ON THE NUMBER OF ELEMENTS INDEPENDENT WITH RESPECT TO AN IDEAL

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Let R denote a commutative noetherian ring, and $\mathfrak{a}$ an ideal of R. Elements $x_1, \ldots, x_n \in R$ are called *independent with respect to* $\mathfrak{a}$, or simply $\mathfrak{a}$-independent, if every form $F \in R[X_1, \ldots, X_n]$ such that $F(x_1, \ldots, x_n) = 0$ has all its coefficients in $\mathfrak{a}$. Valla [**10**] introduced the notation $\sup \mathfrak{a}$ for the maximal number of $\mathfrak{a}$-independent elements in $\mathfrak{a}$. He found that $\sup \mathfrak{a}$ is bounded above by the height of $\mathfrak{a}$ (which we denote by $\operatorname{ht} \mathfrak{a}$) whereas the lower bound grade $\mathfrak{a}$ was established by Rees:

PROPOSITION 1. $\operatorname{grade} \mathfrak{a} \leqslant \sup \mathfrak{a} \leqslant \operatorname{ht} \mathfrak{a}$.

(The grade of $\mathfrak{a}$ is the maximal length of a R-sequence contained in $\mathfrak{a}$.) More precisely Rees provided in [**9**] the following proposition.

PROPOSITION 2. *Let $x_1, \ldots, x_n$ be a R-sequence. Then $x_1, \ldots, x_n$ are independent with respect to the ideal they generate in R.*

The relationship between R-sequences and sequences of elements which are independent with respect to an ideal can best be explained in terms of the Rees ring of an ideal $\mathfrak{a}$. It is the graded ring

$$R(\mathfrak{a}) := \bigoplus_{i \geqslant 0} \mathfrak{a}^i .$$

Let $x_1, \ldots, x_n$ generate $\mathfrak{a}$. Then, as was shown in the proof of Theorem 2.1 of [**9**], $x_1, \ldots, x_n$ are independent with respect to an ideal $\mathfrak{b} \supset \mathfrak{a}$ if and only if the homomorphism $R[X_1, \ldots, X_n] \to R(\mathfrak{a})$, which sends X_i to x_i, induces an isomorphism

$$\phi : (R/\mathfrak{b})[X_1, \ldots, X_n] \to R(\mathfrak{a})/\mathfrak{b}R(\mathfrak{a}) ,$$

and exactly when $x_1, \ldots, x_n$ form a R-sequence, ϕ is already an isomorphism for $\mathfrak{b} = \mathfrak{a}$ (cf. [**6**; Theorem p. 202], and observe that $R(\mathfrak{a})/\mathfrak{a}R(\mathfrak{a})$ is just the associated graded ring of R with respect to $\mathfrak{a}$).

A system of parameters in a local ring R is independent with respect to the maximal ideal $\mathfrak{m}_R$ of R; the most important consequence of this theorem is the analytic independence of a system of parameters in a complete equicharacteristic local ring [**13**; Theorem 2.1 and Corollary 2]. By an elementary localization argument one can further conclude that $\sup \mathfrak{b} = \operatorname{ht} \mathfrak{b}$ for radical ideals $\mathfrak{b}$ [**10**; Theorem 5.4].

We do not believe that $\sup \mathfrak{a}$ can always be expressed in terms of better

Received 4 September, 1979.

J. LONDON MATH. SOC. (2), 22 (1980), 57–62]

understood invariants of $\mathfrak{a}$. There are simple examples of ideals $\mathfrak{a}$ with $\sup \mathfrak{a}^n < \sup \mathfrak{a}$ for some $n > 1$. They demonstrate that it is in general impossible to compute $\sup \mathfrak{a}$ from invariants which do not change if $\mathfrak{a}$ is replaced by a power of itself. Thus one is led to study the "asymptotic stabilization"

$$\sup^\infty \mathfrak{a} := \inf \{\sup \mathfrak{a}^n : n \geqslant 1\},$$

and this is the main object of our note. The final theorem is a bit too technical to be given here. However, two special cases suffice to indicate the nature of our results:

(1) Let R be a local ring, and $\mathfrak{m}_R$ its maximal ideal; then $\sup^\infty \mathfrak{m}_R$ is the minimal Krull dimension of an associated prime ideal in the $\mathfrak{m}_R$-adic completion $\hat{R}$ of R:

$$\sup^\infty \mathfrak{m}_R = \min \{\dim \hat{R}/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} \hat{R}\}.$$

(2) If R is an excellent domain, then $\sup^\infty \mathfrak{a} = \sup \mathfrak{a} = \operatorname{ht} \mathfrak{a}$ for every ideal $\mathfrak{a}$ of R.

We start our investigation by giving an upper bound for $\sup^\infty \mathfrak{a}$. It is a consequence of Proposition 1 and the lemma of Artin–Rees.

PROPOSITION 3. *Let R be a noetherian ring, $\mathfrak{a}$ an ideal of R, and S a flat R-algebra. Then* $\sup^\infty \mathfrak{a} \leqslant \operatorname{ht} (\mathfrak{a}S + \mathfrak{p})/\mathfrak{p}$ *for every associated prime ideal* $\mathfrak{p}$ *of* S.

Proof. According to Proposition 1 it is enough to prove that $\sup^\infty \mathfrak{a} \leqslant \sup^\infty (\mathfrak{a}S + \mathfrak{p})/\mathfrak{p}$. An associated prime ideal $\mathfrak{p}$ is the annihilator of an element $x \in S$. As a consequence of the lemma of Artin–Rees (cf. [**8**; (3.12)], there exists an integer j such that

$$(\mathfrak{a}^i S) : xS \subset \mathfrak{p} + \mathfrak{a}^{i-j} S \qquad (*)$$

for $i > j$. Let $x_1, \ldots, x_n$ be elements of $\mathfrak{a}^{m+j}$, where m is chosen such that $\sup^\infty (\mathfrak{a}S + \mathfrak{p})/\mathfrak{p} = \sup (\mathfrak{a}^m S + \mathfrak{p})/\mathfrak{p}$. In case $n > \sup (\mathfrak{a}^m S + \mathfrak{p})/\mathfrak{p}$, one can find a form $F \in S[X_1, \ldots, X_n]$, which is not contained in $(a^m S + \mathfrak{p})S[X_1, \ldots, X_n]$ and vanishes at $(x_1, \ldots, x_n)$ modulo $\mathfrak{p}$. Then $xF(x_1, \ldots, x_n) = 0$, and at least one coefficient of xF lies outside $\mathfrak{a}^{m+j}S$ because of $(*)$. Since the extension from R to S is flat by hypothesis, every S-relation of elements of R is a linear combination of R-relations. Therefore $x_1, \ldots, x_n$ can not be $\mathfrak{a}^{m+j}$-independent.

As an immediate consequence of Proposition 3 one obtains $\sup^\infty \mathfrak{m}_R \leqslant \min \{\dim \hat{R}/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} \hat{R}\}$ for the maximal ideal of a local ring R. The lemma below is the first step towards the proof of the opposite inequality.

LEMMA. *Let R be a complete local ring with a single associated prime ideal $\mathfrak{p}$, and $x_1, \ldots, x_r$ a system of parameters of R. Then there is a ring $S \subset R$ with the following properties:*

(a) *S is a complete local ring. S is regular or a residue class ring of a regular local ring modulo a power of a prime element;*

(b) *R is a finitely generated torsionfree S-module;*

(c) *the elements $x_1, \ldots, x_r$ form a system of parameters in S;*

(d) *for every non-zero $x \in R$ there exists a $f \in \operatorname{Hom}_S(R, S)$ with $f(x) \neq 0$.*

Proof. We use the structure theorems for complete local rings [5]. If R is equicharacteristic, then we choose $S = K[\![x_1, ..., x_r]\!]$ where K is a coefficient field of R. Otherwise there exist a complete discrete valuation domain V and a local homomorphism $\phi : V[\![X_1, ..., X_1]\!] \to R$, which sends X_i to x_i and induces an isomorphism $V/\mathfrak{m}_V \xrightarrow{\sim} R/\mathfrak{m}_R$. Then we take $S = V[\![X_1, ..., X_n]\!]/\text{Ker}\,\phi$. In both cases S is complete, R is a finitely generated S-module by [8; (30.6)], and S is regular in the equicharacteristic case. The fundamental theorems on integral extensions imply that the dimensions of R and S coincide, that $x_1, ..., x_r$ is a system of parameters in S, and that $\mathfrak{p} \cap S$ is the only associated prime ideal of S. Therefore every non-zero divisor of S is a non-zero divisor of R: the S-module R is torsionfree. Since

$$\dim V[\![X_1, ..., X_r]\!] = \dim R+1 = \dim S+1,$$

Ker ϕ is an ideal of height 1, and has a single prime divisor, as was just shown. Hence Ker ϕ is generated by a power of a prime element in the factorial ring $V[\![X_1, ... X_r]\!]$. This completes the proof of (a), (b) and (c).

Property (d) follows from (a) and (b): Every finitely generated torsionfree module M over S is torsionless, i.e. the natural homomorphism

$$M \to \text{Hom}_S(\text{Hom}_S(M, S), S)$$

is injective. In fact, it suffices that the localization of S with respect to its associated prime ideal is a Gorenstein ring [12; Theorem (A.1)].

Proposition 4. *Let R be a complete local ring with a single associated prime ideal, and $x_1, ..., x_r$ a system of parameters in R. Then, given an integer m, the elements $x_1^i, ..., x_r^i$ are independent with respect to $\mathfrak{m}_R^m$ for i sufficiently large.*

Proof. We choose S as in the preceding lemma. Let $\mathfrak{a}_i$ denote the ideal generated by $x_1^i, ..., x_r^i$ in S, and put

$$N_i := \bigcap \{f^{-1}(\mathfrak{a}_i) : f \in \text{Hom}_S(R, S)\}.$$

A theorem of Chevalley [8; (30.1)] tells us: If $(\mathfrak{b}_i)$ is a descending chain of ideals in a complete local ring such that $\bigcap \mathfrak{b}_i = 0$, then $\mathfrak{b}_i \subset \mathfrak{m}_T^m$ for all sufficiently large i. This theorem remains valid if one replaces $(\mathfrak{b}_i)$ by a descending chain (M_i) of submodules of a finitely generated T-module M and, correspondingly, $\mathfrak{m}_T^m$ by $\mathfrak{m}_T^m M$.

The S-modules N_i of R form a descending chain, Since $\bigcap \mathfrak{a}_i = 0$, property (d) of S, as given by the lemma, guarantees that $\bigcap N_i = 0$, and thus $N_i \subset \mathfrak{m}_S^m R \subset \mathfrak{m}_R^m$ as soon as i is large enough. Let now

$$F = \sum_{j_1+ ... +j_r=d} a_{j_1 ... j_r} X_1^{j_1} ... X_r^{j_r}$$

be a form in $R[X_1, ..., X_r]$ such that $F(x_1^i, ..., x_r^i) = 0$. Then for all $f \in \text{Hom}_S(R, S)$ we obtain

$$f(F(x_1^i, ..., x_r^i)) = \sum_{j_1+ ... +j_r=d} f(a_{j_1 ... j_r}) x_1^{ij_1} ... x_1^{ij_r} = 0.$$

By Proposition 2, all the elements $f(a_{j_1 \dots j_r})$ are contained in $\mathfrak{a}_i$, hence $a_{j_1 \dots j_r} \in N_i \subset \mathfrak{m}_{\hat{R}}^m$ if i is sufficiently large.

The idea, by which we shall now derive the first main result, has already been used in the proof of [4; Satz 6].

THEOREM 1. *Let R be a local ring, and $\mathfrak{n}_1 \cap \dots \cap \mathfrak{n}_s = 0$ a reduced primary decomposition of the zero ideal in $\hat{R}$. Let the images of $x_1, \dots, x_r \in R$ under the natural homomorphism $R \to \hat{R}/\mathfrak{n}_t$ form part of a system of parameters in $\hat{R}/\mathfrak{n}_t$ for $t = 1, \dots, s$. Then, given m, the elements $x_1^i, \dots, x_r^i$ are independent with respect to $\mathfrak{m}_R^m$ for i sufficiently large.*

Proof. Since $\mathfrak{m}_{\hat{R}}^m \cap R = \mathfrak{m}_R^m$, we may assume that $R = \hat{R}$. By the preceding proposition, the elements $x_1^i, \dots, x_r^i$ modulo $\mathfrak{n}_t$ are independent with respect to the k-th power of the maximal ideal of $R/\mathfrak{n}_t$ if i is chosen large enough. Hence $x_1^i, \dots, x_r^i$ themselves are independent with respect to

$$\mathfrak{a}_k := \bigcap_{t=1}^{s} (\mathfrak{m}_R^k + \mathfrak{n}_t).$$

Now we only need to prove that the inclusion $\mathfrak{a}_k \subset \mathfrak{m}_R^m$ holds for large k. This, however, is again a consequence of Chevalley's theorem since we have

$$\bigcap_k \mathfrak{a}_k = \bigcap_{t=1}^{s} \left(\bigcap_k (\mathfrak{m}_R^k + \mathfrak{n}_t) \right) = \bigcap_{t=1}^{s} \mathfrak{n}_t = 0.$$

COROLLARY. *Let R be a local ring. Then*

$$\sup^\infty \mathfrak{m}_R = \min \{\dim \hat{R}/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} \hat{R}\}.$$

Proof. With the notation of Theorem 1, we can find elements $x_1, \dots, x_r \in R$, satisfying the hypothesis of that theorem, by elementary prime avoidance arguments, as long as $r \leqslant \min \{\dim \hat{R}/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} \hat{R}\}$.

The minimal dimension of an associated prime divisor of $\hat{R}$ was thoroughly investigated in [2], and was (under mild restrictions) characterized as the maximal length of so-called quasi-R-sequences [2; (5.5)]. The preceding corollary provides another characterization by intrinsic properties of R.

Now we can compute $\sup^\infty \mathfrak{a}$ for ideals $\mathfrak{a}$ in arbitrary noetherian rings, and simultaneously improve the lower bound of Proposition 1. Brodmann [3] recently showed that the set of prime ideals in R, which are associated to $R/\mathfrak{a}^n R$, is independent of n for n sufficiently large. We denote this collection of prime ideals by $\operatorname{Ass}^\infty \mathfrak{a}$. (To avoid any ambiguity: $\hat{R}_\mathfrak{p}$ is the $\mathfrak{p}R_\mathfrak{p}$-adic completion of $R_\mathfrak{p}$.)

THEOREM 2. *Let R be a noetherian ring, and $\mathfrak{a}$ an ideal in R. Then*

$$\sup \mathfrak{a} \geqslant \min \{\operatorname{ht} (\mathfrak{a}\hat{R}_\mathfrak{p} + \mathfrak{q})/\mathfrak{q} : \mathfrak{p} \in \operatorname{Ass} R/\mathfrak{a}, \quad \mathfrak{q} \in \operatorname{Ass} \hat{R}_\mathfrak{p}\},$$

and

$$\sup^\infty \mathfrak{a} = \min \{\operatorname{ht} (\mathfrak{a}\hat{R}_\mathfrak{p} + \mathfrak{q})/\mathfrak{q} : \mathfrak{p} \in \operatorname{Ass}^\infty \mathfrak{a}, \quad \mathfrak{q} \in \operatorname{Ass} \hat{R}_\mathfrak{p}\}.$$

Proof. In view of Proposition 3 it is enough to prove the inequality. We use an obvious localization argument. The ideal $\mathfrak{a}$ has a reduced primary decomposition: $\mathfrak{a} = \bigcap_{t=1}^{s} \mathfrak{b}_t$, $\mathfrak{b}_t$ being primary to a prime ideal $\mathfrak{p}_t$, and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} = \operatorname{Ass} R/\mathfrak{a}$. Let r denote the number on the right hand side of the inequality. Using Krull's Principal Ideal Theorem and elementary prime avoidance arguments one constructs elements $x_1, \ldots, x_r \in \mathfrak{a}$, which have the following property: their images under each of the natural homomorphism $R \to \hat{R}_\mathfrak{p}/\mathfrak{q}$ form part of a system of parameters in $\hat{R}_\mathfrak{p}/\mathfrak{q}$, $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{a}$, $\mathfrak{q} \in \operatorname{Ass} \hat{R}_\mathfrak{p}$. Then $x_1, \ldots, x_r$ satisfy the requirements of Theorem 1 in every localization $R_\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{a}$. Hence $x_1^i, \ldots, x_r^i$ are $\mathfrak{b}_t R_\mathfrak{p}$-independent, $\mathfrak{b}_t R_{\mathfrak{p}_t}$ containing a power of $\mathfrak{p}_t R_{\mathfrak{p}_t}$, as soon as i is large enough. This suffices to render $x_1^i, \ldots, x_r^i$ independent with respect to $\mathfrak{a}$ since $\mathfrak{a} = \bigcap_{t=1}^{s} \mathfrak{b}_t$ and $\mathfrak{b}_t = \mathfrak{b}_t R_{\mathfrak{p}_t} \cap R$.

Theorem 2 answers many questions raised in [1] and [11]. In particular, we can describe the rings in which $\sup \mathfrak{a} = \operatorname{ht} \mathfrak{a}$ for all ideals.

Corollary 1. *Let R be a noetherian ring. Then* $\sup \mathfrak{a} = \operatorname{ht} \mathfrak{a}$ *for all ideals* $\mathfrak{a}$ *of R if and only if* $\dim \hat{R}_\mathfrak{p}/\mathfrak{q} = \operatorname{ht} \mathfrak{p}$ *for all prime ideals* $\mathfrak{p}$ *of R and all associated prime ideals* $\mathfrak{q}$ *of* $\hat{R}_\mathfrak{p}$.

Proof. If $\sup \mathfrak{a} = \operatorname{ht} \mathfrak{a}$ for all ideals $\mathfrak{a}$ of R, then in particular $\sup^\infty \mathfrak{p} = \operatorname{ht} \mathfrak{p}$, and

$$\sup^\infty \mathfrak{p} \leqslant \operatorname{ht} (\mathfrak{p}\hat{R}_\mathfrak{p} + \mathfrak{q})/\mathfrak{q} \leqslant \operatorname{ht} \mathfrak{p}\hat{R}_\mathfrak{p} = \operatorname{ht} \mathfrak{p}$$

by Proposition 3. On the other hand, if $\dim \hat{R}_\mathfrak{p}/\mathfrak{q} = \operatorname{ht} \mathfrak{p}$ for all prime ideals $\mathfrak{p}$ and all associated prime ideals $\mathfrak{q}$ of $\hat{R}_\mathfrak{p}$, then

$$\begin{aligned} \operatorname{ht} \mathfrak{a} &= \min \{\dim R_\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} R/\mathfrak{a}\} \\ &= \min \{\dim \hat{R}_\mathfrak{p}/\mathfrak{q} : \mathfrak{p} \in \operatorname{Ass} R/\mathfrak{a}, \quad \mathfrak{q} \in \operatorname{Ass} \hat{R}_\mathfrak{p}\} \\ &\leqslant \sup \mathfrak{a} \leqslant \operatorname{ht} \mathfrak{a} \end{aligned}$$

by Proposition 1 and Theorem 2.

Nagata gave an example of a two dimensional local domain R [8; pp. 204, 205] such that $\hat{R}$ has an associated prime ideal $\mathfrak{q}$ with $\dim \hat{R}/\mathfrak{q} = 1$. Thus, even in local domains $\sup \mathfrak{a}$ does not always equal $\operatorname{ht} \mathfrak{a}$. However, in a class of rings, which behave well under completion, we can simplify our formula considerably:

Corollary 2. *Let R be an excellent integral domain. Then for every ideal* $\mathfrak{a}$ *in R one has* $\sup \mathfrak{a} = \operatorname{ht} \mathfrak{a}$.

Proof. The localizations of R are excellent again, and the completions of excellent local domains are (reduced and) equi-dimensional (cf. [7]).

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