POLYHEDRAL ALGEBRAS, ARRANGEMENTS OF TORIC VARIETIES, AND THEIR GROUPS

WINFRIED BRUNS AND JOSEPH GUBELADZE

Abstract. We investigate the automorphism groups of graded algebras defined by lattice polyhedral complexes and of the corresponding projective varieties, which form arrangements of projective toric varieties. These groups are polyhedral versions of the general and projective linear groups. It is shown that for wide classes of complexes they are generated by toric actions, elementary transformations and symmetries of the underlying complex. The main results extend our previous work for single polytopes [BG].

1. Introduction

In our previous paper [BG] we generalized standard properties of the group $\mathrm{GL}_n(k)$ of graded automorphisms of the polynomial ring $k[x_1, \dots, x_n]$ over a field k to the group $\mathrm{gr.aut}(k[S_P])$ of graded automorphisms of a polytopal k-algebra $k[S_P]$ associated with a lattice polytope P. The generators of the k-algebra $k[S_P]$ correspond bijectively to the lattice points in P, and their relations are the binomials representing the affine dependencies of the lattice points. (See Bruns, Gubeladze, and Trung [BGT] for polytopal algebras.) Thus $k[x_1, \dots, x_n]$ can be viewed as the polytopal algebra $k[S_{\Delta_{n-1}}]$ for the unit $(n-1)$-simplex Δ_{n-1}, and the fact that every invertible matrix can be reduced to a diagonal one by elementary row transformations is then a special case of our theorem [BG, Theorem 3.2] that every element of $\mathrm{gr.aut}(k[S_P])$ is a composition of elementary automorphisms, toric automorphisms, and affine symmetries of the polytope. (The symmetries are only needed if $\mathrm{gr.aut}(k[S_P])$ is not connected.) Polytopal algebras and their normalizations are special instances of affine semigroup algebras; more generally, we have described the group of graded automorphisms of an arbitrary normal affine semigroup algebra [BG, Remark 3.3(c)].

In [BG] an application to toric geometry is a description of the automorphism group of a projective toric variety over an algebraically closed field of arbitrary characteristic. Our approach avoids the theory of linear algebraic groups, and for projective toric varieties we have strengthened the classical theorem of Demazure [De] and its recent generalizations by Cox [Co] and Bühler [Bu].

The main issue of this paper is a generalization from the case of a single polytope to algebras $k[\Pi]$ corresponding to *lattice polyhedral complexes* Π of type as general as possible; these algebras will be called *polyhedral algebras*. Thus we are concerned with the graded automorphisms of fiber products of polytopal algebras, labeled naturally by lattice polyhedral complexes. In plain terms, the set of monomials of $k[\Pi]$ is the union of the set of monomials of the algebras $k[S_P]$ where P runs through the facets of Π, and the product of two monomials is their product in $k[S_P]$ if there exists P with both monomials belonging to $k[S_P]$, and zero otherwise. The simplest representatives of such algebras are Stanley-Reisner rings of simplicial complexes, whose graded automorphisms have recently been considered by Müller [Mu].

There is a natural hierarchy of lattice polyhedral complexes

$$\begin{aligned}
&\{\textit{abstract simplicial complexes}\} \subset \\
&\quad\{\textit{boundary lattice polyhedral complexes}\} \subset \\
&\quad\quad\{\textit{Euclidean lattice polyhedral complexes}\} \subset \\
&\quad\quad\quad\{\textit{quasi-Euclidean lattice polyhedral complexes}\} \subset \\
&\quad\quad\quad\quad\{\textit{oriented lattice polyhedral complexes}\} \subset \\
&\quad\quad\quad\quad\quad\{\textit{general lattice polyhedral complexes}\},
\end{aligned}$$

which appears in the subsequent sections; each of these classes constitutes just a small subclass in the next class, as illustrated by examples.

Boundary lattice complexes are obtained as subcomplexes of the set of faces of a single lattice polytope, whereas Euclidean complexes are formed by a collection of lattice polytopes in a Euclidean space $\mathbb{R}^n$ whose lattice structures are induced from the lattice $\mathbb{Z}^n$. For a quasi-Euclidean complex Π we relax the last requirement: the lattice providing the semigroup associated with each face of Π may vary among the facets of the complex. The definition of an oriented lattice polyhedral complex is more technical; roughly speaking, it permits us to define *elementary automorphisms* in terms of so-called *column structures*.

The group $\operatorname{gr.aut}(k[\Pi])$ is a linear algebraic group in a natural way. We will show that the elementary automorphisms together with the toric automorphisms generate its unity component if Π is oriented; if Π is even quasi-Euclidean, then the whole group is generated by elementary automorphisms, diagonal automorphisms and symmetries of the underlying complex. Here an automorphism α is called *diagonal* if each monomial is an eigenvector for α, and the *toric* automorphisms are the members of the unity component of the group of diagonal automorphisms (in the case of a single polytope this group is always connected). Moreover, under a certain combinatorial condition on the complex, one can provide a normal form for the representation of a general automorphism. This is the first main result of the paper (Theorem 5.2).

The combinatorial treatment that was successful in the case of a single polytope [BG] becomes exceedingly complicated for polyhedral complexes. Instead we will invoke Borel's theorem on maximal algebraic tori and other algebro-geometric arguments.

Polytopal algebras are related to graded normal affine semigroup algebras in the same way as polyhedral algebras are related to graded algebras defined by *rational polyhedral complexes* (Section 2). (Combinatorial aspects of algebras defined by certain rational polyhedral complexes have been discussed by Stanley [Sta].) Analogously to the situation of a single polytope, our arguments apply to this class of algebras as well, yielding a description of their graded automorphism groups. An even more general class is constituted by the algebras described in terms of *weak fans* (Section 2). They are analogues of general, non-graded normal affine semigroup rings and are useful in the description of affine charts for *arrangements* of projective toric varieties; see Section 6. The analogy is limited, though: neither is the normalization of a polyhedral algebra combinatorially well-behaved in general, nor do all algebras given by weak fans come from rational polyhedral complexes (i. e. carry a graded structure such that monomials are homogeneous and of positive degree).

The second main result (Theorem 9.1) concerns the automorphism group of an arrangement of projective toric varieties, i. e. the Proj of a polyhedral algebra. Here the situation is more complicated than it was for projective toric varieties themselves: no longer can one give a natural one-to-one 'polyhedral interpretation' of very ample line bundles, which exists for single polytopes (Teissier [Te]). However, using once again Borel's theorem on maximal tori, we show that there are still reasonable polyhedral 'images' of the spaces of global sections for certain very ample line bundles. This suffices for the computation of the unity component of the automorphism group of an arrangement defined by a quasi-Euclidean complex and of the whole group for an arrangement defined by a *projectively quasi-Euclidean* complex Π; such a complex is distinguished by the fact that every complex projectively equivalent to Π is also quasi-Euclidean. Not all quasi-Euclidean complexes are projectively quasi-Euclidean, but in Section 8 we describe two natural big classes of such complexes; one of them includes the simplicial complexes.

In conjunction with [BG] this paper establishes a polyhedral generalization of classical K-theoretical objects – the general linear group $\mathrm{GL}_n(k)$ and its elementary subgroup $E_n(k)$. Naturally there arises a question: is there a further analogy with K-theory that might lead to a *polyhedral* K-theory? Already for low dimensional K-groups this question suggests challenging open problems.

Acknowledgement. The second author was supported by the Alexander von Humboldt Foundation, the MR Project (Contract Offer ERB FMRX CT-97-0107) and by INTAS (93-2618-Ext). Their generous grants are gratefully acknowledged.

2. Polyhedral complexes and polyhedral algebras

A polytope in a real vector space $\mathbb{R}^n$ is the convex hull of finitely many points. The vertices of a lattice polytope belong to the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

We recall that for a lattice polytope $P \subset \mathbb{R}^n$ the sub-semigroup $S_P \subset \mathbb{Z}^{n+1}$ is by definition generated by $\{(x,1) \mid x \in \mathbb{Z}^n \cap P\} \subset \mathbb{Z}^{n+1}$. For a field k the semigroup algebra $k[S_P]$ is called the polytopal algebra of P over k ([BGT], [BG]).

Definition 2.1. A *lattice polyhedral complex* Π consists of

(a) an *abstract (finite) polyhedral complex* Π_X, that is a finite set X of *vertices* and a system Π_X of subsets of X such that $P \cap Q \in \Pi_X$ whenever $P, Q \in \Pi_X$,
(b) an embedding $P \to \mathbb{R}^{n_P}$ for each $P \in \Pi_X$ such that the image of P constitutes the vertex set of an n_P-dimensional lattice polytope $P^* \subset \mathbb{R}^{n_P}$,
(c) an embedding $\iota_{PQ} : P^* \to Q^*$ for each inclusion $P \subset Q$, $P, Q \in \Pi_X$ such that ι_{PQ} is an isomorphism of P^* with a face of Q^* as lattice polytopes.

Furthermore we require the following compatibility conditions:

(i) $\iota_{QR} \circ \iota_{PQ} = \iota_{PR}$ for $P, Q, R \in \Pi_X$, $P \subset Q \subset R$,
(ii) for every element $Q \in \Pi_X$ and each face F of the polytope Q^* there is an element $P \in \Pi_X$ such that $P \subset Q$ and $\iota_{PQ}(P^*) = F$.

(The condition $\dim(P^*) = n_P$ is useful for convenience of notation when we define projectively equivalent polyhedral complexes in Section 7.)

Let Π be a lattice polyhedral complex. For $P \in \Pi_X$ the set of lattice points of P^* will be denoted by $\mathrm{L}(P^*)$. We want to identify lattice points $x \in \mathrm{L}(P^*)$ and $y \in \mathrm{L}(Q^*)$ if $\iota_{PQ}(x) = y$. More precisely, we introduce the equivalence relation $\sim$ on the disjoint union of the sets of lattice points $\mathrm{L}(P^*)$, $P \in \Pi_X$, that is spanned by the relations $x \sim y$ for all x, y such that there exist $P, Q \in \Pi_X$ with $x \in \mathrm{L}(P^*)$, $y \in \mathrm{L}(Q^*)$, $P \subset Q$, and $\iota_{PQ}(x) = y$. The set of equivalence classes with respect to $\sim$ is denoted by $\mathrm{L}(\Pi)$ and they are called *lattice points of* Π. For simplicity of notation we will identify $\mathrm{L}(P^*)$ with its image in $\mathrm{L}(\Pi)$.

Let k be a field and Π a lattice polyhedral complex. Then it is easy to show that there exists a unique k-algebra $k[\Pi]$, satisfying the following conditions:

(1) $k[\Pi]$ is generated by $\mathrm{L}(\Pi)$;
(2) for any element $P \in \Pi_X$ the subalgebra of $k[\Pi]$ generated by $\mathrm{L}(P^*)$ is naturally isomorphic to the polytopal algebra $k[S_{P^*}]$;
(3) if there exists no $Q \in \Pi_X$ such that $x_1, \dots, x_s \in \mathrm{L}(\Pi)$ all belong to $\mathrm{L}(Q^*)$, then $x_1 \cdots x_s = 0$.

The algebra $k[\Pi]$ will be called the *polyhedral algebra of* Π. Condition (2) just means that for each $P \in \Pi_X$ the elements of $\mathrm{L}(P) \subset \mathrm{L}(\Pi)$ satisfy the binomial relations corresponding to their affine relations as lattice points in P^*. Furthermore these binomial relations together with the monomial relations in (3) define $K[\Pi]$.

Convention: The polytopes P^* will simply be denoted by P and they will be called *faces* of Π. We will write $P \prec \Pi$. Moreover, for $P, Q \in \Pi$, $P \subset Q$, we indicate by $P \prec Q$ that P is considered as a face of Q via ι_{PQ}.

The elements of the semigroups S_P, $P \prec \Pi$, will be called *monomials*; elements of the form αx, $\alpha \in k^*$, $x \in S_P$ are called *terms*.

Let Π^{face} be the poset (with respect to $\prec$) of the faces of Π, and Π^{facet} the subset consisting of all faces that can be written as an intersection of *facets*, i. e. maximal faces, of Π. Our conditions imply that we have a contravariant functor to (commutative) k-algebras:

$$\operatorname{alg}_k^{\text{face}} : \Pi^{\text{face}} \to k\text{-alg}$$

for which

$$\operatorname{alg}_k^{\text{face}}(P) = k[S_P] \quad \text{and} \quad \operatorname{alg}_k^{\text{face}}(P \prec Q) = (\text{the 'face projection' } k[Q] \to k[P]).$$

('Face projection' here means the unique k-algebra homomorphism under which $\mathrm{L}(Q) \setminus \mathrm{L}(P)$ is mapped to $0 \in k$ and each $x \in \mathrm{L}(P)$ to itself.) The restriction of $\operatorname{alg}_k^{\text{face}}$ to Π^{facet} will be denote by $\operatorname{alg}_k^{\text{facet}}$.

The following is the universal characterization of $k[\Pi]$:

$$k[\Pi] = \varprojlim \operatorname{alg}_k^{\text{face}} = \varprojlim \operatorname{alg}_k^{\text{facet}}.$$

Definition 2.2. (a) A polyhedral subcomplex of the complex of all faces of some lattice polytope is called a *boundary polyhedral lattice complex.*

(b) A lattice polyhedral complex that can be realized as a polyhedral complex of lattice polytopes (with respect to $\mathbb{Z}^n$) in some real vector space $\mathbb{R}^n$ is called *Euclidean.*

(c) A lattice polyhedral complex Π, realizable as a polyhedral complex of rational polytopes in some real vector space, is called *quasi-Euclidean.*

Proposition 2.3. (a) *If Π is the lattice polyhedral complex of all faces of some lattice polytope P (including P itself), then $k[\Pi] = k[S_P]$.*

(b) *If Π is a lattice simplicial complex consisting of unit lattice simplices, then $k[\Pi]$ is exactly the Stanley-Reisner ring of Π_X. Any Stanley-Reisner ring can be realized in this way.*

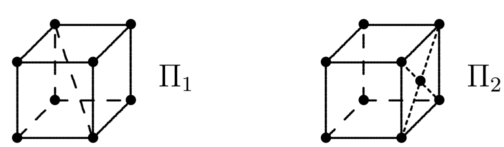
(c) *The inclusions* $\{$*abstract simplicial complexes*$\} \subset \{$*boundary lattice complexes*$\} \subset \{$*Euclidean complexes*$\} \subset \{$*quasi-Euclidean complexes*$\}$ *are strict.*

(See Bruns and Herzog [BH, Ch.5] for Stanley-Reisner rings.)

Proof. The claims (a) and (b) are obvious, as is the first inclusion in (c). It is evidently strict. So we only need to construct a Euclidean, but not boundary, lattice polyhedral complex and a quasi-Euclidean, but not Euclidean, one.

Consider the polyhedral complexes in Figure 1 where Π_1 consists of the 6 two-

FIGURE 1



dimensional facets forming the surface of the lattice unit cube and one more facet given by a space diagonal, and Π_2 has 6 two-dimensional facets of which 5 are lattice unit squares and the 6th has an additional lattice point in its barycenter.

We claim that Π_1 is not a boundary complex. In fact, assume to the contrary that there exists a lattice polytope P in $\mathbb{R}^m$ whose boundary complex contains Π_1. Then there is a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}$ that is positive on Π_1 outside the 1-dimensional facet and is 0 on it. Now observe that the affine hull of Π_1 in $\mathbb{R}^m$ is 3-dimensional and that the 2-dimensional facets of Π_1 must form the boundary of a 3-dimensional parallelepiped Ξ in $\mathbb{R}^m$ – this is an obvious rigidity property of the boundary complex of the unit 3-cube. The space diagonal, except its end-points, consists of interior points of Ξ. Hence any linear form positive on the boundary of Ξ (except the endpoints of the space diagonal) must also be positive in the interior of this diagonal – a contradiction.

It is easy to check that, like in the previous case, a Euclidean realization of Π_2 must form the boundary of some lattice parallelepiped. In particular, the lattice structures of each of the opposite pairs of facets must be naturally isomorphic. But this is not the case for Π_2 and, hence, there is no Euclidean realization of Π_2. (That Π_2 is quasi-Euclidean is obvious.) □

To a lattice polyhedral complex Π one can also associate a semigroup (commutative, with unity) S_Π, which is generated by $\mathrm{L}(\Pi)$ and one extra element ∞ in such a way that

(1) S_P is a sub-semigroup of S_Π for every face $P \in \Pi$,
(2) $x \cdot \infty = \infty \cdot \infty = \infty$ and $x_1 \cdots x_s = \infty$ whenever $x_1 \cdots x_s = 0$ in $k[\Pi]$.

Of course, this definition is independent of the field k. The kernel of the natural surjection $k[S_\Pi] \to k[\Pi]$ is the ideal (∞) $(\dim_k(\infty) = 1)$. Moreover, S_Π is mapped isomorphically to the multiplicative sub-semigroup of $k[\Pi]$ generated by $\mathrm{L}(\Pi)$ and 0.

Observe that $k[\Pi]$ is equipped with a natural grading:

$$k[\Pi] = k \oplus A_1 \oplus A_2 \oplus \cdots, \qquad A_1 = k\mathrm{L}(\Pi).$$

The group of graded k-automorphisms of $k[\Pi]$, denoted by $\Gamma_k(\Pi)$ later on, is called the *polyhedral linear group associated with* Π. Clearly, If Π is a lattice polyhedral complex determined by a lattice polytope P, then $\Gamma_k(\Pi)$ is the polytopal linear group $\Gamma_k(P)$ of [BG]. As for polytopal groups, one observes easily that polyhedral linear groups are affine k-groups: $\Gamma_k(\Pi)$ is a closed subgroup of $\mathrm{GL}_N(k)$, $N = \#\mathrm{L}(\Pi)$, whose defining equations are derived from the relations between the degree 1 monomials of $k[\Pi]$ by use of an obvious, simple algorithm.

The group of semigroup automorphisms of S_Π will be denoted by $\Sigma(\Pi)$. It is a finite group embedded into $\Gamma_k(\Pi)$ in a natural way, and we will identify $\Sigma(\Pi)$ with its image.

Next we introduce the notion of a *rational polyhedral complex*. The corresponding graded algebras are related to polyhedral algebras in the same way as graded normal affine semigroup rings are related to polytopal algebras.

Definition 2.4. A *rational polyhedral complex* Π_{rat} consists of the following data:

(a) an abstract polyhedral complex Π_X,

(b) an embedding $P \to \mathbb{R}^{n_P}$ for each $P \in \Pi_X$ such that the image is the vertex set of a *rational* polytope P^* (with respect to $\mathbb{Q}^{n_P} \subset \mathbb{R}^{n_P}$) whose faces correspond to the sets $\{R \in \Pi_X \mid R \subset P\}$ so that if $P \subset Q$ are two elements of Π_X, then the polytope P^* and the face P' of Q^* corresponding to P are naturally isomorphic as rational polytopes.

Furthermore we require that the isomorphism of P^* and P' induces a bijection between the sets of lattice points of $cP^* \cap \mathbb{Z}^{n_P}$ and $cP' \cap \mathbb{Z}^{n_Q}$ for each $c \in \mathbb{N}$. (Here cP^* and cP' denote the c-th homothetic images.)

It may happen that the faces of a finite rational polyhedral complex Π_{rat} are actually lattice polytopes, but the subscript $-_{\mathrm{rat}}$ emphasizes that we are considering the rational structure.

For a face $P \in \Pi_{\mathrm{rat}}$ we let $C(P)$ denote the finite rational convex cones in $\mathbb{R}^{n_P+1}$ with apex 0 that is spanned by $\{(x,1) \mid x \in P\}$; moreover, we let $\hat{S}_P$ denote the sub-semigroup $\mathbb{Z}^{n_P+1} \cap C(P) \subset \mathbb{Z}^{n_P+1}$. The algebra $k[\Pi_{\mathrm{rat}}]$ is defined as the unique algebra satisfying the following conditions:

(1) $k[\hat{S}_P]$ is a subalgebra of $k[\Pi_{\mathrm{rat}}]$ for every face $P \in \Pi_{\mathrm{rat}}$ and if $P \prec Q$, then $k[\hat{S}_P] \subset k[\hat{S}_Q]$ in a natural way,

(2) $x_1 \cdots x_s = 0$ whenever $x_i \in \hat{S}_{P_i}$ for some faces $P_i \in \Pi_{\mathrm{rat}}$, $i \in [1,s]$, and there is no face $R \in \Pi_{\mathrm{rat}}$ such that $x_1, \dots, x_s \in \hat{S}_R$,

(3) $k[\Pi_{\mathrm{rat}}] = \sum_P k[\hat{S}_P]$ as k-spaces, where P runs through the faces of Π_{rat}.

Here we adopt a convention on terminology and notation similar to that we have introduced for lattice polyhedral complexes. In particular, we can speak of a monomial in $k[\Pi_{\mathrm{rat}}]$.

Proposition 2.5. (a) *The class of affine normal semigroup k-algebras coincides with the class of algebras of type $k[\Pi_{\mathrm{rat}}]$ where Π_{rat} is the rational complex (of all faces) of a rational polytope.*

(b) *For a rational polyhedral complex Π_{rat} and a field k the algebra $k[\Pi_{\mathrm{rat}}]$ carries a graded structure where all monomials are homogeneous of positive degree given by the last component of (the exponent vector of) x in $\mathbb{Z}^{n_P+1}$ for $x \in \hat{S}_P$. The group $\Gamma_k(\Pi_{\mathrm{rat}})$ of graded automorphisms of $k[\Pi_{\mathrm{rat}}]$ is an affine k-group.*

(c) *In general a lattice polyhedral complex Π does not define a rational polyhedral complex in a natural way, i. e. if we pass to the normalizations $\{y \in \mathrm{gp}(S_P) \mid y^m \in S_P$ for some $m \in \mathbb{N}\}$ of the S_P where P runs through the faces of Π, then the new system of semigroups may not satisfy the compatibility condition required in* 2.4.

Proof. (a) just says that all affine normal semigroup k-algebras can be equipped with a graded structure such that monomials become homogeneous elements of positive degree (see [BH, Ch. 6]). (b) is an obvious analogue of the corresponding observations for lattice polyhedral complexes.

For (c) consider a 4-dimensional lattice polytope $P \subset \mathbb{R}^4$ such that

(1) its lattice points span $\mathbb{Z}^4$ (as an additive group),
(2) one of its facets is a 3-simplex $\delta \subset P$ whose vertices are the only lattice points in δ, but do not span the whole 3-dimensional affine sublattice $\mathrm{Aff}(\delta) \cap \mathbb{Z}^4 \subset \mathbb{Z}^4$.

($\mathrm{Aff}(\delta)$ is the affine hull of δ in $\mathbb{R}^4$.) The existence of such P is clear: just take a non-unimodular lattice 3-simplex δ in $\mathbb{R}^3$ whose vertices are the only lattice points in δ, and then complete it to a sufficiently big 4-polytope in the upper halfspace (with respect to an embedding $\mathbb{R}^3 \to \mathbb{R}^4$ as a coordinate hyperplane).

Now consider the lattice polyhedral complex having just two facets: P and a unit 4-simplex Δ (in its own ambient Euclidean space) which meet along δ. Then the normalizations of S_P and that of S_Δ do not agree along the cone spanned by δ. (The complex just considered is quasi-Euclidean, but not Euclidean. In fact, a Euclidean complex defines a rational polyhedral complex, since the semigroups of its faces are derived from the same lattice; the corresponding algebras have been considered by Stanley [Sta].) □

In our general setting the rôle of all normal affine semigroup rings is played by the algebras determined by *weak fans*. These algebras are useful in the description of the affine chart of $\mathrm{Proj}(k[\Pi])$ (see Section 6).

Definition 2.6. A *weak fan* $\mathcal{WF}$ consists of the following data:

(a) an abstract polyhedral complex Π_X,
(b) for each $P \in \Pi_X$ a rational strictly convex polyhedral cone $C_P \subset \mathbb{R}^{n_P}$ whose extremal rays are labeled by the elements of P in such a way that the faces of C_P correspond to the faces of P.

Furthermore we require that this correspondence induces an isomorphism of the lattice structures of C_P (with respect to $\mathbb{Z}^{n_P}$) and that of the corresponding face of C_Q (with respect to $\mathbb{Z}^{n_Q}$) if $P, Q \in \Pi_X$, $P \subset Q$.

Observe that a rational polyhedral complex Π_{rat} defines in a natural way a weak fan $\mathcal{WF}(\Pi_{\mathrm{rat}})$: one just considers the cones $C_P \subset \mathbb{R}^{n_P+1}$, $P \in \Pi_{\mathrm{rat}}$. Likewise, any (finite) fan Φ in the sense of toric geometry (for example, see Fulton [Fu]) gives rise to a weak fan $\mathcal{WF}(\Phi)$.

To a field k and a weak fan $\mathcal{WF}$ one associates a k-algebra $k[\mathcal{WF}]$ by patching the semigroup algebras $k[\mathbb{Z}^{n_P} \cap C_P]$ along the facets of the cones C_P. Again, one has the equalities (in self-explanatory notation):

$$k[\mathcal{WF}] = \varprojlim \mathrm{alg}_k^{\mathrm{face}} = \varprojlim \mathrm{alg}_k^{\mathrm{facet}}.$$

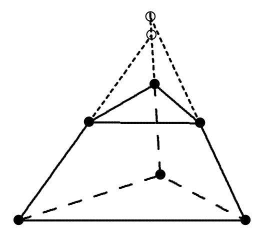
As mentioned already, any normal affine semigroup ring (without non-trivial units) can be equipped with a graded structure so that its monomials become homogeneous and of positive degree. However, not all algebras of type $k[\mathcal{WF}]$ carry a graded structure.

Example 2.7. There exists a complete fan Φ in $\mathbb{R}^3$ such that $\mathcal{WF}(\Phi)$ is not of type $\mathcal{WF}(\Pi_{\text{rat}})$ for some rational polyhedral complex Π_{rat}, i. e. there is no graded structure on $k[\mathcal{WF}(\Phi)]$ (k a field) such that its monomials are homogeneous of positive degree.

First observe that a weak fan $\mathcal{WF}$ is of type $\mathcal{WF}(\Pi_{\text{rat}})$ if and only if $k[\mathcal{WF}]$ carries a graded structure $k[\mathcal{WF}] = k \oplus A_1 \oplus A_2 \oplus \cdots$ such that all monomials are homogeneous of positive degree.

Choose 6 rational non-coplanar points in $\mathbb{R}^3$ as shown in Figure 2 where the top

Figure 2



and bottom triangles are in parallel planes and the 'hidden' quadrangles are flat polygons whereas the frontal quadrangle is not a flat figure. Suppose that $0 \in \mathbb{R}^3$ lies in the interior of the convex hull of these 6 points; then the cones with common apex 0 that are spanned by the 2 triangles and the 3 quadrangles form a complete fan Φ of rational cones in $\mathbb{R}^3$. We claim that $\mathcal{WF}(\Phi)$ is not of type $\mathcal{WF}(\Pi_{\text{rat}})$.

In fact, this could only be the case if all 3 quadrangles were flat. We leave the proof of this general statement to the reader and content ourselves with a concrete example. Choose the 6 points as follows:

$$u = (1,1,1), \quad w = (-1,1,1), \quad y = (0,-1,1),$$
$$v = (1,1,0), \quad x = (-1,1,0), \quad z = (1,-3,0).$$

Then we have the binomial relations

$$ux = vw, \qquad u^4 z = v^5 y^4, \qquad w^2 z = xy^2$$

in $k[\mathcal{WF}(\Phi)]$. It is easy to show by hand that one cannot assign positive degrees to the 6 elements such that these relations become homogeneous.

3. Diagonal and toric automorphisms.

Let Π be a lattice polyhedral complex. The rôle of the embedded torus of an affine toric variety is played by the subgroup of $\Gamma_k(\Pi)$ whose elements multiply the monomials $x \in \mathrm{L}(\Pi)$ by scalars from k^*. This subgroup is denoted by $\mathbb{D}_k(\Pi)$ and its elements are called *diagonal automorphisms*. It becomes a diagonal subgroup of

$\mathrm{GL}_N(k)$, $N = \#\mathrm{L}(\Pi)$, in the natural realization of $\Gamma_k(\Pi)$ as an affine subgroup of $\mathrm{GL}_N(k)$.

One can give a more explicit description of $\mathbb{D}_k(\Pi)$. Consider the finitely generated Abelian group

$$A(\Pi) = \mathbb{Z}^{\mathrm{L}(\Pi)}/U(\Pi)$$

where $\mathbb{Z}^{\mathrm{L}(\Pi)}$ is the free Abelian group generated by the lattice points in Π and $U(\Pi)$ represents the affine relations between the elements of $\mathrm{L}(\Pi)$, i. e. $U(\Pi)$ is generated by all linear combinations

$$\sum_{x\in \mathrm{L}(P)} a_x e_x, \qquad a_x \in \mathbb{Z},\ \sum_{x\in \mathrm{L}(P)} a_x x = 0,$$

where P runs through the facets of Π and e_x represents the base element corresponding to x. (Here $x \in \mathrm{L}(P)$ is to be considered as an element of $\mathbb{R}^{n_P+1}$ with last coordinate 1 so that $\sum_{x\in \mathrm{L}(P)} a_x x = 0$ implies $\sum_{x\in \mathrm{L}(P)} a_x = 0$.)

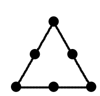
Let $\gamma \in \mathbb{D}_k(\Pi)$ and set $\lambda_x = \gamma(x)/x$ for all $x \in \mathrm{L}(\Pi)$. Then it is clear that $\sum_{x\in \mathrm{L}(P)} a_x x = 0$ implies $\prod_{x\in \mathrm{L}(P)} \lambda_x^{a_x} = 1$, and, conversely, every choice of $\lambda_x \in k^*$, $x \in \mathrm{L}(\Pi)$, satisfying these relations induces a diagonal automorphism of $k[\Pi]$. Therefore one has

Lemma 3.1. *For every lattice polyhedral complex* Π *and any field* k

$$\mathbb{D}_k(\Pi) = \mathrm{Hom}_{\mathbb{Z}}(A(\Pi), k^*).$$

Clearly, $\mathbb{D}_k(\Pi)$ contains a distinguished copy of k^* – the automorphisms which multiply the elements of $\mathrm{L}(\Pi)$ by a fixed scalar. When k^* is considered as a subgroup of $\Gamma_k(\Pi)$, we always mean the subgroup just specified.

In general $A(\Pi)$ is not torsionfree, not even if Π is Euclidean. Consider for example the complex Π below. It is easy to see that $A(\Pi) \approx \mathbb{Z}^3 \oplus (\mathbb{Z}/(2))$. Therefore $\mathbb{D}(\Pi)$ is not connected (if $\mathrm{char}\, k \neq 2$).



It is well known that the connected component of the diagonalizable group $\mathbb{D}_k(\Pi)$ is a torus. We denote it by $\mathbb{T}_k(\Pi)$ and call its elements *toric automorphisms*. Moreover we set $\Lambda(\Pi) = A(\Pi)/(\text{torsion})$. Then it is easy to show

Lemma 3.2. $\mathbb{T}_k(\Pi) \approx \mathrm{Hom}_{\mathbb{Z}}(\Lambda(\Pi), k^*) \approx \mathrm{Hom}_{\mathbb{Z}}(A(\Pi), \mathbb{Z}) \otimes k^*$.

Remark 3.3. (a) Let Π be a quasi-Euclidean complex, $\Pi \subset \mathbb{R}^n$, and $\mathrm{L}(\Pi) \subset \mathbb{Z}^n$. Then the elements $x \in \mathrm{L}(\Pi) \subset \mathbb{Z}^n$ satisfy all the affine relations that define $A(\Pi)$. Consequently one has an induced $\mathbb{Z}$-linear map $\Lambda(\Pi) \to \mathbb{Z}^n$. It is clear that the residue classes $\bar{e}_x$, $x \in \mathrm{L}(\Pi)$, span a quasi-Euclidean complex isomorphic to Π (in the vector space $\Lambda(\Pi) \otimes \mathbb{R}$). This realization is the *maximal embedding* of Π; every other embedding into a vector space factors through it.

(b) While $A(\Pi)$ may have torsion if Π is quasi-Euclidean, the subgroup generated by the elements $\bar{e}_x$, $x \in \mathrm{L}(P)$, is torsionfree for every face $P \prec \Pi$. In fact, the map described in part (a) sends this subgroup isomorphically onto the group $A(P)$.

For an arbitrary complex this does not necessarily hold; for example, it fails for the 'Möbius strip' Π_7 below (see Example 4.1).

The next lemma describes the subgroup of those elements of $\Gamma_k(\Pi)$ that map monomials to terms.

Lemma 3.4. (a) *If $\gamma \in \Gamma_k(\Pi)$ maps monomials to terms, then $\gamma = \delta \circ \sigma$ for some $\delta \in \mathbb{D}_k(\Pi)$ and $\sigma \in \Sigma(\Pi)$.*

(b) *For $\delta \in \mathbb{D}(\Pi)$ and $\sigma \in \Sigma(\Pi)$ one has $\sigma^{-1} \circ \delta \circ \sigma \in \mathbb{D}(\Pi)$; moreover, the subgroup of $\Gamma_k(\Pi)$ generated by $\mathbb{D}(\Pi)$ and $\Sigma(\Pi)$ is their semi-direct product.*

Proof. (a) is checked as easily as in the case of a single polytope treated in [BG, Section 4], and (b) is obvious. □

The next lemma provides a crucial argument.

Lemma 3.5. *Suppose Π is quasi-Euclidean and k is an infinite field.*

(a) *For any pair of different monomials $x_1, x_2 \in k[\Pi]$ there exists $\tau \in \mathbb{T}_k(\Pi)$ such that $\tau(x_1) = a_1 x_1$ and $\tau(x_2) = a_2 x_2$ for some distinct elements $a_1, a_2 \in k^*$.*

(b) *$\mathbb{T}_k(\Pi)$ is a maximal torus of $\Gamma_k(\Pi)$.*

Proof. We may assume that Π consists of rational polytopes in $\mathbb{R}^n$. By homothetic blowing up we can further assume that all the lattice points of Π have integral coordinates in $\mathbb{R}^n$. Clearly, we have a natural action of the torus $\mathbb{Z}^n \otimes k^* = (k^*)^n$ on $k[\Pi]$ – the restriction of the action on $k[\mathbb{Z}^n]$ to the monomials of $k[\Pi]$. This gives rise to an algebraic homomorphism $\phi : (k^*)^n \to \mathbb{D}_k(\Pi)$. By reasons of connectivity, $\phi((k^*)^n)$ is contained in $\mathbb{T}_k(\Pi)$. Now (a) becomes obvious.

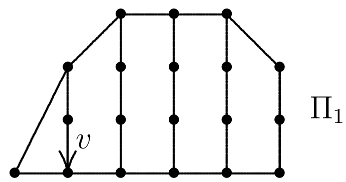
Assume there is a torus $T \subset \Gamma_k(\Pi)$ that contains $\mathbb{T}_k(\Pi)$. Then $\alpha^{-1} \circ \beta \circ \alpha(x) = \beta(x)$ for all $\alpha \in \mathbb{T}_k(\Pi)$, $\beta \in T$ and $x \in k[\Pi]$. By running α through $\mathbb{T}_k(\Pi)$ and x through the monomials of $k[\Pi]$, and using (a), we conclude that β must map monomials to terms, i. e. $\beta \in \mathbb{D}_k(\Pi) \rtimes \Sigma(\Pi)$ by 3.4. But, since k is infinite, there is no torus in $\mathbb{D}_k(\Pi) \rtimes \Sigma(\Pi)$ strictly containing the unity component $\mathbb{T}_k(\Pi)$. Hence $T = \mathbb{T}_k(\Pi)$. □

4. Column structures and elementary automorphisms.

We recall from [BG] that a non-zero element $v \in \mathbb{Z}^n$ is called a *column vector* for a lattice polytope $P \subset \mathbb{R}^n$ if there exists a facet $F \prec P$ such that $x + v \in P$ for every lattice point $x \in P \setminus F$ [BG]. The pair (P, v) is a *column structure* and the facet F its *base facet*. We use the notation P_v for F. Figure 3 illustrates this notion.

Let (P, v) be a column structure. Then for any $x \in S_P$ there is a uniquely determined non-negative integer $\mathrm{ht}_v(x)$ such that $x + \mathrm{ht}_v(x)v \in S_P$ and $x + (\mathrm{ht}_v(x)+1)v \notin$

FIGURE 3



S_P [BG, Lemma 2.2]. Clearly, if $C(P)$ denotes the cone in $\mathbb{R}^{n+1}$ spanned by S_P and $C(P_v)$ is its facet corresponding to the facet $P_v \prec P$ then $x + \operatorname{ht}_v(x)v \in C(P_v)$.

Let k be a field. The element $v \in \mathbb{Z}^n$ can be thought of as an element of the quotient field Q. F.$(k[S_P])$ after the identification of $\mathbb{Z}^n$ with $\mathbb{Z}^n \oplus 0$ $(\subset \mathbb{Z}^{n+1})$. Choose $\lambda \in k$. Then the semigroup homomorphism

$$S_P \to \mathrm{Q.\,F.}(k[S_P]), \qquad x \mapsto (1+\lambda v)^{\operatorname{ht}_v(x)}x,$$

gives rise to a k-algebra homomorphism $k[S_P] \to \mathrm{Q.\,F.}(k[S_P])$. This homomorphism is actually a graded automorphism of $k[S_P]$ [BG, Section 3]. We denote it by e_v^λ and call it an *elementary automorphism* of $k[S_P]$. If P is a unimodular lattice n-simplex, then $\Gamma_k(P)$ is just $\mathrm{GL}_{n+1}(k)$ and the e_v^λ are exactly the standard elementary matrices [BG, Section 3]; this explains our terminology.

Now we extend these notions to lattice polyhedral complexes Π. For $x \in \mathrm{L}(\Pi)$ we let $\operatorname{Supp}(x)$ denote the set off all *facets* of Π that contain x – the set of supporting facets.

Consider the set of all column structures (P, v), $P \prec \Pi$, satisfying the condition

$$(\#_1) \qquad \operatorname{Supp}(x + v) \subset \operatorname{Supp}(x)$$

for every lattice point $x \in P \setminus P_v$. Here the sum $x + v$ is understood 'locally', i. e. with respect to the column structure (P, v).

We have the following relation on this set of column structures: $(P, v) \sim (Q, w)$ if $Q \prec P$ and $w = v$ on Q. Consider the equivalence relation spanned by $\sim$. Among the corresponding equivalence classes $[P, v]$ there are distinguished ones, namely those satisfying the condition:

($\#_2$) If $(Q, w) \in [P, v]$ and $R \prec \Pi$ is a face such that (Q, w) restricts to a column structure on $Q \cap R$, then there is a column structure (R, u) satisfying ($\#_1$) and restricting to the same column structure on $Q \cap R$.

Observe that ($\#_2$) is equivalent to the condition:

($\#_2'$) $[P, v]$ induces (i. e. contains) a column structure on at least one facet and if $(Q, w) \in [P, v]$ is a column structure for some *facet* $Q \prec \Pi$ and $R \prec \Pi$ is another facet such that (Q, w) restricts to a column structure on $Q \cap R$ then there is a column structure (R, u) satisfying ($\#_1$) and restricting to the same column structure on $Q \cap R$.

A *column vector for* Π is defined as such a distinguished equivalence class. For a column vector V the pair (Π, V) will be called a *column structure* (on Π).

We let Col(Π) denote the set of column structures on Π.

Example 4.1. The Figures 4 and 5 show several polyhedral complexes and their column structures.

FIGURE 4

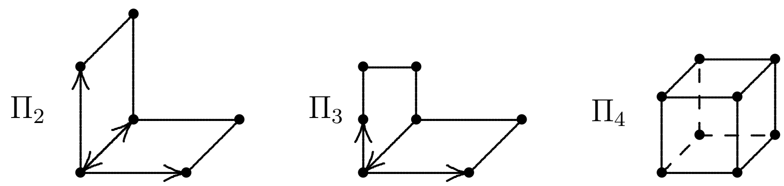
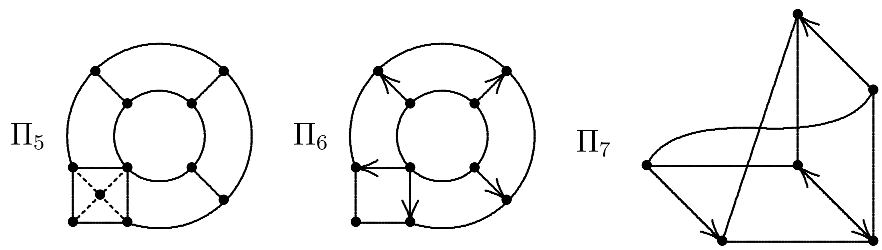


FIGURE 5



- Π_2 and Π_3 have two 2-dimensional facets, $\#\mathrm{Col}(\Pi_2) = 4$ and $\#\mathrm{Col}(\Pi_3) = 3$,
- Π_4 is the boundary of the unit lattice cube, $\#\mathrm{Col}(\Pi_4) = 0$,
- Π_5 has five 2-dimensional facets four of which are unit squares and the fifth is a lattice square with a lattice point in its barycenter, $\#\mathrm{Col}(\Pi_5) = 0$,
- Π_6 has five unit squares as facets, as shown in the picture, $\#\mathrm{Col}(\Pi_6) = 1$,
- Π_7 is a Möbius strip consisting of 3 unit squares, $\#\mathrm{Col}(\Pi_7) = 1$, and the only column structure on Π_7 includes 2 'opposite' column structures on 3 edges.

Next we introduce the notion of an *oriented polyhedral complex*. This includes the class of quasi-Euclidean polyhedral complexes.

Definition 4.2. A lattice polyhedral complex Π is called oriented if $(P, v) \in V$ and $(P, w) \in V$ imply $v = w$ for any column structure (Π, V).

Lemma 4.3. *Every quasi-Euclidean lattice polyhedral complex is oriented, but not conversely.*

Proof. Assume Π is a quasi-Euclidean complex realized by a polyhedral complex of rational polytopes in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let (Π, V) be a column structure and $(P, v), (P, w) \in V$. Then there is a finite sequence of column structures

$$(P, v) = (P_0, v_0), (P_1, v_1), \dots, (P_s, v_s) = (P, w),$$

where the P_i are faces of Π such that for each $i \in [1, s-1]$ either P_i is a face of P_{i+1} and $v_i = v_{i+1}$ on P_i or P_{i+1} is a face of P_i and $v_i = v_{i+1}$ on P_{i+1}. In particular, the v_i define the same vector in $\mathbb{R}^n$. Hence all quasi-Euclidean lattice polyhedral complexes are oriented.

An example of an oriented, but not quasi-Euclidean lattice polyhedral complex is provided by Π_5 above. In fact, easy geometric arguments show that if it were quasi-Euclidean, then the two adjacent edges of the square with barycenter would have to coincide. (One just uses that any affine realization of a unit lattice square must be a parallelogram.) □

Let k be a field, Π an oriented lattice polyhedral complex and V its column vector. For any element $\lambda \in k$ we define the map

$$e_V^\lambda : \mathrm{L}(\Pi) \to k[\Pi]$$

as follows. For $x \in \mathrm{L}(\Pi)$ there are two possibilities: either there is a column structure (P, v) such that $x \in \mathrm{L}(P)$, $P \prec \Pi$ and $V = [P, v]$, or such a column structure does not exist. In the first case we put

$$e_V^\lambda(x) = e_v^\lambda(x),$$

where e_v^λ is the corresponding elementary automorphism of $k[S_P]$ ($\subset k[\Pi]$), and in the second case x is mapped to itself. It follows from the definitions of a column vector and an oriented complex that this map is well defined. We claim that it gives rise to a (uniquely determined) graded k-algebra homomorphism of $k[\Pi]$. One only needs to check the following implication

$$\forall x_1, \dots, x_s \in \mathrm{L}(\Pi) \quad x_1 \cdots x_s = 0 \Rightarrow e_V^\lambda(x_1) \cdots e_V^\lambda(x_s) = 0;$$

in fact, $e_V(\lambda)$ respects the binomial relations since it restricts to an automorphism on $k[S_P]$ for each $P \prec \Pi$. Straightforward arguments show that condition $(\#_1)$ together with

$$\bigcap_1^s \mathrm{Supp}(x_i) = \emptyset$$

implies that none of the monomials in the canonical k-linear expansion of $e_V^\lambda(x_i)$ shares a supporting facet with those in the k-linear expansion of $e_V^\lambda(x_j)$ for $i \neq j$. This means that $e_V^\lambda(x_1) \cdots e_V^\lambda(x_s) = 0$, as claimed.

Next we define a pairing

$$\mathbb{D}(\Pi) \times \mathrm{Col}(\Pi) \to k^*, \quad (\delta, V) \mapsto \delta(V),$$

for an oriented polyhedral complex Π. Choose $\delta \in \mathbb{D}_k(\Pi)$, $V = [P, v] \in \mathrm{Col}(\Pi)$, and a face $P \prec \Pi$. Then δ restricts to a toric automorphism of $k[S_P]$. The latter extends to a toric automorphism of $k[\mathrm{gp}(S_P)]$. In particular, the image of $v \in \mathrm{gp}(S_P)$ under this automorphism equals $a_v v$ for some $a_v \in k^*$. We set $\delta(V) = a_v$. It is easily checked that this is a well defined mapping. Moreover, we have the equality $(a \cdot \delta)(v) = a(\delta(v))$ for δ and v as above and $a \in k^*$ ($\subset \mathbb{D}_k(\Pi)$).

Lemma 4.4. *Let* (Π, V) *be a column structure, where* Π *is an oriented lattice polyhedral complex. Then*

(a) $e_V^\lambda \in \Gamma_k(\Pi)$*, and the assignment* $\lambda \mapsto e_V^\lambda$ *defines an embedding of algebraic groups* $\mathbb{A}_k^1 \to \Gamma_k(\Pi)$*;*

(b) *the equation*

$$\delta \circ e_V^\lambda \circ \delta^{-1} = e_V^{\delta(V)\lambda}$$

holds for all $\delta \in \mathbb{D}_k(\Pi)$ *and all elementary automorphism* e_V^λ *of* $k[\Pi]$*.*

Proof. (a) follows from the analogous fact for a single polytope [BG, Lemma 3.1] and (b) is immediate from direct calculation. □

Let $\mathbb{E}_k(\Pi)$ denote the subgroup of $\Gamma_k(\Pi)$, generated by the elementary automorphisms. By Lemma 4.4(a) $\mathbb{E}_k(\Pi)$ is a connected subgroup of $\Gamma_k(\Pi)$ (see Borel [Bo, Proposition 2.2]). Therefore, we arrive at the following

Lemma 4.5. $\mathbb{E}_k(\Pi)$ *is a connected affine* k*-subgroup of the connected component of unity* $\Gamma_k(\Pi)^0 \subset \Gamma_k(\Pi)$*.*

Remark 4.6. One can define the notion of a column structure for a rational polyhedral complex Π_{rat} and the appropriate notion of an elementary automorphism for algebras of type $k[\Pi_{\mathrm{rat}}]$ in a natural way (along the lines of the definition for a single polytope [BG, Remark 3.3(c)]). One just has to work with monomials of arbitrary degrees. Then all the facts we have observed for lattice polyhedral complexes remain true in this situation as well. The details are left to the reader.

Remark 4.7. One could introduce the notion of *commutative* lattice polyhedral complexes which are more general than the oriented ones and for which one can still define the notion of an elementary automorphism so that the exact analogue of Lemma 4.4(a) is valid. (But we are not able to prove the analogue of Theorem 5.2 below for them.) Namely, a lattice polyhedral complex Π is called k-commutative (k a field) if for every column structure (Π, V) and every face $P \prec \Pi$ the following implication holds:

$$\forall \lambda, \mu \in k \quad ((P,v),(P,w) \in V) \Rightarrow (e_v^\lambda \text{ and } e_w^\mu \text{ commute}).$$

If Π is k-commutative for all fields, then it is called commutative.

We do not know whether k-commutative complexes are always commutative.

Observe that the complex Π_6 of Example 4.1 is commutative (easy) and its only column structure includes 2 column structures on one of the facets. In particular, Π_6 is not oriented. On the other hand the complex Π_7, Example 4.1, is apparently not a k-commutative lattice polyhedral complex for any field k: looking at the edges on which the only global column structure on Π_7 induces two 'opposite' column structures we get non-commutativity since $e_{12}^1 e_{21}^1 \neq e_{21}^1 e_{12}^1$, where e_{12}^1 and e_{21}^1 are the standard elementary matrices in $\mathrm{GL}_2(k)$.

5. The main result: affine case

Before we state the first main result let us single out the following class of polytopes.

Definition 5.1. A polytope P is *facet-separated* if for every facet $F \prec P$ there is a facet $G \prec P$ such that $F \cap G = \emptyset$.

Typical representatives of non-facet-separated polytopes are *pyramids* – the polytopes whose vertices all but one live in some affine proper subspace of the ambient Euclidean space. However, starting from dimension 4, facet-separated polytopes and pyramids do not exhaust the class of all polytopes.

Theorem 5.2. *Let k be a field and Π be a lattice polyhedral complex.*

(a) *If Π is oriented and* $\operatorname{char}(k) = 0$*, then the unity component $\Gamma_k(\Pi)^0 \subset \Gamma_k(\Pi)$ consists precisely of those elements $\gamma \in \Gamma_k(\Pi)$ which admit a representation of type $\gamma = \varepsilon \circ \tau$ for some $\varepsilon \in \mathbb{E}_k(\Pi)$ and $\tau \in \mathbb{T}(\Pi)$; we have* $\dim \Gamma_k(\Pi) = \#\mathrm{Col}(\Pi) + \operatorname{rank}(\Lambda(\Pi))$.

(b) *If Π is quasi-Euclidean and* $\operatorname{char}(k) = 0$*, then every element $\gamma \in \Gamma_k(\Pi)$ admits a representation of type $\gamma = \varepsilon \circ \delta \circ \sigma$ for some $\varepsilon \in \mathbb{E}(\Pi)$, $\delta \in \mathbb{D}(\Pi)$ and $\sigma \in \Sigma(\Pi)$; furthermore $\mathbb{T}(\Pi)$ is a maximal torus of $\Gamma_k(\Pi)$.*

(c) *If all facets of Π are facet-separated polytopes then the exact analogues of* (a) *and* (b) *hold for any infinite field k; moreover, for any enumeration* $\mathrm{Col}(\Pi) = \{V_1, \dots, V_s\}$ *and every element $\gamma \in \Gamma_k(\Pi)^0$ (under the hypothesis of* (a)*) and $\gamma \in \Gamma_k(\Pi)$ (under the hypothesis of* (b)*) there is a representation*

$$\gamma = e_{V_1}^{\lambda_1} \circ \cdots \circ e_{V_s}^{\lambda_s} \circ \delta \circ \sigma,$$

where $\lambda_1, \dots, \lambda_s \in k$, $\delta \in \mathbb{D}_k(\Pi)$ and $\sigma \in \Sigma(\Pi)$.

Remark 5.3. The proof we present below yields the same result for algebras of type $k[\Pi_{\mathrm{rat}}]$ where Π_{rat} is a rational polyhedral complex of the appropriate type (see Remark 4.6).

We need some preparation. Throughout this section k is a field and Π is an oriented lattice polyhedral complex.

A convention: for an element $z \in k[\Pi]$ let $\mathrm{Supp}(z)$ denote the set of the facets $P \prec \Pi$ such that S_P contains a monomial appearing in the canonical k-linear expansion of z. (This notation is compatible with the previous one for lattice points).

For any face $P \in \Pi$ the canonical split epimorphism

$$k[\Pi] \to k[S_P], \quad x \mapsto 0 \text{ for } x \in \mathrm{L}(\Pi) \setminus \mathrm{L}(P),$$

will be denoted by π_P. Thus π_P is split by the inclusion $\iota_P : k[S_P] \to k[\Pi]$. Note that

$$(\dagger) \qquad \mathrm{Supp}(z) = \{P \mid \pi_P(z) \neq 0\}.$$

Lemma 5.4. *$\{\mathrm{Ker}(\pi_P) \mid P \prec \Pi$ a facet$\}$ is the set of minimal prime ideals of $k[\Pi]$.*

The proof is straightforward.

Lemma 5.5. *Let $\gamma \in \Gamma_k(\Pi)$. Then there is a permutation of the set of facets $P \in \Pi$, say ρ_γ, such that $\gamma(\mathrm{Ker}(\pi_P)) = \mathrm{Ker}(\pi_{\rho_\gamma(P)})$ for all facets P of Π. The assignment $\gamma \mapsto \rho_\gamma$ defines a group homomorphism from $\Gamma_k(\Pi)$ to the permutation group of the set of facets of Π. Its kernel is a closed subgroup of $\Gamma_k(\Pi)$ containing $\Gamma_k(\Pi)^0$.*

Proof. The first and second assertion follow immediately from Lemma 5.4, and that the kernel of the assignment $\gamma \mapsto \rho_\gamma$ is a closed subgroup eventually boils down to the statement that the stabilizer of a vector subspace is a closed subgroup of a linear algebraic group acting algebraically on a finite-dimensional vector space. □

Since γ maps $\mathrm{Ker}(\pi_P)$ onto $\mathrm{Ker}(\pi_{\gamma_\rho(P)})$, it induces a (unique) isomorphism γ_P fitting into the commutative diagram

$$\begin{array}{ccc} k[\Pi] & \xrightarrow{\ \gamma\ } & k[\Pi] \\ {\scriptstyle \pi_P}\downarrow & & \downarrow{\scriptstyle \pi_{\rho_\gamma(P)}} \\ k[S_P] & \xrightarrow[\ \gamma_P\ ]{} & k[S_{\rho_\gamma(P)}]. \end{array}$$

One obviously has $\gamma_P = \pi_{\rho_\gamma(P)} \circ \gamma \circ \iota_P$.

More generally, let $P_1, \dots, P_s \prec \Pi$ be facets and $Q = P_1 \cap \dots \cap P_s$. We set

$$\rho_\gamma(Q) = \rho_\gamma(P_1) \cap \dots \cap \rho_\gamma(P_s).$$

As above, γ induces an isomorphism γ_Q fitting into the same commutative diagram as above where we only replace P by Q; furthermore $\gamma_Q = \pi_Q \circ \gamma \circ \iota_Q$.

Lemma 5.6. *Suppose γ_P maps monomials to terms for every facet $P \prec \Pi$. Then γ does so as well.*

Proof. Let z be a monomial, $\mathrm{Supp}(z) = \{P_1, \dots, P_s\}$, and $Q = P_1 \cap \dots \cap P_s$. Then $z \in k[S_Q]$, and therefore $\pi_Q(z) \neq 0$. It follows that $\pi_{\rho_\gamma(Q)}(\gamma(z)) \neq 0$ as well. Therefore the canonical k-linear expansion of $\gamma(z)$ must contain a monomial x with $\mathrm{Supp}(x) = \{\rho_\gamma(P_1), \dots, \rho_\gamma(P_s)\}$.

By equation (†) above we likewise have $\mathrm{Supp}(\gamma(z)) = \{\rho_\gamma(P_1), \dots, \rho_\gamma(P_s)\}$. Now the hypothesis implies that x is the only monomial appearing in the k-linear expansion of $\gamma(z)$. □

We also need several facts from [BG, Lemma 4.1, 4.2, 4.3 and Theorem 3.2(b)]). For the reader's convenience we collect them in the following proposition.

Let P be a lattice polytope. As usual, $\bar{S}_P$ stands for the normalization of the semigroup S_P, i. e. $\bar{S}_P = \{x \in \mathrm{gp}(S_P) \mid cx \in S_P \text{ for some } c \in \mathbb{N}\}$. Then $k[\bar{S}_P]$ is a Noetherian normal domain. For any facet $F \prec P$ one has the monomial height 1 prime ideal

$$\mathrm{Div}(F) \subset k[\bar{S}_P]$$

generated by the monomials of $k[\bar{S}_P]$ that do not belong to the facet of the cone $C(\bar{S}_P)$ corresponding to F.

One more observation: since any graded automorphism of $k[S_P]$ extends to a unique graded automorphism and $k[S_P]$ and $k[\bar{S}_P]$ coincide in degree 1, the two rings $k[S_P]$ and $k[\bar{S}_P]$ have the same group $\Gamma_k(P)$ of graded automorphisms.

Proposition 5.7. (a) *An automorphism $\gamma \in \Gamma_k(P)$ inducing a permutation of the set $\{\mathrm{Div}(F) \mid F \prec P$ a facet$\}$ maps monomials to terms.*

(b) *Let $v_1, \dots, v_s$ be column vectors of P with the common base facet $F = P_{v_i}$, $\lambda_1, \dots, \lambda_s \in k$, and $G \neq F$ another facet. Then*

$$e_{v_1}^{\lambda_1} \circ \dots \circ e_{v_s}^{\lambda_s}(\mathrm{Div}(F)) = (1 + \lambda_1 v_1 + \dots + \lambda_s v_s)\mathrm{Div}(F),$$
$$e_{v_1}^{\lambda_1} \circ \dots \circ e_{v_s}^{\lambda_s}(\mathrm{Div}(G)) = \mathrm{Div}(G).$$

(c) *Let $F \prec P$ be a facet, $\lambda_1, \dots, \lambda_s \in k \setminus \{0\}$ and $v_1, \dots, v_s \in \mathrm{gp}(S_P)$ ($\subset$ Q. F.$(k[\bar{S}_P])$) be pairwise different nonconstant Laurent monomials of degree 0. Suppose $(\lambda_1 v_1 + \dots + \lambda_s v_s)\mathrm{Div}(F) \subset k[\bar{S}_P]$. Then $v_1, \dots, v_s$ are column vectors for P with the common base facet F.*

(d) *The connected component of unity $\Gamma_k(P)^0 \subset \Gamma_k(P)$ consists of those graded automorphisms of $k[S_P]$ which induce (by extension to $k[\bar{S}_P]$) the identity map on the divisor class group $\mathrm{Cl}(k[\bar{S}_P])$.*

We will also need the following facts.

Lemma 5.8. *Let $M \subset \mathbb{Z}^m$ be a finite system of Laurent monomials of $k[\mathbb{Z}^m]$ (k is a field and $m \in \mathbb{N}$) and $f, g \in k[\mathbb{Z}^m]$. Assume the k-subspaces of $k[\mathbb{Z}^m]$ generated by $\{xf \mid x \in M\}$ and $\{xg \mid x \in M\}$ coincide. Then $f = ag$ for some $a \in k^*$.*

Proof. The case $\#(M) = 1$ is trivial, and for the general case we use induction as follows. There is a $\mathbb{Z}$-linear form ϕ such that ϕ attains its maximal value on each of the following polytopes in a single point: the Newton polytopes $\mathrm{N}(f)$, $\mathrm{N}(g)$ and the convex hull $P(M)$ of M; in $P(M)$ let $\phi(z)$ be the maximum. Then z is a vertex of $P(M)$, hence $z \in M$, and we can pass to $M \setminus \{z\}$. □

Lemma 5.9. *Let G be an algebraic $\mathbb{C}$-group and $X \subset G$ be a Zariski closed subset with $\dim X < \dim G$. Then there is an element $g \in G$ such that none of the powers of g is in X.*

Proof. Passing to G^0 we may assume that G is connected and therefore irreducible. For any natural number c the algebraic mapping $\mathrm{pow}_c : G \to G$, $g \mapsto g^c$, is not globally degenerate since it is not degenerate in a small neighborhood of $1 \in G$ (the differential at 1 is the multiplication by c on the tangent space). In particular,

$\operatorname{pow}_c^{-1}(X) \subset G$ is a Zariski closed subset of dimension strictly less then $\dim G$. (Otherwise we would have $\operatorname{pow}_c^{-1}(X) = G$, and pow_c would be degenerate everywhere.) Therefore, $\bigcup_1^\infty \operatorname{pow}_c^{-1}(X) \subset G$ is a proper subset. □

Proof of Theorem 5.2(a). Choose $\gamma \in \Gamma_k(\Pi)^0$. By Lemma 5.5 $\rho_\gamma = 1_{\text{facet}}$. As seen above, γ induces a graded k-automorphisms $\gamma_P : k[S_P] \to k[S_P]$ for each facet $P \prec \Pi$. More generally, for a finite system of facets $P_1, \dots, P_s \prec \Pi$ there is a graded automorphism $\gamma_Q : k[S_Q] \to k[S_Q]$, $Q = P_1 \cap \dots \cap P_s$ induced by γ. We let $\bar\gamma_Q$ denote the unique automorphic extension of γ_Q to $k[\bar S_Q]$. Clearly, the assignment $\gamma \mapsto \gamma_Q$ defines an algebraic group homomorphism $\Gamma_k(\Pi) \to \Gamma_k(Q)$. In particular, if $\gamma \in \Gamma_k(\Pi)^0$ then, by Proposition 5.7(d), the automorphism $\bar\gamma_Q$ induces the identity map on $\operatorname{Cl}(k[\bar S_Q])$.

For a pair of faces $P_1 \prec P_2 \prec \Pi$

$$\pi_{P_2P_1} : k[S_{P_2}] \to k[S_{P_1}], \quad \mathrm{L}(P_2) \setminus \mathrm{L}(P_1) \to 0,\ x \mapsto x \text{ for } x \in \mathrm{L}(P_1).$$

will denote the 'face' projection. Further, we let $\bar S_{P_2P_1}$ denote the sub-semigroup of $\bar S_{P_2}$ that corresponds to the face $P_1 \prec P_2$ and let $\bar\pi_{P_2P_1}$ denote the corresponding face projection from $k[\bar S_{P_2}]$ to $k[\bar S_{P_2P_1}]$. In particular, $\pi_{P_2P_1}$ and $\bar\pi_{P_2P_1}$ coincide on $\mathrm{L}(P_2)$. By Proposition 2.5(c) the inclusion $\bar S_{P_1} \subset \bar S_{P_2P_1}$ may be strict.

Step 1. Let P be a face of Π, F facet of P and $\gamma \in \Gamma_k(\Pi)^0$. By Lemma 5.7(d) $\bar\gamma_P$ leaves the class of $\operatorname{Div}(F) \subset k[\bar S_P]$ invariant, i. e.

$$\bar\gamma_P(\operatorname{Div}(F)) = d \operatorname{Div}(F) \tag{1}$$

for some $d \in \mathrm{Q.F.}(k[\bar S_P])$. Since $\bar\gamma_P$ is a graded automorphism, d must be a homogeneous element of degree 0. Moreover, since $\operatorname{Div}(F)$ is a monomial ideal, d is a sum of degree 0 Laurent terms of $\mathrm{Q.F.}(k[\bar S_P])$. Say $d = a_1\mu_1 + \dots + a_s\mu_s$, where $a_1, \dots, a_s \in k^*$ and $\mu_1, \dots, \mu_s$ are pairwise different degree 0 Laurent monomials of $\operatorname{gp}(S_P)$. Assume that $\mu_i \neq 1$. Then by Proposition 5.7(c) (P, μ_i) is a column structure.

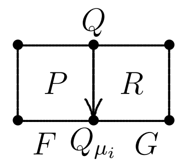
We claim that (P, μ_i) gives rise to a column vector for Π.

First we must show that if μ_i is a column vector for some face $Q \prec P$ and there is a face $R \prec \Pi$ containing Q, then there is a column structure on R restricting to the same column structure on Q. By enlarging Q to the intersection $P \cap R$ we may assume without loss of generality that $Q = P \cap R$.

We have a column structure (Q, μ_i) with the base facet Q_{μ_i} (of Q). There clearly exists a facet $G \prec R$ such that $G \cap Q = Q_{\mu_i}$. Fix any such a facet G (below it will become clear that G is unique) and consider the height 1 prime ideal $\operatorname{Div}(G) \subset k[\bar S_R]$. (Figure 6 illustrates the relation between P, Q, Q_{μ_i}, R, F, and G.) By the same reasons as for P one has

$$\bar\gamma_R(\operatorname{Div}(G)) = (b_1\nu_1 + \dots + b_t\nu_t)\operatorname{Div}(G) \tag{2}$$

FIGURE 6



for *uniquely* determined pairwise different degree 0 Laurent monomials $\nu_1, \dots, \nu_t \in \operatorname{gp}(S_Q)$ and $b_1, \dots, b_t \in k^*$. We have the following set-theoretical inclusions:

$$\operatorname{Div}(Q_{\mu_i}) \subset \bar\pi_{PQ}(\operatorname{Div}(F)) \quad \text{and} \quad \operatorname{Div}(Q_{\mu_i}) \subset \bar\pi_{RQ}(\operatorname{Div}(G)),$$

where $\operatorname{Div}(Q_{\mu_i})$ is the corresponding height 1 prime ideal of $k[\bar S_Q]$.

We also know that there is a representation

$$\bar\gamma_Q(\operatorname{Div}(Q_{\mu_i})) = (c_1\kappa_1 + \cdots + c_r\kappa_r)\operatorname{Div}(Q_{\mu_i}), \tag{3}$$

where $\kappa_1, \dots, \kappa_r$ are pairwise different degree 0 Laurent monomials from $\operatorname{gp}(S_Q)$ and $c_1, \dots, c_r \in k^*$.

By the construction of γ_Q and γ_P we have

$$\pi_{PQ} \circ \gamma_P = \gamma_Q \quad \text{and} \quad \pi_{RQ} \circ \gamma_R = \gamma_R.$$

It is clear that S_P coincides with $\bar S_P$ in degree 1, and similarly this holds for S_R and S_Q. Hence the equalities (1), (2) and (3) imply

$$\bar\pi_{PQ} \circ \bar\gamma_P(\operatorname{Div}(F)_1) = (\textstyle\sum_j a_{i_j}\mu_{i_j})\pi_{PQ}(\operatorname{Div}(F)_1) = (c_1\kappa_1 + \cdots + c_r\kappa_r)\operatorname{Div}(Q_{\mu_i})_1$$
$$\bar\pi_{RQ} \circ \bar\gamma_R(\operatorname{Div}(G)_1) = (\textstyle\sum_l b_{k_l}\nu_{k_l})\pi_{RQ}(\operatorname{Div}(G)_1) = (c_1\kappa_1 + \cdots + c_r\kappa_r)\operatorname{Div}(Q_{\mu_i})_1,$$

where $\operatorname{Div}(-)_1$ refers to the corresponding degree 1 homogeneous component, and the summations are considered for

$$\mu_{i_j} \in \operatorname{gp}(S_Q) \cap \{\mu_1, \dots, \mu_s\} \quad \text{and} \quad \nu_{k_l} \in \operatorname{gp}(S_Q) \cap \{\nu_1, \dots, \nu_t\};$$

of course, the first intersection is taken in $\operatorname{gp}(S_P)$ ($\supset \operatorname{gp}(S_Q)$) and the second one in $\operatorname{gp}(S_R)$ ($\supset \operatorname{gp}(S_Q)$). By Lemma 5.8 we see that in the representation $\bar\gamma_Q(\operatorname{Div}(G)) = (b_1\nu_1 + \cdots + b_t\nu_t)\operatorname{Div}(G)$ one of the ν_k is μ_i.

Next we show that each μ_i satisfies the condition

$$\operatorname{Supp}(x\mu_i) \subset \operatorname{Supp}(x)$$

for every $x \in \mathrm{L}(P) \setminus \mathrm{L}(F)$. Assume to the contrary that there are a point $x \in \mathrm{L}(P) \setminus \mathrm{L}(F)$ and a facet $T \prec \Pi$ such that $\mu_i x \in T$ and $x \notin T$. We have

$$z = x(a_1\mu_1 + \cdots + a_s\mu_s) \in \bar\gamma_P(\operatorname{Div}(F)).$$

Since $k[\bar S_P]$ and $k[S_P]$ have the same degree 1 components, $z \in \operatorname{Div}(F) \cap k[S_P]$; by assumption $T \in \operatorname{Supp}(\gamma(z))$. Let I be the annihilator of $\operatorname{Ker}(\pi_T)$; then I is spanned by the monomials $\mu \in S_T$ that do not belong to S_R for any other facet $R \prec \Pi$. We have $\gamma(z) \cdot I \neq 0$. On the other hand, $z \cdot I = 0$. Thus we get the desired contradiction, because $\gamma(I) = I$, as follows from $\gamma(\operatorname{Ker}(\pi_T)) = \operatorname{Ker}(\pi_T)$.

Finally, assume (P, μ_i) restricts to a column structure on $P \cap R$ for some $R \prec \Pi$. Then we know already that there is a column structure (R, ν) restricting to the same column structure on $P \cap R$. But what we have shown is more. Namely,

(4) the column vector ν for the face $R \prec \Pi$ is derived from γ exactly in the same way as μ_i for P.

Thus the above arguments apply to the column structure (R, ν) as well, yielding condition $(\#_1)$ for it.

Step 2. We fix an enumeration of the facets of Π, say $P_1, P_2, \dots$. For each facet $P_p \prec \Pi$ we also fix an enumeration of the facets of P_p, say $F_{p1}, F_{p2}, \dots$. Consider a total ordering of the pairs (p, q). Then for each (p, q) the subgroup

$$\Gamma_{pq} = \{\gamma \in \Gamma_k(\Pi) \mid \bar{\gamma}_{P_r}(\mathrm{Div}(F_{rs})) = \mathrm{Div}(F_{rs}) \text{ for all } (r, s) \le (p, q)\} \subset \Gamma_k(\Pi)$$

is (Zariski) closed. In fact, it is the intersection of the stabilizers of finitely many vector subspaces. By Lemma 5.6 and Proposition 5.7(a) we have the equality

$$\mathbb{D}_k(\Pi) = \Gamma_{pq_{\max}}$$

where $(p, q)_{\max}$ is the maximal pair. Now we enlarge the set of pairs (p, q) by one element $(0, 0)$, declare it as the smallest element of the new system and set $\Gamma_{00} = \Gamma_k(\Pi)$. We then have the sequence of affine groups

$$(*) \qquad \mathbb{D}_k(\Pi) = \Gamma_{pq_{\max}} \subset \ldots \subset \Gamma_{pq} \subset \ldots \subset \Gamma_{rs} \subset \ldots \subset \Gamma_{00} = \Gamma_k(\Pi)$$

for $(p, q) > (r, s)$.

Claim. If $(p, q) > (r, s)$ are consecutive pairs and $\gamma \in \Gamma_{rs}$, then $E \circ \gamma^c \in \Gamma_{pq}$ for some natural number c and some element $E \in \mathbb{E}_k(\Pi)$.

Without loss of generality we can assume (by passing to some power) that $\gamma \in \Gamma_k(\Pi)^0$.

Recall that we have identified $\Gamma_k(P)$ with the corresponding closed subgroup of $\mathrm{GL}_N(k)$, $N = \#\mathrm{L}(P) - 1$ (see Section 1). There is a finitely generated *subring* $\Lambda \subset k$ such that $\gamma, \gamma^{-1} \in \mathrm{GL}_N(\Lambda)$. Let k_0 denote any residue field of Λ. Thus k_0 is a finite field. Let γ_0 denote the reduction of γ in $\mathrm{GL}_N(k_0)$, a finite group. So there exists a natural number c such that $(\gamma_0)^c$ is the identity map of $k_0[\Pi]$, i. e. the identity matrix of $\mathrm{GL}_N(k_0)$. We will show that c is the desired number.

First we want to show that if $P \prec \Pi$ is any facet and $F \prec P$ is a facet of P, then

$$(\bar{\gamma}_P)^c(\mathrm{Div}(F)) = (1 + a_1 m_1 + \cdots + a_n m_n)\mathrm{Div}(F)$$

for some degree zero non-constant monomials $m_1, \dots, m_n \in \mathrm{gp}(S_P)$ and $a_1, \dots, a_n \in k^*$. (We do not exclude the case $n = 0$.)

All we need for this assertion is that $((\gamma_0)_P)^c$ is the identity map of $k_0[P]$ and

$$(\bar{\gamma}_P)^c(\mathrm{Div}(F)) = (\mu_1 + \cdots + \mu_n)\mathrm{Div}(F)$$

for some degree zero Laurent terms $\mu_1, \dots, \mu_n \in \mathrm{Q.F.}(k[S_P])$ (see the previous step). Now assume to the contrary that none of the μ_i is an element of k^*. Looking at the homogeneous degree 1 component (as we did in Step 1) we get

$$(\gamma_P)^c(\mathrm{Div}(F)_1) = (\mu_1 + \dots + \mu_n)\mathrm{Div}(F)_1.$$

By Proposition 5.7(c) each of the $\mu_i \neq 1$ is a column vector for P with base facet F. The corresponding semigroup homomorphisms $\mathrm{ht}_{\mu_i} : S_P \to \mathbb{Z}_+$ are all the same. Let $x \in \mathrm{L}(P)$ be any point with the maximal possible value of $\mathrm{ht}_{\mu_i}(x)$. Clearly, $x \in \mathrm{Div}(F)$. By our assumption none of the elements of $(\mu_1 + \dots + \mu_n)\mathrm{Div}(F)_1$ may involve the monomial x in its canonical k-linear expansion. But this contradicts the condition that $\mathrm{Div}(F)_1$ and $(\mu_1 + \dots + \mu_n)\mathrm{Div}(F)_1$ have the same images in $k_0[S_P]$.

In particular we have

$$(\bar\gamma_{P_p})^c(\mathrm{Div}(F_{pq})) = (1 + a_1m_1 + \dots + a_nm_n)\mathrm{Div}(F_{pq})$$

for $a_1, \dots, a_n$ and $m_1, \dots, m_n$ as above. By Step 1 each of the monomials m_i defines a column structure on Π. Consider the automorphism

$$E = e_{V_1}^{a_1} \circ \dots \circ e_{V_n}^{a_n} \in \mathbb{E}_k(\Pi),$$

where $V_1 = [P, m_1], \dots, V_n = [P, m_n]$. By Proposition 5.7(b) we get

$$(\overline{E^{-1} \circ \gamma^c})_{P_p}(\mathrm{Div}(F_{pq})) = \mathrm{Div}(F_{pq}).$$

Clearly, $(\overline{\gamma^c})_{P_t}(\mathrm{Div}(F_{tu})) = \mathrm{Div}(F_{tu})$ for $(t, u) < (p, q)$. Therefore, by (4) in Step 1 and 5.7(b) $(\overline{E^{-1} \circ \gamma^c})_{P_r}$ also leaves $\mathrm{Div}(F_{rs})$ untouched for any pair $(r, s) < (p, q)$. The claim has been proved.

We record a property of the column structures V_i that will be important below:

(5) (p, q) is the smallest (with respect to $<$) among all pairs (t, u) such that V_i contains (P_t, v) with base facet F_{tu}.

This follows from the construction of V_i in Step 1: if $\gamma(\mathrm{Div}(F_{tu}) = \mathrm{Div}(F_{tu}))$, then V_i cannot contain a column vector with base facet F_{tu}.

Step 3. For two subsets $A, B \subset G$ of a group G we let $A \cdot B$ denote the subset $\{ab \mid a \in A, b \in B\} \subset G$ and $\big(AB\big)$ the subgroup of G generated by A and B.

Consider the special case $k = \mathbb{C}$. Assume $(p, q) > (r, s)$ are consecutive pairs. We will show the equality of the two connected groups

$$\big(\mathbb{E}_k(\Pi)\Gamma_{rs}^0\big) = \big(\mathbb{E}_k(\Pi)\Gamma_{pq}^0\big), \qquad (**)$$

where $-^0$ refers to the corresponding unity component. That these groups are in fact connected follows from Lemma 4.5 and [Bo, Proposition 2.2].

Consider the partition into right cosets

$$\Gamma_{pq} = \Gamma_{pq}^0 g_1 \cup \dots \cup \Gamma_{pq}^0 g_t \qquad g_1, \dots, g_t \in \Gamma_{pq}.$$

We have

$$\mathbb{E}_k(\Pi)\cdot\Gamma_{pq}\subset Yg_1\cup\cdots\cup Yg_t,$$

where $Y=\Big(\mathbb{E}_k(\Pi)\Gamma^0_{pq}\Big)$. By Step 2 there is a natural number c for any $\gamma\in\Gamma^0_{rs}$ such that

$$\gamma^c\in(\Gamma^0_{rs}\cap Yg_1)\cup\cdots\cup(\Gamma^0_{rs}\cap Yg_t).$$

Omitting some g_i if necessary we get a disjoint union $Yg_1\cup\cdots\cup Yg_t=Y\cup Yg_{l_2}\cup Yg_{l_3}\cup\cdots$ of right cosets of Y. By Lemma 5.11

$$\dim\Gamma^0_{rs}=\dim(\Gamma^0_{rs}\cap Y)\cup(\Gamma^0_{rs}\cap Yg_{l_2})\cup(\Gamma^0_{rs}\cap Yg_{l_3})\cdots.$$

Hence, by the irreducibility of Γ^0_{rs} and the fact that $\Gamma^0_{rs}\cap Y\neq\emptyset$, we arrive at the inclusion $\Gamma^0_{rs}\subset Y$. Therefore, $\Big(\mathbb{E}_k(\Pi)\Gamma^0_{rs}\Big)\subset Y$. The opposite inclusion is obvious, hence the equality $(**)$.

The equality $(**)$ and the sequence $(*)$ in Step 2 imply

$$\Gamma_k(\Pi)^0=\Big(\mathbb{E}_k(\Pi)\mathbb{T}_k(\Pi)\Big).$$

Now the same equality holds for an arbitrary subfield $k\subset\mathbb{C}$ because of the following general observations. Since $\Gamma_{\mathbb{C}}(\Pi)$ is defined over k, so is its unity component $\Gamma_{\mathbb{C}}(\Pi)^0$ [Bo, Proposition 1.2]. By the Lemmas 3.2 and 4.4(b) the connected subgroup $\Big((\mathbb{E}_{\mathbb{C}}(\Pi)\mathbb{T}_{\mathbb{C}}(\Pi)\Big)\subset\Gamma_{\mathbb{C}}(\Pi)$ is likewise defined over k. If the two irreducible affine k-varieties were different, then they would remain so after the scalar extension $k\to\mathbb{C}$, which is not the case.

Consider the case of an arbitrary field k of characteristic 0. If $\gamma\in\Gamma_k(\Pi)$, then γ is defined over a finitely generated subfield $k_0\subset k$. Choosing any embedding $k_0\to\mathbb{C}$ we fall in the previous case.

Finally, by Lemma 4.4(b) we have the equality $\Big(\mathbb{E}_k(\Pi)\mathbb{T}_k(\Pi)\Big)=\mathbb{E}_k(\Pi)\cdot\mathbb{T}_k(\Pi)$.

Step 4. We have to compute the dimension. As in Step 3 we may assume $k=\mathbb{C}$. For each facet $P\prec\Pi$ fix an interior monomial $x\in\operatorname{int}(S_P)$, i. e. a monomial corresponding to an interior point of the cone C_P. Let (P,v) be a column structure defining a column vector V_1 for Π. Assume $\operatorname{Col}(\Pi)=\{V_1,V_2,\dots,V_s\}$. Then for arbitrary elements $\lambda_1,\lambda_2,\dots,\lambda_s\in k^*$ the set of monomials appearing in the canonical k-linear expansion of $e^{\lambda_1}_{V_1}(x)$ is not covered by those appearing in the k-linear expansions of $e^{\lambda_2}_{V_2}(x)$, $e^{\lambda_3}_{V_3}(x)$ and so on (just look at the projection of x through v into the base facet P_v). This shows that we have $\#\operatorname{Col}(\Pi)$ linearly independent tangent vectors at $1\in\Gamma_k(\Pi)$. Since the tangent vectors corresponding to the elements of $\mathbb{T}_k(\Pi)$ clearly belong to a complementary dimension and $\Gamma_k(\Pi)^0$ is a smooth variety, by Lemma 3.2 we conclude

$$\dim\Gamma_k(\Pi)^0\geq\#\operatorname{Col}(\Pi)+\operatorname{rank}(\Lambda(\Pi)).$$

The opposite inequality is derived as follows. For any pair (r,s) we let $\mathbb{E}_{rs}$ denote the subgroup of $\mathbb{E}_k(\Pi)$ generated by elementary automorphisms of type e^{λ}_V where

$V = [P_r, v]$ and such that (r,s) is the smallest (with respect to $<$) pair for which F_{rs} appears as a base facet of V (in particular $(P_r)_v = F_{rs}$). Let $\{V_1, \dots, V_m\}$ denote the set of column vectors for Π that contribute to $\mathbb{E}_{rs}$. Essentially the same arguments as in the proof of Lemma 3.1 in [BG] show that the assignment

$$(\lambda_1, \dots, \lambda_m) \mapsto e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_m}^{\lambda_m}$$

establishes the isomorphism of the abelian affine groups $\mathbb{A}_k^m$ and $\mathbb{E}_{rs}$.

We claim that for all consecutive pairs $(r,s) < (p,q)$ there is an element $g_{pq} \in \Gamma_k(\Pi)$ such that the subset

$$(\mathbb{E}_{pq} \cdot \Gamma_{pq}^0) g_{pq} \subset \Gamma_k(\Pi)$$

contains a Zariski open subset of Γ_{rs}^0. In fact, by the property (5) of the automorphism E in Step 2, Lemma 5.9 and the irreducibility of Γ_{pq}^0 we have

$$\Gamma_{rs}^0 \subset \overline{\mathbb{E}_{pq} \cdot \Gamma_{pq}},$$

where the bar on the right hand side means the Zariski closure (in $\Gamma_k(\Pi)$). Now the claim follows from the facts that Γ_{pq} decomposes into finite number of right cosets of Γ_{pq}^0 and that for each of these cosets, say $\Gamma_{pq}^0 g$, the subset $(\mathbb{E}_{pq} \cdot \Gamma_{pq}^0) g \subset \Gamma_k(\Pi)$ is constructibe (and, hence, contains a Zariski open set of its closure). In particular we have the equality $\dim(\mathbb{E}_{pq} \cdot \Gamma_{pq}^0) \geq \dim \Gamma_{rs}$. In view of the sequence $(*)$ of the groups Γ_{tu} we get

$$\dim \Gamma_k(\Pi) \leq \dim \mathbb{T}_k(\Pi) + \Sigma_{(p,q) \neq (0,0)} \dim \mathbb{E}_{pq} = \operatorname{rank}(\Lambda(\Pi)) + \#\operatorname{Col}(\Pi). \qquad \square$$

Proof of Theorem 5.2(b). Suppose Π is a quasi-Euclidean lattice polyhedral complex, $\operatorname{char}(k) = 0$ and $\gamma \in \Gamma_k(\Pi)$. By Lemma 3.5(b) $\mathbb{T}_k(\Pi)$ is a maximal torus of $\Gamma_k(\Pi)$. By Theorem 5.2(a) and Lemma 3.4(a) it suffices to show that there is an element $\alpha \in \Gamma_k(\Pi)^0$ such that $\alpha \circ \gamma$ maps monomials to terms.

Consider the closed subgroup

$$\mathbb{D} = \gamma \mathbb{D}_k(\Pi) \gamma^{-1} \subset \Gamma_k(\Pi).$$

Its unity component is $\mathbb{D}^0 = \gamma \mathbb{T}_k(\Pi) \gamma^{-1}$. In particular, $\mathbb{D}^0$ is a maximal torus of $\Gamma_k(\Pi)$. By [Bo, Corollary 11.3(1)] there is an element $\alpha \in \Gamma_k(\Pi)^0$ such that $\alpha^{-1} \mathbb{T}_k(\Pi) \alpha = \mathbb{D}^0$. We get

$$(\alpha \circ \gamma)^{-1} \mathbb{T}_k(\Pi) (\alpha \circ \gamma) = \mathbb{T}_k(\Pi). \tag{1}$$

We claim that α is the desired element.

Assume to the contrary that there is a monomial $x \in k[\Pi]$ such that in the canonical k-linear expansion of $\alpha \circ \gamma(x)$ there occur two distinct monomials $y_1, y_2 \in k[\Pi]$.

By Lemma 3.5(a) there exist $\tau \in \mathbb{T}_k(\Pi)$ and two distinct elements $a_1, a_2 \in k^*$ such that $\tau(y_1) = a_1 y_1$ and $\tau(y_2) = a_2 y_2$. Therefore, there does not exist $a \in k^*$ for which $\tau \circ \alpha \circ \gamma(x) = a \cdot (\alpha \circ \gamma(x))$, or equivalently, there does not exist $a \in k^*$ such that $(\alpha \circ \gamma)^{-1} \circ \tau \circ (\alpha \circ \gamma)(x) = ax$ and this contradicts (1). $\square$

Proof of Theorem 5.2(c). Let P be a facet-separated lattice polytope and $F \prec P$ be one of its facets. Suppose $m_1, \dots, m_s \in \operatorname{gp}(S_P)$ are pair-wise different Laurent monomials of degree 0 and $a_1, \dots, a_s \in k^*$ such that $(a_1m_1 + \cdots + a_sm_s)\operatorname{Div}(F)$ is a height 1 prime ideal of $k[\bar{S}_P]$ (notation as in Step 1); we claim that either one of the m_i is $1 \in k$ or

$$(a_1m_1 + \cdots + a_sm_s)\operatorname{Div}(F) = \operatorname{Div}(G)$$

for some facet $G \prec P$.

Indeed, if none of the m_i is $1 \in k$ then $(a_1m_1+\cdots+a_sm_s)\operatorname{Div}(F) \subset \operatorname{Div}(G)$ for any facet $G \prec P$ such that $F \cap G = \emptyset$, which exists by assumption. We get an inclusion of two height 1 prime ideals. Hence the desired equality.

Note that the lower dimensional faces of a facet-separated polytopes need not be facet-separated again. Therefore one has to modify Step 1 slightly by working only with the facets of Π and condition $(\#_2')$. The fact just proved implies that in Step 2 above one can then take $c = 1$. Thus the restriction to fields of characteristic 0, which entered the proof only via Lemma 5.9, becomes superfluous, and all the arguments go through for an arbitrary field. We only need k to be infinite in order to be able to apply Lemma 3.5.

The existence of the normal forms, claimed in Theorem 5.2(c), follows immediately since we deal just once with each column vector during the whole process and since the process can be carried out in an arbitrary order of the column vectors. □

6. Arrangements of projective toric varieties

In this section we develop some notions similar to those in [BG, Section 5], generalized from single polytopes to the new situation of polyhedral complexes. (For standard facts on toric varieties we refer to Danilov [Da], Fulton [Fu], Oda [Oda].)

Throughout this section k denotes an algebraically closed field.

A lattice polytope $P \subset \mathbb{R}^n$ is called *very ample* if for every vertex $v \in P$ the affine semigroup in $\mathbb{Z}^n$, defined by the $\dim(P)$-dimensional cone spanned by P at its corner v and then shifted by $-v$, is generated by the set

$$\{x - v \mid x \in \mathbb{Z}^n \cap P\}.$$

All normal lattice polytopes (i. e. those for which $k[S_P]$ is normal) are very ample, but not conversely [BG, Example 5.5].

A lattice polyhedral complex is called *very ample* if all its faces are very ample. Observe that it would suffice to require very ampleness only for the facets: the property is inherited by the lower-dimensional faces.

Suppose Π is a very ample lattice polyhedral complex and $z \in \Pi$ is a vertex. Then we define the *associated weak fan* $\Pi(z)$ as follows. Consider the faces $P \prec \Pi$ containing z. For any such face we have the rational polyhedral cone $C(P, z) \subset \mathbb{R}^{n_P}$ spanned by P at its vertex z and shifted (in $\mathbb{R}^{n_P}$) by $-z$. Due to the very ampleness of Π the system of these cones forms a weak fan in a natural way which we denote by $\Pi(z)$. Therefore, the cones of $\Pi(z)$ are naturally labeled by the faces $P \prec \Pi$ such that $z \in P$. We will denote them by $\mathcal{W}_P^{(z)}$ correspondingly.

Now assume we are given a finite system of vertices $z_1, \dots, z_k \prec \Pi$. For each face $P \prec \Pi$ we define the convex (but not necessarily strictly convex) rational cone $C(P, z_1, \dots, z_s) \subset \mathbb{R}^{n_P}$ as follows. If $\{z_1, \dots, z_s\}$ is not a subset of P, then we put $C(P, z_1, \dots, z_s) = \{0\} \subset \mathbb{R}^{n_P}$. If $\{z_1, \dots, z_s\} \subset P$ then there are two possibilities – either there is a supporting halfspace for P (in $\mathbb{R}^{n_P}$) that contains $\{z_1, \dots, z_s\}$ in its boundary, or such does not exist. In the first case we let $C(P, z_1, \dots, z_s)$ be the intersection (in $\mathbb{R}^{n_P}$) of all these supporting halfspaces for P, shifted after that by one of the $-z_i$ (all these parallel translates coincide). In the second case we put $C(P, z_1, \dots, z_n) = \mathbb{R}^{n_P}$.

Observe that if $P \prec Q$, then $C(P, z_1, \dots, z_s)$ is a face (in the obvious sense) of $C(Q, z_1, \dots, z_s)$. In particular, we can patch the semigroup algebras

$$k[C(P, z_1, \dots, z_s) \cap \mathbb{Z}^{n_P}], \quad P \prec \Pi,$$

using these 'face identifications' for all pairs $P \prec Q$ as we did for weak fans in Section 2. The resulting k-algebra will be denoted by $k[\Pi(z_1, \dots, z_n)]$. It is a common localization of the $k[\Pi(z_i)]$.

In the following we will use the notations $\mathcal{Z}_\Pi = \operatorname{Proj}(k[\Pi])$ and $\mathcal{Z}_P = \operatorname{Proj}(k[S_P])$ for $P \prec \Pi$. Thus $\mathcal{Z}_P$ is a normal projective toric variety (the normality follows from the very ampleness of P), and all normal projective toric varieties arise in this way. For each face $P \prec \Pi$ we have fixed an embedded torus of $\mathcal{Z}_P$ – namely, the one that respects the monomial structure of S_P. Let $\mathbb{T}(\mathcal{Z}_P)$ denote this torus. Thus $\mathbb{T}(\mathcal{Z}_P) = \mathbb{T}_k(P)/k^*$. If $P \prec Q \prec \Pi$ then we have the closed embeddings $\mathcal{Z}_P \subset \mathcal{Z}_Q \subset \mathcal{Z}_\Pi$ given by the 'face' projections of the corresponding homogeneous rings. We get a diagram of toric varieties and a corresponding diagram of their embedded tori,

$$\mathcal{D}_\Pi = \{\mathcal{Z}_P \subset \mathcal{Z}_Q \mid P \prec Q \prec \Pi\} \quad \text{and} \quad \mathcal{D}_T = \{\mathbb{T}(\mathcal{Z}_Q) \xrightarrow{\text{rest}} \mathbb{T}(\mathcal{Z}_P) \mid P \prec Q \prec \Pi\}$$

where 'rest' denotes the restriction map. We set

$$\mathbb{D}(\mathcal{Z}_\Pi) = \varprojlim \mathcal{D}_T \quad \text{and} \quad \mathbb{T}(\mathcal{Z}_P) = \mathbb{D}(\mathcal{Z}_\Pi)^0.$$

In general, $\mathbb{D}(\mathcal{Z}_\Pi) \neq \mathbb{T}(\mathcal{Z}_\Pi)$, as can be seen as follows: if Π' is the cone over Π (adding exactly one more lattice point corresponding to a new variable), then $\mathbb{D}(\mathcal{Z}_{\Pi'}) = \mathbb{D}(\Pi)$, and in Section 3 we have given an complex Π with $\mathbb{D}(\Pi) \neq \mathbb{T}(\Pi)$.

One has the following easily verified description of $\mathcal{Z}_\Pi$ (see also [BG, Section 5]).

Proposition 6.1. *Let Π be a very ample lattice polyhedral complex.*

(a) *The projective variety $\mathcal{Z}_\Pi \subset \mathbb{P}^N$, $N = \#\mathrm{L}(\Pi) - 1$, is obtained by patching the* *affine schemes* $\operatorname{Spec}(k[\Pi(z)])$ *along their open subschemes*

$$\operatorname{Spec}(k[\Pi(z, z_1, \dots, z_s)]) \subset \operatorname{Spec}(k[\Pi(z)]),$$

where $z, z_1, \dots, z_s$ *are vertices of* Π.

(b) *The irreducible components of* $\mathcal{Z}_\Pi$ *are precisely the normal projective toric varieties* $\mathcal{Z}_P = \operatorname{Proj}(k[\bar{S}_P]) \subset \mathbb{P}^{N_P}$, $N_P = \#\mathrm{L}(P) - 1$, *where* P *runs through the facets of* Π. *Moreover,*

$$\mathcal{Z}_\Pi = \varinjlim \mathcal{D}_\Pi.$$

(c) $\mathbb{D}(\mathcal{Z}_\Pi)$ *is a diagonalizable group and, hence,* $\mathbb{T}(\mathcal{Z}_\Pi)$ *is a torus; they act algebraically on* $\mathcal{Z}_\Pi$ *so that for each face* $P \prec \Pi$ *the action restricts to the original one of* $\mathbb{T}(\mathcal{Z}_P)$ *on* $\mathcal{Z}_P$.

$\mathbb{D}(\mathcal{Z}_\Pi)$ is diagonalizable since it is a subgroup of the product of the $\mathbb{T}(\mathcal{Z}_P)$, $P \prec \Pi$.

Projective varieties of type $\mathcal{Z}_\Pi$ with Π very ample are called *arrangements of projective toric varieties* and the affine charts, described in Proposition 6.1(a), will be called Π-*affine charts*.

One easily observes the exact sequence of algebraic groups

$$0 \longrightarrow (k^*)^{\pi_0(\Pi)} \longrightarrow \Gamma_k(\Pi) \xrightarrow{\mathrm{pr}_\Pi} \operatorname{Aut}_k(\mathcal{Z}_\Pi),$$

where pr_Π is the canonical anti-homomorphism and $\pi_0(\Pi)$ refers to the (number of) connected components of Π (viewed as a CW-complex in a natural way).

It is clear that $\mathbb{D}_k(\Pi)$ is mapped to $\mathbb{D}(\mathcal{Z}_\Pi)$ by pr_Π. However, $\mathrm{pr}_\Pi(\mathbb{D}_k(\Pi))$ is in general smaller than $\mathbb{D}(\mathcal{Z}_\Pi)$; likewise $\mathrm{pr}_\Pi(\Gamma_k(\Pi))$ need not exhaust $\operatorname{Aut}_k(\mathcal{Z}_\Pi)$.

Example 6.2. Let Π be the complex of three unit segments forming the boundary of a triangle. Then $\mathcal{Z}_\Pi$ is an arrangement of three copies of the projective line $\mathbb{P}^1_k$ meeting each other pairwise in three different points. It follows from Theorem 5.2 and easy observations that $\Gamma_k(\Pi) = \Gamma_k(\Pi)^0 = \mathbb{T}_k(\Pi)$. But one has $\mathbb{T}_k(\Pi)/k^* = (k^*)^2$ and $\mathbb{D}(\mathcal{Z}_\Pi) = (k^*)^3$.

Next we introduce the notion of *projectively equivalent* lattice polyhedral complexes. Recall that the *normal fan* $\mathcal{N}(P)$ of a polytope $P \subset \mathbb{R}^n$ is defined as the complete fan in the dual space $(\mathbb{R}^n)^* = \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ given by the system of cones

$$(\{\phi \in (\mathbb{R}^n)^* \mid \operatorname{Max}_P(\phi) = F\},\ F \text{ a face of } P).$$

Two polytopes $P, Q \subset \mathbb{R}^n$ are called *projectively equivalent* if $\mathcal{N}(P) = \mathcal{N}(Q)$. In other words, P and Q are projectively equivalent if and only if they have the same dimension, the same combinatorial type, and the faces of P are parallel translates of the corresponding ones of Q.

Lattice polyhedral complexes Π and Π' are called *projectively equivalent* if the following conditions are satisfied:

(a) there is an isomorphism between the underlying abstract polyhedral complexes $\psi : \Pi_X \to \Pi_{X'}$, i. e. there is a bijection ψ between the vertex sets X and X' inducing a bijection of the polyhedral complexes,
(b) $n_P = n_{\psi(P)}$,
(c) the lattice polytopes P^* and $(\psi(P))^* \subset \mathbb{R}^{n_P}$, are projectively equivalent for all $P \subset \Pi_X$ so that if $F \subset P^*$ and $G \subset (\psi(P))^*$ correspond each other under this projective equivalence, then $F = Q^*$ and $G = (\psi(Q))^*$ for some $Q \subset \Pi_X$.

(Here we use the same notation as in the definition of a lattice polyhedral complex in Section 2.) The isomorphism $\psi : \Pi_X \to \Pi_{X'}$ is called a *projective equivalence.*

The next lemma explains the name 'projectively equivalent'.

Lemma 6.3. *Let* Π *and* Π' *be projectively equivalent very ample lattice polyhedral complexes. Then there is a natural isomorphism* $\mathcal{Z}_\Pi \approx \mathcal{Z}_{\Pi'}$ *transforming the* Π*-affine chart into the* Π'*-affine chart; furthermore the sets* $\operatorname{Col}(\Pi)$ *and* $\operatorname{Col}(\Pi')$ *of column vectors are in natural one-to-one corrsepondence.*

Proof. The isomorphism $\mathcal{Z}_\Pi \approx \mathcal{Z}_{\Pi'}$ exists due to the tautological identification of the two affine charts. The claim on column vectors follows easily from the analogous fact for single polytopes [BG, Section 2]. □

We will need the following standard

Lemma 6.4. *Let* V *be a* k*-variety and* G *a connected* k*-group acting algebraically on* V*. Then* G *leaves the irreducible components of* V *invariant.*

Let Π be an oriented lattice polyhedral complex. An automorphism of $\mathcal{Z}_\Pi$ is called *elementary* if it is of type $\operatorname{pr}_\Pi(e_V^\lambda)$ for some elementary automorphism $e_V^\lambda \in \Gamma_k(\Pi)$ $(\lambda \in k)$. For a column vector $V \in \operatorname{Col}(\Pi)$ the assignment

$$e_V^\lambda \mapsto \operatorname{pr}_\Pi(e_V^\lambda),\ \ \lambda \in k,$$

defines an algebraic homomorphism $\mathbb{A}_k^1 \to \operatorname{Aut}(\mathcal{Z}_\Pi)$. It follows from the exact sequence above that this is an injective mapping. The subgroup of $\operatorname{Aut}_k(\mathcal{Z}_\Pi)$ generated by the elementary automorphisms will be denoted by $\mathbb{E}(\mathcal{Z}_\Pi)$. Thus $\mathbb{E}(\mathcal{Z}_\Pi) = \operatorname{pr}_\Pi(\mathbb{E}_k(\Pi))$ is a connected group, spanned by one-parameter unipotent subgroups forming affine lines in $\mathbb{E}(\mathcal{Z}_\Pi)$.

Lemma 6.5. *Let* Π *and* Π' *be two projectively equivalent, oriented, and very ample polyhedral complexes. Then*
(a) $\mathbb{E}(\mathcal{Z}_\Pi) = \mathbb{E}(\mathcal{Z}_{\Pi'})$*;*
(b) $\delta \circ \varepsilon \circ \delta^{-1}$ *is an elementary automorphism for any* $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ *and any elementary automorphism* ε *of* $\mathcal{Z}_\Pi$*.*

(Here $\mathbb{E}(\mathcal{Z}_\Pi)$ and $\mathbb{E}(\mathcal{Z}_{\Pi'})$ are regarded as subgroups of the same group $\operatorname{Aut}_k(\mathcal{Z}_\Pi)$ by virtue of Lemma 6.3.)

Proof. (a) It is enough to show that if $V \in \mathrm{Col}(\Pi)$ and $V' \in \mathrm{Col}(\Pi')$ are corresponding column vectors (in the sense of Lemma 6.3) and $\lambda \in k$ then $e_V^\lambda \in \Gamma_k(\Pi)$ and $e_{V'}^\lambda \in \Gamma_k(\Pi')$ define the same elements in $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$. In fact, we get two elements from the unity component $\mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$ and, hence, by Lemma 6.4 they both leave the irreducible components of $\mathcal{Z}_\Pi$ invariant. Therefore, by Proposition 6.1(b) the problem reduces to the special case of single polytopes and here [BG, Lemma 5.1] applies.

(b) follows from the case of a single polytope, which is covered by 4.4(b), and patching arguments. □

Next we define the finite subgroup $\Sigma(\Pi)_{\mathrm{Proj}} \subset \mathrm{Aut}_k(\mathcal{Z}_\Pi)$ for a very ample lattice polyhedral complex, which generalizes the symmetry group of the normal fan $\mathcal{N}(P)$ of a polytope P [BG, Section 5].

For each vertex $z \prec \Pi$ and each face $P \prec \Pi$ we have introduced the corresponding weak fan $\Pi(z)$ and normal fan $\mathcal{N}(P)$ (the latter defined in the dual space $(\mathbb{R}^{n_P})^*$). The cone of $\mathcal{N}(P)$, corresponding to a face $F \prec P$, will be denoted by $\mathcal{N}_F^{(P)}$.

For a face $P \prec \Pi$ and a vertex $z \prec P$ we have

$$\mathcal{N}_z^{(P)} = (\mathcal{W}_P^{(z)})^*,$$

where the star on the right hand side denotes the dual cone in $(\mathbb{R}^{n_P})^*$.

It follows that if $\mathcal{W}_P^{(z)} \in \Pi(z)$, $\mathcal{W}_Q^{(y)} \in \Pi(y)$ ($y, z \prec \Pi$ vertices, $z \prec P \prec \Pi$, $y \prec Q \prec \Pi$) and

$$\alpha : \mathbb{Z}^{n_P} \cap \mathcal{W}_P^{(z)} \to \mathbb{Z}^{n_Q} \cap \mathcal{W}_Q^{(y)}$$

is a semigroup homomorphism, then one has the corresponding naturally defined semigroup homomorphism

$$\alpha^* : (\mathbb{Z}^{n_Q})^* \cap \mathcal{N}_y^{(Q)} \to (\mathbb{Z}^{n_P})^* \cap \mathcal{N}_z^{(P)},$$

and vice versa. Moreover, $\alpha^{**} = \alpha$ and $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$. In particular, isomorphisms are mapped to isomorphisms.

We recall that an isomorphism of complete fans in Euclidean spaces means a integral linear isomorphism of the ambient spaces transforming one fan into the other.

We shall say that two weak fans are *isomorphic* if their underlying abstract polyhedral complexes are isomorphic and the corresponding affine semigroups are isomorphic in such a way that the involved isomorphisms agree on common 'face' sub-semigroups.

Now an element of $\Sigma(\Pi)_{\mathrm{Proj}}$ by definition is a triple $(\rho, \mathcal{A}, \mathcal{B})$, where

(1) ρ is an automorphism of the abstract polyhedral complex X_Π,
(2) $\mathcal{A}$ is a set of isomorphisms $\alpha_P^{(z)} : \mathbb{Z}^{n_P} \cap \mathcal{W}_P^{(z)} \longrightarrow \mathbb{Z}^{n_{\rho(P)}} \cap \mathcal{W}_{\rho(P)}^{(\rho(z))}$, where z runs through the vertices and P through the faces of Π with $z \prec P$,
(3) $\mathcal{B}$ is a set of isomorphisms $\beta_z^{(P)} : (\mathbb{Z}^{n_{\rho(P)}})^* \cap \mathcal{N}_{\rho(z)}^{(\rho(P))} \longrightarrow (\mathbb{Z}^{n_P})^* \cap \mathcal{N}_z^{(P)}$, z and P as above,

so that the following conditions are satisfied:

(4) for each vertex $z \prec \Pi$ the subset $\{\alpha_P^{(z)} \mid z \prec P \prec \Pi\} \subset \mathcal{A}$ establishes an isomorphism between the weak fans $\Pi(z)$ and $\Pi(\rho(z))$,
(5) for each face $P \prec \Pi$ the subset $\{\beta_z^{(P)} \mid z \prec P$ a vertex$\} \subset \mathcal{B}$ establishes an isomorphism between the normal fans $\mathcal{N}(\rho(P))$ and $\mathcal{N}(P)$,
(6) $\mathcal{B} = \{\alpha^* \mid \alpha \in \mathcal{A}\}$.

The group structure of $\Sigma(\Pi)_{\mathrm{Proj}}$ is defined by taking the appropriate compositions. It follows readily from Proposition 6.1(a) that we can consider the finite group $\Sigma(\Pi)_{\mathrm{Proj}}$ as a subgroup of $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$ in a natural way, provided Π is very ample – the elements of $\Sigma(\Pi)_{\mathrm{Proj}}$ can naturally be thought of as automorphisms of the Π-affine chart on $\mathcal{Z}_\Pi$. Now a straightforward verification shows the following

Lemma 6.6. *Let Π be a very ample lattice polyhedral complex.*

(a) *If Π' is a very ample lattice polyhedral complex, projectively equivalent to Π, then $\Sigma(\Pi)_{\mathrm{Proj}}$ and $\Sigma(\Pi')_{\mathrm{Proj}}$ coincide (in the sense of Lemma 6.3);*
(b) *Let $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$, $\sigma \in \Sigma(\Pi)_{\mathrm{Proj}}$, and $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$; then $\sigma^{-1} \circ \delta \circ \sigma \in \mathbb{D}(\mathcal{Z}_\Pi)$ and $\sigma^{-1} \circ \varepsilon \circ \sigma \in \mathbb{E}(\mathcal{Z}_\Pi)$;*
(c) pr_Π *embeds* $\Sigma(\Pi)$ *into* $\Sigma(\Pi)_{\mathrm{Proj}}$.

7. Very ample line bundles on arrangements

In this section we first give an overview of known results ([Oda, Ch. 2], Teissier [Te]) on very ample line bundles on projective toric varieties. The generalization to arrangements of toric varieties, discussed later on, will be needed in the proof of Theorem 9.1.

Let n be a natural number and $P \subset \mathbb{R}^n$ be a very ample lattice n-polytope. We let $\mathcal{P}$ denote the set of lattice polytopes $Q \subset \mathbb{R}^n$, which are very ample and projectively equivalent to P. Then $\mathcal{P}$ carries the following semigroup structure (without unity):

$$Q + R = \{q + r \mid q \in Q, r \in R\}, \quad Q, R \in \mathcal{P}.$$

Thus $Q + R$ is the Minkowski sum of Q and R (very ampleness is preserved by Minkowski sums). Any element $Q \in \mathcal{P}$ defines a normal projective toric variety $\mathcal{Z}_Q = \mathrm{Proj}(k[\bar{S}_Q])$ (Proposition 6.1(b)) and the very ample line bundle $\mathcal{L}_Q$, the preimage of the structural line bundle $\mathcal{O}(1)$ under the natural closed embedding

$$\mathcal{Z}_Q \to \mathbb{P}_k^N, \; N = \#\mathrm{L}(Q) - 1.$$

We shall identify all the $\mathcal{Z}_Q$ for $Q \in \mathcal{P}$ via the natural isomorphism mentioned in Lemma 6.3.

The torus $(k^*)^n = \mathrm{Hom}(\mathbb{Z}^n, k^*)$ operates on all the algebras $K[\bar{S}_Q]$; furthermore it can be identified with the embedded torus $\mathbb{T}(\mathcal{Z}_P)$ of $\mathcal{Z}_P$. Thus the line bundle $\mathcal{L}_Q$ carries an $\mathbb{T}(\mathcal{Z}_P)$-equivariant structure, i. e. an action of the embedded torus $\mathbb{T}(\mathcal{Z}_P)$ which is compatible with the structural projection $\mathcal{L} \to \mathcal{Z}_P$ and is fiber-wise linear. (This action is obtained as the restriction of the action of $\mathbb{T}(\mathcal{Z}_P) \subset (k^*)^N$ on $\mathcal{O}(1)$.)

Of course, any such action can be modified by a character χ of $\mathbb{T}(\mathcal{Z}_P)$, i. e. one replaces the linear map $\tau_x : \mathcal{L}_x \to \mathcal{L}_{\tau(x)}$ by $\chi(\tau)\tau_x$.

Furthermore, any equivariant $\mathbb{T}(\mathcal{Z}_P)$-structure on $\mathcal{L}_Q$ induces a corresponding action of $\mathbb{T}(\mathcal{Z}_P)$ on the canonical algebra

$$\mathcal{O}(\mathcal{L}_Q) = \bigoplus_{i\geq 0} H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i}),$$

given by $\tau(f) = \tau^{-1} \circ f \circ \tau$ for $\tau \in \mathbb{T}(\mathcal{Z}_P)$ and a global section $f : \mathcal{Z}_P \to L_Q^{\otimes i}$ ($i \in \mathbb{N}$). With respect to this action $H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i})$ decomposes into a direct sum of one-dimensional representations of $\mathbb{T}(\mathcal{Z}_P)$.

Lemma 7.1. (a) *With respect to the equivariant structure induced by the action of $(k^*)^n$ on $k[\bar{S}_Q]$, the characters of $(k^*)^n = \mathbb{T}(\mathcal{Z}_P)$ corresponding to the one-dimensional representations of $\mathbb{T}(\mathcal{Z}_P)$ in $H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i})$ are pairwise different; under the identification* $\operatorname{Hom}((k^*)^n, k) = \mathbb{Z}^n$ *they are the lattice points of the i-th homothetic blow up iQ of Q.*

(b) *Any two equivariant structures on $\mathcal{L}_Q$ differ by a character of $\mathbb{T}(\mathcal{Z}_P)$. Thus the decomposition of $H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i})$ is independent of the equivariant structure, and if one multiplies the equivariant structure in* (a) *by χ, than Q has to be replaced by $Q - \chi$.*

By letting k^* act trivially on $\mathcal{Z}_P$ we can extend the action of $\mathbb{T}(\mathcal{Z}_P)$ to an action of $k^{n+1} = \mathbb{T}_k(P) = \mathbb{T}(\mathcal{Z}_P) \times k^*$ on $\mathcal{Z}_P$. Moreover, any $\mathbb{T}(\mathcal{Z}_P)$-equivariant structure on $\mathcal{L}_Q$ can be extended to an action of $\mathbb{T}_k(P)$ if we let k^* act on $\mathcal{L}_Q$ by fiber-wise multiplication. This gives rise to an action of $\mathbb{T}_k(P)$ on the canonical algebra of $\mathcal{L}_Q$; of course, $\mathbb{T}_k(P)$ also acts naturally on $k[\bar{S}_Q]$.

Lemma 7.2. *For the equivariant structure on $\mathcal{L}_Q$ induced by the action of $(k^*)^n$ on $k[\bar{S}_Q]$ we have a graded k-algebra isomorphism $\mathcal{O}(\mathcal{L}_Q) \approx k[\bar{S}_Q]$ that respects the two $\mathbb{T}_k(P)$-actions.*

The assignment $Q \mapsto \mathcal{L}_Q$ induces a mapping $\mathcal{P} \to \operatorname{Pic}(\mathcal{Z}_P)$ which obviously factors through the quotient $\mathcal{P}/\sim$ where $Q \sim R$ if and only if R is a parallel translate of Q. This equivalence relation defines a congruence on the semigroup $\mathcal{P}$.

Lemma 7.3. *Let* $\operatorname{VALB}(\mathcal{Z}_P)$ ($\operatorname{EVALB}(\mathcal{Z}_P)$) *denote the sets of isomorphism classes of (equivariant) very ample line bundles on $\mathcal{Z}_P$. One has a commutative diagram*

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \operatorname{EVALB}(\mathcal{Z}_P) \\ \downarrow & & \downarrow \\ \mathcal{P}/\sim & \longrightarrow & \operatorname{VALB}(\mathcal{Z}_P) \end{array}$$

where the horizontal maps are semigroup isomorphisms and the right vertical map 'forgets' the equivariant structure.

We can summarize this discussion as follows. For an ample line bundle $\mathcal{L}$ and a $\mathbb{T}(\mathcal{Z}_P)$-equivariant structure on $\mathcal{L}$, the decomposition of the canonical algebra of $\mathcal{L}$ into one-dimensional representations depends only on $\mathcal{L}$. Furthermore the representations appearing in $H^0(\mathcal{Z}_P, \mathcal{L})$ can naturally be labeled by the lattice points of Q where $Q \in \mathcal{P}$ is chosen such that $\mathcal{L} \approx \mathcal{L}_Q$. We denote them by $V_{\mathcal{L},x}$, $x \in \mathrm{L}(Q)$. These observations will be used in the next definition.

In what follows, the subalgebra of the canonical algebra of a line bundle $\mathcal{L}$ generated by its global sections will be called the *subcanonical algebra* of $\mathcal{L}$ and it will be denoted by $\mathrm{Alg}(\mathcal{L})$.

Definition 7.4. Let $\mathcal{L}$ be a very ample line bundle on $\mathcal{Z}_P$, $\mathcal{L} \approx \mathcal{L}_Q$. A system of global sections $(f_x)_{x \in \mathrm{L}(Q)} \subset H^0(\mathcal{Z}_P, \mathcal{L})$ is called *polytopal* if $f_x \in V_{\mathcal{L},x}$ for all x and there is a k-algebra isomorphism between $\mathrm{Alg}(\mathcal{L})$ and $k[S_Q]$, mapping f_x, $x \in \mathrm{L}(Q)$, to $x \in k[S_Q]$.

Roughly speaking, the next lemma says that two polytopal systems of sections in a line bundle only differ by a toric automorphism of $\mathcal{Z}_P$.

Lemma 7.5. *Let $\mathcal{L}$ and $\mathcal{L}'$ be very ample line bundles on $\mathcal{Z}_P$, $\mathcal{L} \approx \mathcal{L}' \approx \mathcal{L}_Q$ for some $Q \in \mathcal{P}$. Suppose $(f_x)_{x \in \mathrm{L}(Q)}$ and $(f'_x)_{x \in \mathrm{L}(Q)}$ are polytopal systems of global sections. Then there is a unique commutative diagram with vertical structural projections*

$$\begin{array}{ccc} \mathcal{L}' & \xrightarrow{T} & \mathcal{L} \\ \big\downarrow & & \big\downarrow \\ \mathcal{Z}_P & \xrightarrow[\tau]{} & \mathcal{Z}_P \end{array}$$

such that T is an algebraic fiber-wise linear map, $\tau \in \mathbb{T}(\mathcal{Z}_P)$ and $T^{-1} \circ f_x \circ \tau = f'_x$ for all $x \in \mathrm{L}(Q)$.

Proof. Fix $\mathbb{T}(\mathcal{Z}_P)$-equivariant structures on $\mathcal{L}$ and $\mathcal{L}'$. Observe that for any commutative diagram of the type considered the mapping T is automatically equivariant. It follows that T is $\mathbb{T}_k(P)$-equivariant for the induced $\mathbb{T}_k(P)$-structures on $\mathcal{L}$ and $\mathcal{L}'$ as well. It is also clear, that such a mapping T induces a graded k-algebra isomorphism between the canonical algebras $\mathcal{O}(\mathcal{L})$ and $\mathcal{O}(\mathcal{L}')$ which respects the $\mathbb{T}_k(P)$ actions – the diagram above is a pull-back diagram with equivariant horizontal isomorphisms.

Conversely, if we are given a graded $\mathbb{T}_k(P)$-equivariant isomorphism between the two canonical algebras, then this isomorphism gives rise (by projectivization) to a commutative diagram of the type considered.

Therefore, in view of Lemma 7.2, Lemma 7.5 is equivalent to the following obvious claim, which finishes the proof: for a lattice polytope Q and two systems of degree 1 terms $\alpha_x x$ and $\beta_x x$, $\alpha_x, \beta_x \in k^*$, $x \in \mathrm{L}(Q)$, satisfying the same relations in $k[\bar{S}_Q]$

as the $x \in \mathrm{L}(Q)$, there is a unique toric automorphism $\tau \in \mathbb{T}_k(Q)$ transforming one system into the other. □

For a very ample polyhedral complex Π we let $[\Pi]$ denote the class of such lattice polyhedral complexes Π' that there exists a projective equivalence $\psi : \Pi_X \to \Pi_{X'}$ (see above) and a system of semigroup isomorphisms $\phi_P : S_P \to S_{\psi(P)}$, $P \prec \Pi$, compatible on 'face' sub-semigroups.

Let Π be a very ample polyhedral complex. The set

$$\Big\{[\Pi'] \mid \Pi' \text{ a very ample lattice polyhedral complex, projectively equivalent to } \Pi\Big\}$$

carries a natural semigroup structure (without unity). Assume $[\Pi_1]$ and $[\Pi_2]$ belong to it. Then a face of Π_1 and the corresponding face of Π_2 can be realized as projectively equivalent very ample lattice polytopes in the same Euclidean space. The pairwise Minkowski sums naturally form a very ample lattice polyhedral complex Π_3 which is projectively equivalent to Π (one uses fixed projective equivalences $\Pi_X \xrightarrow{\approx} \Pi_{X_1}$, $\Pi_X \xrightarrow{\approx} \Pi_{X_2}$ and systems of the corresponding semigroup isomorphisms). It is clear that the class $[\Pi_3]$ is well defined. We put $[\Pi_1] + [\Pi_2] = [\Pi_3]$.

Assume $\mathcal{L}$ is a very ample line bundle on $\mathcal{Z}_\Pi$. For each face $P \prec \Pi$ the restricted line bundle $\mathcal{L}|_{\mathcal{Z}_P}$ is very ample on $\mathcal{Z}_P$, and we pick a lattice polytope $Q(\mathcal{L}, P)$ such that $\mathcal{L}|_{\mathcal{Z}_P} \approx \mathcal{L}_{Q(\mathcal{L},P)}$. It is clear from the discussion preceding Lemma 7.5 and the obvious isomorphisms

$$\mathcal{L}|_{\mathcal{Z}_P} \approx (\mathcal{L}|_{\mathcal{Z}_{P'}})|_{\mathcal{Z}_P}$$

for any faces $P \prec P' \prec \Pi$, that the polytopes $Q(\mathcal{L}, P)$ naturally form a very ample lattice polyhedral complex, which is projectively equivalent to Π. The class of this complex will be denoted by $\Pi(\mathcal{L})$. (Different choices of the the polytopes $Q(\mathcal{L}, P)$ give rise to the same class.) If $\mathcal{L}'$ is another very ample line bundle on $\mathcal{Z}_\Pi$, then $\Pi(\mathcal{L}) + \Pi(\mathcal{L}') = \Pi(\mathcal{L} \otimes \mathcal{L}')$ (this reduces to the case of single polytopes; see 7.3).

Observe that for *any* very ample lattice polyhedral complex Π', which is projectively equivalent to Π, there is a very ample line bundle $\mathcal{L}$ on $\mathcal{Z}_\Pi$ such that $[\Pi'] = \Pi(\mathcal{L})$. In fact, the desired line bundle is provided by the restriction of $\mathcal{O}(1)$ under the canonical closed embedding $\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi'} \to \mathbb{P}^N_k$, $N = \#\mathrm{L}(\Pi') - 1$.

Definition 7.6. Let Π and $\mathcal{L}$ be as above. A system $\bar{f} = \{f_1, \dots, f_s\}$ of global sections of $\mathcal{L}$ is called *polyhedral* if it satisfies the following conditions:

(i) $f_i \neq 0$ for all i, and for each face $P \prec \Pi$ the set of restrictions

$$\{f_i|_{\mathcal{Z}_P} \mid \ f_i \in \bar{f} \text{ and } f_i|_{\mathcal{Z}_P} \neq 0\}$$

is a polytopal system of global sections of the line bundle $\mathcal{L}|_{\mathcal{Z}_P}$ on $\mathcal{Z}_P$;

(ii) if $f_i|_{\mathcal{Z}_P} \neq 0$ and $f_i|_{\mathcal{Z}_Q} \neq 0$ for faces P and Q of Π, then $f_i|_{\mathcal{Z}_P \cap \mathcal{Z}_Q} \neq 0$ (in particular $\mathcal{Z}_P \cap \mathcal{Z}_Q \neq \emptyset$).

Caution. In general a very ample line bundle on $\mathcal{Z}_\Pi$ does not have a polyhedral system of global sections; see Example 9.3(a) below.

Lemma 7.7. *Let Π be a very ample complex and $\mathcal{L}$ be a very ample line bundle on $\mathcal{Z}_\Pi$ possessing a polyhedral system of global sections $\bar f$.*

(a) *$\bar f$ is a basis of the k-vector space $H^0(\mathcal{Z}_\Pi, \mathcal{L})$.*

(b) *Let $\Pi' \in \Pi(\mathcal{L})$. Then there is a k-algebra isomorphism $\Theta : \operatorname{Alg}(\mathcal{L}) \to k[\Pi']$ mapping the elements of $\bar f$ to elements of $\mathrm{L}(\Pi')$. Moreover, for every face $Q \prec \Pi'$ there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{Alg}(\mathcal{L}) & \xrightarrow{\Theta} & k[\Pi'] \\ {\scriptstyle \mathrm{rest}_Q}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\ \operatorname{Alg}(\mathcal{L}|_{\mathcal{Z}_Q}) & \longrightarrow & k[S_Q], \end{array}$$

where $\mathcal{Z}_Q$ denotes the projective toric subvariety of $\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi'}$ naturally associated to Q, rest_Q is the restriction map and, as usual, π_Q is the corresponding face-projection.

(c) *If another very ample line bundle $\mathcal{L}'$ on $\mathcal{Z}_\Pi$ also has a polyhedral system of global sections, then so does $\mathcal{L} \otimes \mathcal{L}'$.*

Proof. We may assume $\Pi = \Pi'$. The essential point is that the elements of $\bar f$ correspond uniquely to the lattice points $x \in \mathrm{L}(\Pi)$: if $f_i|_{\mathcal{Z}_P}$ corresponds to $x \in \mathrm{L}(P)$, P a face of Π, then $f_i|_{\mathcal{Z}_Q}$ corresponds to the same lattice point x for all faces Q with $x \in \mathrm{L}(Q)$, as follows from condition (i) in the definition above. Condition (ii) implies that $f_i|_{\mathcal{Z}_Q} = 0$ if $x \notin \mathrm{L}(Q)$.

Now (a) is easily verified, and (b) and (c) follow from the analogous observations for single polytopes. It is important for (c) that in the case of a single polytope P the family $(f_i \otimes f'_j)$ formed from polytopal systems of global sections (f_i) and (f'_j) for $\mathcal{L}$ and $\mathcal{L}'$ has a unique extension to a polytopal system of global sections for $\mathcal{L} \otimes \mathcal{L}'$. Therefore a patching argument yields (c) for polyhedral complexes as well. $\square$

The next lemma extends 7.5 to polyhedral complexes.

Lemma 7.8. *Let $\mathcal{L}$ and $\mathcal{L}'$ be very ample line bundles on $\mathcal{Z}_\Pi$, where Π is a very ample lattice polyhedral complex. Assume that $\Pi(\mathcal{L}) = \Pi(\mathcal{L}')$ and that $\mathcal{L}$ and $\mathcal{L}'$ both have polyhedral systems of global sections. Then there is an element $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ such that $\mathcal{L}' = \delta^*(\mathcal{L})$.*

Remark 7.9. It is in general not true that $\mathcal{L} \approx \mathcal{L}'$ under the assumptions of Lemma 7.8. Moreover, the failure of the analogue of Lemma 7.3 for line bundles with polyhedral systems of global sections is measured precisely by the difference between $\mathbb{D}(\mathcal{Z}_\Pi)$ and the image of $\mathbb{D}_k(\Pi)$ in it. Consider, for instance, the lattice polyhedral complex Π of Example 6.2. Let $\mathcal{L}$ be the very ample line bundle on $\mathcal{Z}_\Pi$ obtained by the restriction of $\mathcal{O}(1)$ under the standard embedding $\mathcal{Z}_\Pi \to \mathbb{P}^3_k$. Now choose some $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ and set $\mathcal{L}' = \delta^*(\mathcal{L})$. It is clear that $\Pi(\mathcal{L}) = \Pi(\mathcal{L}') = \Pi$. Moreover, $\mathcal{L}$ has a polyhedral system of global sections and, hence, so does $\mathcal{L}'$. But $\mathcal{L}$ and $\mathcal{L}'$ cannot be isomorphic line bundles for any δ, for otherwise any element of $\mathbb{D}(\mathcal{Z}_\Pi)$ would be liftable (via pr_Π) to an element of $\Gamma_k(\Pi)$, which is not the case according to Example 6.2. Indeed, for a k-variety $\mathcal{Z}$ and a very ample line bundle $\mathcal{L}$ on it the group of automorphisms $\alpha \in \mathrm{Aut}_k(\mathcal{Z})$, that are liftable to $\mathrm{gr.\,aut}_k(\mathrm{Alg}(\mathcal{L}))$, coincides with the group of automorphisms $\beta \in \mathrm{Aut}_k(\mathcal{Z})$ preserving $\mathcal{L}$ [Ha, II.6]

Proof of Lemma 7.8. Let $\bar{f} = \{f_1, f_2, \dots\}$ and $\bar{g} = \{g_1, g_2, \dots\}$ be polyhedral systems of global sections of $\mathcal{L}$ and $\mathcal{L}'$. Then for any face $P \prec \Pi$ the restrictions of the f_i and g_j form polytopal systems of global sections of $\mathcal{L}|_{\mathcal{Z}_P}$ and $\mathcal{L}'|_{\mathcal{Z}_P}$. By Lemma 7.3 we have $\mathcal{L}|_{\mathcal{Z}_P} \approx \mathcal{L}'|_{\mathcal{Z}_P}$. Therefore, by Lemma 7.5 for each $P \prec \Pi$ there is a unique commutative diagram

$$\begin{array}{ccc} \mathcal{L}|_{\mathcal{Z}_P} & \xrightarrow{T_P} & \mathcal{L}'|_{\mathcal{Z}_P} \\ \downarrow & & \downarrow \\ \mathcal{Z}_P & \xrightarrow[\tau_P]{} & \mathcal{Z}_P, \end{array}$$

where $\tau_P \in \mathbb{T}(\mathcal{Z}_P)$ and T_P is an algebraic fiber-wise linear map. The uniqueness of these squares guarantees that we can patch them to a commutative square

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{D} & \mathcal{L}' \\ \downarrow & & \downarrow \\ \mathcal{Z}_\Pi & \xrightarrow[\delta]{} & \mathcal{Z}_\Pi, \end{array}$$

where $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ and D is an algebraic fiber-wise linear map. Hence the claim. □

8. Projectively quasi-Euclidean complexes

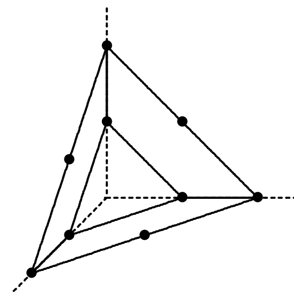
The following class of lattice polyhedral complexes is relevant in the description of $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$.

Definition 8.1. A lattice polyhedral complex Π is *projectively quasi-Euclidean* if it is quasi-Euclidean and every lattice polyhedral complex projectively equivalent to Π is quasi-Euclidean as well.

Below we describe two big classes of projectively quasi-Euclidean complexes. However the following example shows that not all quasi-Euclidean complexes are projectively quasi-Euclidean, not even boundary ones.

Example 8.2. Consider the boundary lattice polyhedral complex in $\mathbb{R}^3$ as shown in the figure. It has three trapezoid facets with vertex sets

$$\{(1,0,0),(2,0,0),(0,1,0),(0,2,0)\},$$
$$\{(0,1,0),(0,2,0),(0,0,1),(0,0,2)\},$$
$$\{(0,0,1),(0,0,2),(1,0,0),(2,0,0)\}.$$



Now we change the last facet with the trapezoid spanned by $\{(0,0,2),(0,0,3),(2,0,0),(3,0,0)\}$ and leave the first two trapezoids untouched. It is clear that the new system of trapezoids again defines a lattice polyhedral complex, which is projectively equivalent to the original one. But the latter complex is not quasi-Euclidean. In fact, if it were so then any (rational) Euclidean realization would fit into $\mathbb{R}^3$ and the two triangles, spanned respectively by the short and long edges of the trapezoids, would be homothetic. This would imply that the ratios of lengths of the two parallel edges in our trapezoids are all the same. This, of course, is not the case: we have the ratios $\frac{1}{2}$, $\frac{1}{2}$, $\frac{2}{3}$.

Let $P \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a (not necessarily lattice) polytope. We shall say that P is *affine-normal* if for any polytope $Q \subset \mathbb{R}^n$ that is projectively equivalent to P there exists an affine automorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that ψ transforms P into Q and respects faces corresponding to each other under normal equivalence. Clearly, such an affine automorphism is *uniquely* determined if $\dim(P) = n$, and a face of an affine-normal polytope is affine-normal as well.

Recall that a polytope $P \subset \mathbb{R}^{n_P}$ is called a *join* of two polytopes $Q \subset \mathbb{R}^{n_Q}$ and $R \subset \mathbb{R}^{n_R}$ if there are affine embeddings $\phi_Q : \mathbb{R}^{n_Q} \to \mathbb{R}^{n_P}$ and $\phi_R : \mathbb{R}^{n_R} \to \mathbb{R}^{n_P}$ such that:

(1) $\mathrm{Im}(\phi_Q) \cap \mathrm{Im}(\phi_R) = \emptyset$,
(2) the affine hull of $\mathrm{Im}(\phi_Q) \cup \mathrm{Im}(\phi_R)$ is an $(n_Q + n_R + 1)$-dimensional affine subspace of $\mathbb{R}^{n_P}$,
(3) P is the convex hull of $\phi_Q(Q) \cup \phi_R(R)$.

In particular, a join of a polytope P and a point is a pyramid of dimension $\dim(P)+1$ with base P. A join of Q and R is denoted by $\mathrm{J}(Q,R)$. It is easily observed that a join is unique up to a non-degenerate affine transformation.

Lemma 8.3. *If $P \subset \mathbb{R}^{n_P}$ and $Q \subset \mathbb{R}^{n_Q}$ are affine-normal polytopes, then so are their product $P \times Q \subset \mathbb{R}^{n_P+n_Q}$ and any join $\mathrm{J}(P,Q)$.*

Proof. Suppose $R \subset \mathbb{R}^{n_P+n_Q}$ is projectively equivalent to $P \times Q$. Without loss of generality we can assume that R is obtained from $P \times Q$ by a parallel translatation of one of the facets – the general case is obtained by induction over the set of facets. The facets of $P \times Q$ are of type either $F \times Q$ or $P \times G$ for some facets $F \prec P$ and $G \prec Q$. Consider the case $F \times Q$. Then R must have the type $P' \times Q$ for some P' projectively equivalent to P. Let $\psi : \mathbb{R}^{n_P} \to \mathbb{R}^{n_P}$ be the affine automorphism transforming P into P'. Then $\psi \times 1 : \mathbb{R}^{n_P+n_Q} \to \mathbb{R}^{n_P+n_Q}$ is the desired affine automorphism.

As for joins, it just suffices to observe that if R is projectively equivalent to $\mathrm{J}(P, Q)$ then $R = \mathrm{J}(P', Q')$ for some $P' \subset \mathbb{R}^{n_P}$ and $Q' \subset \mathbb{R}^{n_Q}$, projectively equivalent to P and Q respectively. □

In particular we see that all simplices, cubes, their joins, or more generally, joins of products of simplices etc., are affine-normal polytopes.

The *incidence graph* of a lattice polyhedral complex Π is defined as the graph, whose vertices are labeled by facets of Π and in which two vertices are connected by an edge if and only if the corresponding facets share a face.

Proposition 8.4. *A quasi-Euclidean lattice polyhedral complex is projectively quasi-Euclidean in either of the cases:*

(a) *the facets of* Π *are affine-normal polytopes,*
(b) *the incidence graph of* Π *is a tree.*

Proof. (a). Let Π be realized as a complex of rational polytopes in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Assume Π' is a lattice polyhedral complex projectively equivalent Π. Then each face $P \prec \Pi$ is projectively equivalent to the corresponding face $P' \prec \Pi'$ (they both are polytopes in $\mathbb{R}^{n_P}$). Let

$$\{\psi_P : \mathbb{R}^{n_P} \to \mathbb{R}^{n_P} \mid \ P \prec \Pi \text{ a face}\}$$

be the corresponding system of affine automorphisms transforming faces of Π' into those of Π. Since these maps are unique (as pointed out above), they are compatible on common faces. Therefore we can patch them to get a global bijective transformation

$$\Psi : \Pi \to \Pi'$$

as CW-complexes, which is face-wise affine. Observe that we are done once we know that $\Psi^{-1}(\mathrm{L}(\Pi'))$ consists of rational points of $\mathbb{R}^n$. But this follows readily from the facts that the vertices of Π are rational and that the ψ_P preserve barycentric coordinates.

For (b) one has an even stronger result – all lattice polyhedral complexes whose incidence graphs are trees are Euclidean. In fact, we can construct the Euclidean realization adding facets step by step and, at each step, sufficiently many new dimensions. □

In particular all simplicial complexes are projectively quasi-Euclidean.

9. The main result: projective case

Theorem 9.1. *Let k be an algebraically closed field and Π be a very ample lattice polyhedral complex.*

(a) *If $\operatorname{char}(k) = 0$ and Π is quasi-Euclidean, then the unity component $\operatorname{Aut}_k(\mathcal{Z}_\Pi)^0 \subset \operatorname{Aut}_k(\mathcal{Z}_\Pi)$ consists precisely of those elements $\alpha \in \operatorname{Aut}_k(\mathcal{Z}_\Pi)^0$ that admit a representation $\alpha = \varepsilon \circ \tau$ for some $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$ and $\tau \in \mathbb{T}(\mathcal{Z}_\Pi)$.*
(b) *If $\operatorname{char}(k) = 0$ and Π is projectively quasi-Euclidean, then every element $\alpha \in \operatorname{Aut}_k(\mathcal{Z}_\Pi)$ admits a representation $\varepsilon \circ \delta \circ \sigma$ for some $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$, $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ and $\sigma \in \Sigma(\Pi)_{\text{Proj}}$.*
(c) *If all facets of Π are facet-separated polytopes, then the exact analogues of* (a) *and* (b) *hold in arbitrary characteristic; moreover, the elements of $\operatorname{Aut}_k(\mathcal{Z}_\Pi)$ have a normal form analogous to that in Theorem* 5.2(c).

Before setting out for the proof we formulate one more auxiliary result.

Lemma 9.2. *Let $P \subset \mathbb{R}^n$ be a very ample lattice polytope and $\mathcal{L}$ be a very ample line bundle on $\mathcal{Z}_P$. Then $\alpha^*(\mathcal{L}) \approx \mathcal{L}$ for every element $\alpha \in \operatorname{Aut}_k(\mathcal{Z}_P)^0$.*

Proof. It follows from Lemma 5.2 and Theorem 5.3 of [BG] that the natural anti-homomorphism $\operatorname{gr.aut}_k(\operatorname{Alg}(\mathcal{L}))^0 \to \operatorname{Aut}_k(\mathcal{Z}_P)^0$ is surjective. But an automorphism liftable to $\operatorname{gr.aut}_k(\operatorname{Alg}(\mathcal{L}))$ preserves $\mathcal{L}$ as an element of $\operatorname{Pic}(\mathcal{Z}_P)$. □

Proof of Theorem 9.1. (a) Let $\mathcal{L}$ be the very ample line bundle on $\mathcal{Z}_\Pi$ obtained by the restriction of $\mathcal{O}(1)$ under the canonical embedding

$$\mathcal{Z}_\Pi \to \mathbb{P}_k^N, \ N = \#\mathrm{L}(\Pi) - 1.$$

Let $\alpha \in \operatorname{Aut}_k(\mathcal{Z}_\Pi)^0$. By Proposition 6.1(b) and Lemma 6.4 α restricts to an element of $\operatorname{Aut}_k(\mathcal{Z}_P)$ for each facet $P \prec \Pi$. It is clear that $\alpha|_{\mathcal{Z}_P} \in \operatorname{Aut}_k(\mathcal{Z}_P)^0$. By Lemma 9.2 we have an isomorphism

$$\mathcal{L}|_{\mathcal{Z}_P} \approx \alpha^*(\mathcal{L})|_{\mathcal{Z}_P} \tag{1}$$

of very ample line bundles on $\mathcal{Z}_P$. By Lemma 7.3, $\Pi(\mathcal{L}) = \Pi(\alpha^*(\mathcal{L}))$. Assume we have shown that $\alpha^*(\mathcal{L})$ has a polyhedral system of global sections. Then Lemma 7.8 yields $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ with $\alpha^*(\mathcal{L}) = \delta^*(\mathcal{L})$. In particular, the element $\alpha \circ \delta^{-1} \in \operatorname{Aut}_k(\mathcal{Z}_\Pi)$ leaves the line bundle $\mathcal{L}$ invariant. But then (as mentioned in Remark 7.9) the image of the canonical anti-homomorphism

$$\Gamma_k(\Pi) = \operatorname{gr.aut}(\operatorname{Alg}(\mathcal{L})) \xrightarrow{\operatorname{pr}_\Pi} \operatorname{Aut}_k(\mathcal{Z}_\Pi)$$

contains $\alpha \circ \delta^{-1}$; in fact $\operatorname{Alg}(\mathcal{L}) = k[\Pi]$ by definition. So Theorem 5.2(b) applies:

$$\alpha = \sigma \circ \delta' \circ \varepsilon \circ \delta$$

for some $\delta' \in \mathbb{D}(\mathcal{Z}_\Pi)$, $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$ and $\sigma \in \Sigma(\Pi)_{\mathrm{Proj}}$ (of course, we use that $\mathbb{D}_k(\Pi)$ maps to $\mathbb{D}(\mathcal{Z}_\Pi)$ and $\Sigma(\Pi)$ to $\Sigma(\Pi)_{\mathrm{Proj}}$). Next, Lemma 6.5(b) implies that

$$\mathbb{E}(\mathcal{Z}_\Pi) \cdot \mathbb{D}(\mathcal{Z}_\Pi) = \Big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{D}(\mathcal{Z}_\Pi)\Big) \quad \text{and} \quad \mathbb{E}(\mathcal{Z}_\Pi) \cdot \mathbb{T}(\mathcal{Z}_\Pi) = \Big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\Big).$$

(See Step 3 in the proof of Theorem 5.2 for this notation.) Using that $\mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$ has finite index in $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$ and $\Big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\Big)$ has finite index in $\Big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{D}(\mathcal{Z}_\Pi)\Big)$, we easily conclude that the connected subgroup $\Big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\Big) \subset \mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$ has finite index and, hence, coincides with $\mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$. Using the equality $\mathbb{E}(\mathcal{Z}_\Pi) \cdot \mathbb{T}(\mathcal{Z}_\Pi) = \Big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\Big)$ once more, we are done.

So everything amounts to showing that $\alpha^*(\mathcal{L})$ has a polyhedral system of global sections. (Observe that so far we did not use the quasi-Euclideaness of Π.) We solve this problem by first fixing polytopal global sections of $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$ for each facet P of Π, and then correcting these systems so that they agree on the intersections $\mathcal{Z}_P \cap \mathcal{Z}_Q$. It is here where we need the quasi-Euclideaness of Π – we will make use of Borel's theorem on maximal tori in the very same way as in the proof of Theorem 5.2(b).

For the facets $P \prec \Pi$ we let $\bar{f}_P$ be arbitrary polytopal systems of global sections of $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$. Fix a disjoint system of lattice polytopes $\hat{P}$ isomorphic to the P. By (1) we can think of the elements of $\bar{f}_P$ as lattice points of the polytopes $\hat{P}$. Let $P, Q \prec \Pi$ be facets and $\{x_1, \dots, x_s\} = \mathrm{L}(P) \cap \mathrm{L}(Q)$. Suppose $\{x_{P1}, \dots, x_{Ps}\} \subset \bar{f}_P$ and $\{x_{Q1}, \dots, x_{Qs}\} \subset \bar{f}_Q$ are the corresponding elements. By Lemma 7.7(b) (applied to the special case of a single polytope) these two systems restrict to polytopal systems of the same line bundle $\alpha^*(\mathcal{L})|_{\mathcal{Z}_{P\cap Q}}$ on $\mathcal{Z}_{P\cap Q}$. We will denote them by $\{x_{PQ1}, \dots, x_{PQs}\}$ and $\{x_{QP1}, \dots, x_{QPs}\}$. In particular, there is a unique toric automorphism $\tau_{PQ} \in \mathbb{T}_k(P \cap Q)$ transforming $\{x_{PQ1}, \dots, x_{PQs}\}$ into $\{x_{QP1}, \dots, x_{QPs}\}$ (by the natural action).

By the same token we get a system of elements $\tau_{PQ} \in \mathbb{T}_k(P \cap Q)$ for all facets $P, Q \prec \Pi$, satisfying the conditions

$$\tau_{PQ} = \tau_{QP}^{-1} \quad \text{and} \quad \tau_{PQ} = \tau_{PR} \circ \tau_{RQ}$$

on $P \cap Q$ and $P \cap Q \cap R$ respectively. The system τ_{PQ} will be called the *twisted structure* corresponding to the family $\bar{f}_P$, and it will be denoted by $\mathcal{T}$.

For any system of toric automorphisms $\{\tau_P \in \mathbb{T}_k(P) \mid P \prec \Pi \text{ a facet}\}$ we get a new family of polytopal global sections of the $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$ – we just apply these toric automorphisms to the $\bar{f}_P$ correspondingly. Therefore, we obtain a new twisted structure. Our goal is to show that there is such a family $\{\tau_P \in \mathbb{T}_k(P) \mid P \prec \Pi \text{ a facet}\}$ that the resulting twisted structure totally consists of the identity automorphisms. To this end we consider the subcanonical algebra $\mathrm{Alg}(\alpha^*(\mathcal{L}))$. First of all we have the k-algebra isomorphism

$$k[\Pi] = \mathrm{Alg}(\mathcal{L}) \xrightarrow{\alpha^*} \mathrm{Alg}(\alpha^*(\mathcal{L})). \tag{2}$$

Consider a point $x \in \mathrm{L}(\Pi)$ and the corresponding set of lattice points

$$\{x_P \in \mathrm{L}(\hat{P}) \mid \ P \in \mathrm{Supp}(x)\}.$$

('Supp' has the same meaning as in Section 4.) By Lemma 7.7(b) (applied to the special situation of a single polytope) each of the x_P restricts to the zero global section of $\mathcal{L}|_{\mathcal{Z}_{P\cap R}}$ whenever $R \notin \mathrm{Supp}(x)$. On the other hand it follows from the observations above that for $P, Q \in \mathrm{Supp}(x)$ there is a uniquely determined element $c_{QP}^{(x)} \in k^*$ such that $c_{QP}^{(x)} x_{QP} = x_{PQ}$, where x_{PQ} denotes the restriction of x_P to $\mathcal{Z}_{P\cap Q}$ and similarly for x_{QP}. Therefore, we can patch the sections $c_{QP}^{(x)} x_Q$, $P, Q \in \mathrm{Supp}(x)$, which are defined on $\mathcal{Z}_Q$, and extend them by 0 on $\mathcal{Z}_R$ for $R \notin \mathrm{Supp}(x)$ to obtain a global section of $\alpha^*(\mathcal{L})$.

It follows that the quotient algebra

$$k[\Pi, \mathcal{T}] = k[S_{\hat{P}_1}] \times_k \cdots \times_k k[S_{\hat{P}_r}]/(\{c_{QP}^{(x)} x_Q - x_P \mid P, Q \in \mathrm{Supp}(x)\})$$

maps naturally to $\mathrm{Alg}(\alpha^*(\mathcal{L}))$ as a graded algebra, where $\{P_1, \dots, P_r\}$ is the set off all facets of Π (and the x_P are identified with the corresponding elements in the fiber product over k). The isomorphism (2) and Hilbert function arguments show that this mapping is actually a graded k-algebra isomorphism. The twisted structure $\{\tau_{PQ} \mid \ P, Q \prec \Pi \text{ facets}, P \cap Q \neq \emptyset\}$ is, of course, encoded in the scalars $c_{QP}^{(x)}$ – one has $\tau_{PQ} = 1$ if and only if $c_{QP}^{(x)} = 1$ for all $x \in \mathrm{L}(P) \cap \mathrm{L}(Q)$. We will in the following identify $\mathrm{Alg}(\alpha^*(\mathcal{L}))$ with $k[\Pi, \mathcal{T}]$.)

The algebra $k[\Pi, \mathcal{T}]$ can be thought of as a 'twisted' polyhedral algebra built up of the the same polytopal 'facet' algebras as $k[\Pi]$, but the identifications along common faces are carried out according to the twisted structure $\mathcal{T}$. The residue class in $k[\Pi, \mathcal{T}]$ of a term of $k[S_{\hat{P}_i}]$, $i \in [1, r]$, will be called a *twisted term*. (There is no appropriate notion of a twisted monomial.)

Let $\mathcal{S}_{\mathcal{T}}$ denote the multiplicative semigroup consisting of the twisted terms and 0, and $\mathcal{S}$ the corresponding one formed by ordinary terms and 0 (the latter live in $k[\Pi]$). One observes easily that there is a natural isomorphism

$$\Psi : \mathcal{S}_{\mathcal{T}}/k^* \xrightarrow{\approx} \mathcal{S}/k^* \approx S_\Pi.$$

Now we define the action of $\mathbb{T}_k(\Pi)$ on $k[\Pi, \mathcal{T}]$ by first setting

$$\tau(z) = \frac{\tau(z')}{z'} z, \quad z \in \mathcal{S}_{\mathcal{T}},\ z' \in \mathcal{S},\ \Psi([z]) = [z'],$$

and then extending it to the whole algebra $k[\Pi, \mathcal{T}]$ by k-linearity. The crucial point is that this action is well-defined. Since Π is quasi-Euclidean, by Lemma 3.5(a) we get an (evidently algebraic) embedding of affine groups

$$\mathbb{T}_k(\Pi) \to \mathrm{gr.aut}_k(k[\Pi, \mathcal{T}]).$$

Let $\mathbb{T}_1$ denote the image of $\mathbb{T}_k(\Pi)$. But we have yet another embedding of the same torus into $\operatorname{gr.aut}_k(k[\Pi,\mathcal{T}])$, namely

$$\mathbb{T}_k(\Pi) \to \operatorname{gr.aut}_k(k[\Pi,\mathcal{T}]), \quad \tau \mapsto \alpha^* \circ \tau \circ (\alpha^*)^{-1}.$$

Let $\mathbb{T}_2$ be the image of the second embedding. By Lemma 3.5(b) we know that $\mathbb{T}_2$ is a maximal torus. Hence $\mathbb{T}_1$ is maximal as well, and the two tori are conjugate in $\operatorname{gr.aut}_k(k[\Pi,\mathcal{T}])$ ([Bo, Corollary 11.3(1)]. Suppose that $\beta^{-1} \circ \mathbb{T}_1 \circ \beta = \mathbb{T}_2$ for some $\beta \in \operatorname{gr.aut}_k(k[\Pi,\mathcal{T}])$. Then

$$(\beta \circ \alpha^*)^{-1}\mathbb{T}_1(\beta \circ \alpha^*) = \mathbb{T}_k(\Pi).$$

Using Lemma 3.5(a) and the very same arguments as in the proof of Theorem 5.2(b), one concludes that the isomorphism

$$\beta \circ \alpha^* : k[\Pi] \xrightarrow{\approx} k[\Pi,\mathcal{T}]$$

maps terms to terms. In particular, for each facet $P \prec \Pi$ we get two polytopal systems of global sections of the line bundle $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$, namely $\beta \circ \alpha^*(\mathrm{L}(P))$ and $\bar{f}_P$. Then there must exist an element $\tau_P \in \mathbb{T}_k(P)$ transforming $\bar{f}_P$ into $\beta \circ \alpha^*(\mathrm{L}(P))$. A straightforward verification shows that this is the desired family of toric automorphisms.

(b) Let Π and k be as in the theorem. We again start with the very ample line bundle $\mathcal{L}$ on $\mathcal{Z}_\Pi$ obtained by restriction of $\mathcal{O}(1)$ under the closed embedding $\mathcal{Z}_\Pi \to \mathbb{P}^N_k$, $N = \#\mathrm{L}(\Pi)-1$. Choose $\alpha \in \operatorname{Aut}_k(\mathcal{Z}_\Pi)$. Since Π is a projectively quasi-Euclidean complex and any representative of $\Pi(\alpha^*(\mathcal{L}))$ is projectively equivalent to Π we see that $\Pi(\alpha^*(\mathcal{L}))$ consists of quasi-Euclidean lattice polyhedral complexes. Now the same arguments as in the proof of (a) show that the very ample line bundle $\alpha^*(\mathcal{L})$ has a polyhedral system of global sections (though we may have $\Pi(\alpha^*(\mathcal{L})) \neq \Pi(\mathcal{L})$ now). Consider $\Pi_1 \in \Pi(\alpha^*(\mathcal{L}))$. Then Lemma 7.8 shows that there is an element $\delta_1 \in \mathbb{D}(\mathcal{Z}_\Pi)$ with

$$\delta_1^*(\mathcal{L}_1) = \alpha^*(\mathcal{L})$$

for the line bundle $\mathcal{L}_1$ on $\mathcal{Z}_\Pi$ obtained by the restriction of $\mathcal{O}(1)$ under the closed embedding

$$\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi_1} \to \mathbb{P}^{N_1}_k,\ N_1 = \#\mathrm{L}(\Pi_1) - 1.$$

(We identify $\mathcal{Z}_\Pi$ and $\mathcal{Z}_{\Pi_1}$ via Lemma 6.3.) By Lemma 7.3 we have $\Pi(\mathcal{L}_1) = \Pi(\alpha^*(\mathcal{L}))$. Now we carry out the same process for Π_1 and $\mathcal{L}_1$ as we did for Π and $\mathcal{L}$, and so on. We will get a sequence of projectively equivalent very ample lattice polyhedral complexes $\Pi_0 = \Pi$, Π_1, Π_2, ... such that the very ample line bundles $\mathcal{L}_i$ on $\mathcal{Z}_\Pi$, obtained by restrictions of the $\mathcal{O}(1)$ under the closed embeddings $\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi_i} \to \mathbb{P}^{N_i}_k$, $N_i = \#\mathrm{L}(\Pi_i) - 1$, satisfy the following conditions for $i \geq 0$:

(1) $\alpha^*(\mathcal{L}_i)$ has a polyhedral system of global sections,
(2) $\Pi_i \in \Pi(\mathcal{L}_i)$,

(3) $\Pi(\mathcal{L}_{i+1}) = \Pi(\alpha^*(\mathcal{L}_i))$,
(4) $\#\mathrm{L}(\Pi_i) = \#\mathrm{L}(\Pi),\ i \geq 0$,

Equation (4) holds because

$$\#\mathrm{L}(\Pi_{i+1}) = \dim_k H^0(\mathcal{Z}_\Pi, \mathcal{L}_{i+1}) \\ = \dim_k H^0(\mathcal{Z}_\Pi, \alpha^*(\mathcal{L}_i)) = \dim_k H^0(\mathcal{Z}_\Pi, \mathcal{L}_i) = \#\mathrm{L}(\Pi_i).$$

Easy inductive arguments guarantee that the number of the classes $[\Pi']$ such that Π' is projectively equivalent to Π and $\#\mathrm{L}(\Pi') = \#\mathrm{L}(\Pi)$ is *finite*. Therefore, by the conditions (2), (3), and (4) there exist natural numbers p and q such that

$$\Pi(\mathcal{L}_{i+qj}) = \Pi(\mathcal{L}_i)$$

for all $i \geq p$ and all $j \geq 0$. Consider the very ample line bundle

$$\hat{\mathcal{L}} = \mathcal{L}_p \otimes \cdots \otimes \mathcal{L}_{p+q-1},$$

which has a polyhedral system of global sections by Lemma 7.7(c). We fix a complex $\hat{\Pi} \in \Pi(\hat{\mathcal{L}})$.

By Lemma 7.7(c) and condition (1) above the line bundle

$$\alpha^*(\hat{\mathcal{L}}) = \alpha^*(\mathcal{L}_p) \otimes \cdots \otimes \alpha^*(\mathcal{L}_{p+q-1})$$

has a polyhedral system of global sections as well, and we have the equalities

$$\Pi(\alpha^*(\hat{\mathcal{L}})) = \Pi(\mathcal{L}_p) + \cdots + \Pi(\mathcal{L}_{p+q-1}) = \Pi(\hat{\mathcal{L}}).$$

So by Lemma 7.8 there exists $\delta \in \mathbb{D}(\mathcal{Z}_{\hat{\Pi}})$ such that $(\alpha \circ \delta)^*(\hat{\mathcal{L}}) = \hat{\mathcal{L}}$. In this situation the automorphism $\alpha \circ \delta \in \mathrm{Aut}_k(\mathcal{Z}_{\hat{\Pi}})$ is in the image of the canonical anti-homomorphism

$$\Gamma_k(\hat{\Pi}) = \mathrm{gr.\,aut}(\mathrm{Alg}(\hat{\mathcal{L}})) \xrightarrow{\mathrm{pr}} \mathrm{Aut}_k(\mathcal{Z}_{\hat{\Pi}}).$$

Since Π is a projectively quasi-Euclidean complex and $\hat{\Pi}$ is projectively equivalent to Π, we can apply Theorem 5.2(b) to obtain the equality

$$\alpha \circ \delta = \sigma \circ \delta' \circ \varepsilon$$

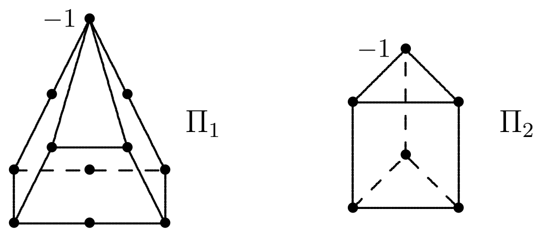
for some $\sigma \in \mathrm{pr}(\Sigma(\hat{\Pi})) \subset \Sigma(\hat{\Pi})_{\mathrm{Proj}} = \Sigma(\Pi)_{\mathrm{Proj}}$ (Lemma 6.6(a),(c)), $\delta' \in \mathrm{pr}(\mathbb{D}_k(\hat{\Pi})) \subset \mathbb{D}(\mathcal{Z}_\Pi)$, $\varepsilon \in \mathbb{E}(\mathcal{Z}_{\hat{\Pi}}) = \mathbb{E}(\mathcal{Z}_\Pi)$ (Lemma 6.5(a)). Now the Lemmas 6.5(b) and 6.6(b) complete the proof.

(c) It is clear that the arguments presented above apply to part (c) as well, once one has observed that a polytope projectively equivalent to a facet-separated polytope is itself facet separated. □

Example 9.3. In the previous proof a 'twisted' structure was derived from $\alpha^*(\mathcal{L})$, and then 'untwisted' to a polyhedral system of global sections of $\alpha^*(\mathcal{L})$. In general a twisted structure cannot be untwisted: (a) it may happen that a line bundle $\mathcal{L}$ on $\mathcal{Z}_\Pi$ does not have a polyhedral system of global sections; at least in the quasi-Euclidean case $\operatorname{Alg}(\mathcal{L})$ and $k[\Pi]$ are then not isomorphic as graded algebras, as can be shown by arguments similar to those in the proof of 5.2(b); (b) if we define a 'twisted polyhedral algebra' abstractly for a given polyhedral complex Π, it may even happen that $\operatorname{Proj}(k[\Pi])$ is not isomorphic to $\operatorname{Proj}(k[\Pi, \mathcal{T}])$:

(a) The polyhedral complex Π_1 in Figure 7 consists of 4 polygons: a 2×1 rectangle as the bottom and three trapezoids. (The two triangles are open.) Let $\mathcal{T}$ denote the

FIGURE 7



'abstract' twisted structure indicated in the figure. Then it is easy to see that the second Veronese algebras of $k[\Pi]$ and $k[\Pi, \mathcal{T}]$ coincide. Therefore they define the same projective varieties, namely $\mathcal{Z}_\Pi$. However, the line bundle coming from $k[\Pi, \mathcal{T}]$ does not have a polyhedral system of sections.

(b) The polyhedral complex Π_2 consists of 3 unit squares forming a triangular box without bottom and lid. In this case $k[\Pi]$ and $k[\Pi, \mathcal{T}]$ even define different projective varieties.

References

[Bo] A. Borel, *Linear algebraic groups*, Springer, 1991.
[BG] W. Bruns and J. Gubeladze *Polytopal linear groups*, preprint.
[BGT] W. Bruns, J. Gubeladze, and N. V. Trung, *Normal polytopes, triangulations, and Koszul algebras*, J. Reine Angew. Math. **485** (1997), 123–160.
[BH] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge University Press, 1993.
[Bu] D. Bühler, *Homogener Koordinatenring und Automorphismengruppe vollständiger torischer Varietäten*, Diplomarbeit, Basel, 1996.
[Co] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebr. Geom. **4** (1995), 17–50.
[Da] V. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.
[De] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 507-588.
[Fu] W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.
[Ha] R. Hartshorne, *Algebraic geometry*, Springer, 1977.
[Mu] G. Müller, *Automorphism group of a Stanley-Reisner ring*, preprint.
[Oda] T. Oda, *Convex Bodies and algebraic geometry (An introduction to the theory of toric varieties)*, Springer, 1988.

[Sta] R. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties, and related results*. Commutative algebra and combinatorics, Adv. Stud. Pure Math. 11 (1987), 187-213.

[Te] B. Teissier, *Variétés toriques et polytopes*, Séminaire Bourbaki n^0 565, Springer Lecture Notes **901**(1981), 71-84.

Universität Osnabrück, FB Mathematik/Informatik, 49069 Osnabrück, Germany
E-mail address: Winfried.Bruns@mathematik.uni-osnabrueck.de

A. Razmadze Mathematical Institute, Alexidze St. 1, 380093 Tbilisi, Georgia
E-mail address: gubel@imath.acnet.ge