}_+ M$ form a totally unimodular configuration: every linearly independent subset of d elements is a basis of $(\mathbb{Z}^d)^*$.

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UNIVERSITÄT OSNABRÜCK, FB MATHEMATIK/INFORMATIK, 49069 OSNABRÜCK, GERMANY
E-mail address: Winfried.Bruns@mathematik.uni-osnabrueck.de