RECTANGULAR SIMPLICIAL SEMIGROUPS

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In [3] Bruns, Gubeladze, and Trung define the notion of polytopal semigroup ring as follows. Let P be a lattice polytope in $\mathbb{R}^n$, i. e. a polytope whose vertices have integral coordinates, and K a field. Then one considers the embedding $\iota\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$, $\iota(x) = (x, 1)$, and chooses S_P to be the semigroup generated by the lattice points in $\iota(P)$; the K-algebra $K[S_P]$ is called a *polytopal semigroup ring*. Such a ring can be characterized as an affine semigroup ring that is generated by its degree 1 elements and coincides with its normalization in degree 1.

The object of this paper are the polytopal semigroup rings given by the simplices $\Delta(\lambda_1, \dots, \lambda_n) \subset \mathbb{R}^n$ with vertices

$$(0, 0, \dots, 0),\ (\lambda_1, 0, \dots, 0),\ (0, \lambda_2, \dots, 0), \dots,\ (0, 0, \dots, \lambda_n).$$

for positive integers $\lambda_1, \dots, \lambda_n$. Thus $\Delta = \Delta(\lambda_1, \dots, \lambda_n)$ is a rectangular n-dimensional simplex with a rectangular corner at the origin $0 \in \mathbb{R}^n$. In the following we denote $(\lambda_1, \dots, \lambda_n)$ by $\boldsymbol{\lambda}$. (See Lagarias and Ziegler [7] for the use of rectangular simplices in the study of certain extremal properties of lattice polytopes.)

We are mainly interested in the normality of $K[S_\Delta]$. For simplicity, let us say that $\boldsymbol{\lambda}$ is normal if $K[S_\Delta]$ is (this makes sense since normality is a property of the semigroup). While it seems impossible to give an easy criterion for the normality of $\boldsymbol{\lambda}$, we will find a surprising number of necessary and of sufficient conditions that allow one to produce plenty of normal and non-normal examples. The most surprising rule is that the normality of $\boldsymbol{\lambda}$ only depends on the residue class of each λ_i modulo the least common multiple of the λ_j with $j \neq i$ (see Theorem 1.6 for the precise statement). Furthermore the normality of $\boldsymbol{\lambda}$ is closely related to certain properties of the semigroup Λ of $\mathbb{Q}_+$ generated by $1/\lambda_1, \dots, 1/\lambda_n$. For example, if $\lambda_1, \dots, \lambda_n$ are pairwise prime, then the normality of $\boldsymbol{\lambda}$ is equivalent to the fact that each $x \in \Lambda$ with $x \leq c \in \mathbb{Z}_+$ can be written in the form $x = x_1 + \dots + x_c$ with $x_i \in \Lambda$, $x_i \leq 1$ (Proposition 1.3).

As a consequence of the results of [3] we also show the following (Proposition 2.4): let $g = \gcd(\lambda_1, \dots, \lambda_n)$; if $g \geq n - 1$, then $\Delta(\boldsymbol{\lambda})$ is normal, and if $g \geq n$, then $K[S_{\Delta(\ )}]$ is a Koszul ring.

By a theorem of Hochster [6], the normality of $K[S_\Delta]$ implies the Cohen–Macaulay property. However, among the non-normal $K[S_\Delta]$ there exist Cohen–Macaulay as well as non-Cohen–Macaulay rings, and we will exhibit examples for both cases (see Example 2.2 and Proposition 2.3).

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1. Normality

We begin with a standard reformulation of the normality condition for polytopes in terms of the multiples $cP = \{cx : x \in P\}$ for $c \in \mathbb{N}$.

Proposition 1.1. *Let $P \subset \mathbb{R}^n$ be a lattice polytope for which $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$. Then P is normal if and only if for any $c \in \mathbb{N}$ and any $x \in cP \cap \mathbb{Z}^n$ there exist elements $x_1, \dots, x_c \in P \cap \mathbb{Z}^n$ such that $x_1 + \dots + x_c = x$.*

Proof. Let C denote the cone in $\mathbb{R}^{n+1}$ with vertex at the origin $0 \in \mathbb{R}^{n+1}$ which is spanned by $\{(z, 1) : z \in P\} \subset \mathbb{R}^{n+1}$. One easily oberves that the two conditions 'S_P is normal' and $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$ together are equivalent to the single equality

$$S_P = C \cap \mathbb{Z}^{n+1}.$$

Now the proof is straightforward. □

Proposition 1.2. *The rectangular simplex $\Delta(\boldsymbol{\lambda})$ is normal if and only if for every $c \in \mathbb{N}$ and all $(b_1, \dots, b_n) \in \mathbb{Z}^n_+$, satisfying the condition*

$$b_1/\lambda_1 + \dots + b_n/\lambda_n \le c,$$

there exist $(a_{1j}, \dots, a_{nj}) \in \mathbb{Z}^n_+$, $j = 1, \dots, c$, such that

$$a_{1j}/\lambda_1 + \dots + a_{nj}/\lambda_n \le 1 \quad \textit{for all } j = 1, \dots, c$$

and

$$b_i = \sum_{j=1}^{c} a_{ij} \quad \textit{for all } i = 1, \dots, n.$$

Proof. Since for any $c \in \mathbb{N}$ and any $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ the inequality

$$x_1/\lambda_1 + \dots + x_n/\lambda_n \le c$$

is equivalent to the inclusion $x \in c\Delta(\boldsymbol{\lambda})$, the proposition will follow from the previous one as soon as the equality $\operatorname{gp}\big(S_{\Delta(\ )}\big) = \mathbb{Z}^{n+1}$ has been established. In turn, the latter equality is equivalent to the condition that $\Delta(\boldsymbol{\lambda}) \cap \mathbb{Z}^n$ generates $\mathbb{Z}^n$ as a group; finally, this condition holds because $\Delta(\boldsymbol{\lambda})$ contains the standard basis of $\mathbb{Z}^n$. □

One should note that it is enough to consider those $\mathbf{b} = (b_1, \dots, b_n)$ that satify the inequalities $b_i < \lambda_i$. If, say, $b_1 \ge \lambda_1$, then we write $\mathbf{b} = (\lambda_1, 0, \dots, 0) + (b_1', b_2, \dots, b_n)$ with $b_1' = b_1 - \lambda_1$, and $b_1/\lambda_1 + b_2/\lambda_2 + \dots + b_n/\lambda_n \le c - 1$. Furthermore, only the numbers $c < n$ are relevant. Namely, if $b_i < \lambda_i$, then $b_1/\lambda_1 + \dots + b_n/\lambda_n < n$, and if $c = n$, then we decompose $\mathbf{b}$ into the sum $\mathbf{b} = (b_1, 0 \dots, 0) + (0, b_2, \dots, b_n)$ and use the inequalities $b_1/\lambda_1 < 1$ and $b_2/\lambda_2 + \dots + b_n/\lambda_n < n - 1$. (In [3], 1.3.4 it has been shown that the degrees of the generators of the normalization of an arbitrary polytopal semigroup ring $K[S_P]$ are bounded by $\dim P - 1$.)

If $\boldsymbol{\lambda}$ is normal, then 1.2 in particular implies the following property of $\boldsymbol{\lambda}$ that we call 1-*normality* since it is a 'one-dimensional' condition: let Λ be the subsemigroup of $\mathbb{Q}_+$ generated by $1/\lambda_1, \dots, 1/\lambda_n$; if $x \le c$ ($c \in \mathbb{N}$) for some $x \in \Lambda$, then there exist $y_1, \dots, y_c \in \Lambda$ with $y_i \le 1$ such that $x = y_1 + \dots + y_c$. Somewhat surprisingly, for the most interesting $\boldsymbol{\lambda}$ normality follows from 1-normality.

Proposition 1.3. *Suppose that $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ has pairwise coprime entries λ_i. Then $\boldsymbol{\lambda}$ is normal if (and only if) it is 1-normal.*

Proof. We consider the natural surjection $\pi \colon \mathbb{Z}^n \to \mathrm{gp}(\Lambda)$ (notation as above) that maps the i-th basis element e_i to $1/\lambda_i$. The hypothesis on $\boldsymbol{\lambda}$ implies that $\operatorname{Ker}\pi$ is generated by the elements $\lambda_i e_i - \lambda_j e_j$; equivalently,

$$\operatorname{Ker}\pi = \{(a_1\lambda_1, \dots, a_n\lambda_n) \colon a_1 + \dots + a_n = 0\}.$$

With the notation of 1.2, we choose $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n_+$, satisfying the condition

$$x = b_1/\lambda_1 + \dots + b_n/\lambda_n,$$

As observed above, in proving the normality of $\boldsymbol{\lambda}$ we may restrict ourselves to the case in which $b_i < \lambda_i$ for all i. By hypothesis $x = y_1 + \dots + y_c$ with $y_i \in \Lambda$, $y_i \le 1$. There exist $(a_{1j}, \dots, a_{nj}) \in \mathbb{Z}^n_+$, $j = 1, \dots, c$, such that $y_j = a_{1j}/\lambda_1 + \dots + a_{nj}/\lambda_n$, and it only remains to verify the equation $b_i = \sum_{j=1}^c a_{ij}$ for all $i = 1, \dots, n$. Set $a_i = \sum_{j=1}^c a_{ij}$; then $(a_1, \dots, a_n) \in \mathbf{b} + \operatorname{Ker}\pi$. However, the description of $\operatorname{Ker}\pi$ above immediately yields that $\mathbf{b} + \operatorname{Ker}\pi$ contains exactly one vector with non-negative entries, namely $\mathbf{b}$ itself. □

The next proposition gives a useful necessary condition for 1-normality, and thus for normality.

Proposition 1.4. *Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, and set $L = \mathrm{lcm}(\lambda_1, \dots, \lambda_n)$, $L_i = L/\lambda_i$, and $d = \gcd(L_1, \dots, L_n)$. If $\boldsymbol{\lambda}$ is 1-normal, then $1 - d/L$ is an element of the semigroup Λ generated by $1/\lambda_1, \dots, 1/\lambda_n$; equivalently, $L - d$ is an element of the semigroup generated by $L_1, \dots, L_n$.*

Proof. The essential point is that d is the smallest positive element of $\mathrm{gp}(L_1, \dots, L_n)$. Division by L yields that d/L is the smallest positive element of $\mathrm{gp}(\Lambda)$. Clearly there exists an integer N such that $N - d/L \in \Lambda$. First, the 1-normality of $\boldsymbol{\lambda}$ implies that $N - d/L = y_1 + \dots + y_N$ with $y_i \in \Lambda$, $y_i \le 1$. Second, one observes that every element $x \in \Lambda$ with $x < 1$ satisfies the inequality $x \le 1 - d/L$. Therefore $N - 1$ of the y_i must be equal to 1, and exactly one equals $1 - d/L$. □

The previous proposition immediately yields that $(2, 3, 5)$ is not (1-)normal: in fact, 29 is not in the semigroup generated by $6, 10, 15$. For use in 2.1 below let us say that $\boldsymbol{\lambda}$ is *almost 1-normal* if it satisfies the condition established by 1.4. (We will see below that almost 1-normality is strictly weaker than 1-normality.)

Remark 1.5. The condition of being almost 1-normal is of geometric significance. For simplicity, suppose that $\lambda_1, \dots, \lambda_n$ are pairwise prime. Then the only lattice points of the facet F of $\Delta(\boldsymbol{\lambda})$ opposite to 0 are its vertices. It is easy to show that $\boldsymbol{\lambda}$ is almost 1-normal if and only if $\Delta(\boldsymbol{\lambda})$ contains a unimodular lattice simplex σ of which F is a facet. (A lattice simplex is unimodular if it has normalized volume 1.) Then the vertex of σ opposite to F is $(b_1, \dots, b_n, 1)$ with $L - 1 = \sum b_i L_i$ (notation as in the proof of 1.4).

The most remarkable property of normality is given by the following theorem; it roughly says that the normality of $\boldsymbol{\lambda}$ depends only on the residue class of λ_i modulo the least common multiple of the λ_j with $i \neq j$

Theorem 1.6. *Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and set $\ell = \operatorname{lcm}(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n)$. Then $\boldsymbol{\lambda}$ is normal if and only if $\boldsymbol{\lambda}' = (\lambda_1, \dots, \lambda_i + \ell, \dots, \lambda_n)$ is normal.*

Proof. Since normality is stable under any permutation of $\lambda_1, \dots, \lambda_n$, we may assume $i = n$. We set $L_i = \ell\lambda_n/\lambda_i$ (so that $L_n = \ell$) and $L'_i = \ell(\lambda_n+\ell)/\lambda_i$ for $i = 1, \dots, n-1$, $L'_n = \ell$. Consider the linear forms $\varphi, \varphi' \colon \mathbb{Z}^{n+1} \to \mathbb{Z}$ given by

$$\varphi(z_1, \dots, z_{n+1}) = \ell\lambda_n z_{n+1} - (L_1 z_1 + \dots + L_n z_n),$$
$$\varphi'(z_1, \dots, z_{n+1}) = \ell(\lambda_n + \ell) z_{n+1} - (L'_1 z_1 + \dots + L'_n z_n).$$

Then $z \in \mathbb{Z}_+^{n+1}$ belongs to the normalization $\overline{S}$ of $S = S_{\Delta(\ )}$ if and only if $\varphi(z) \geq 0$, and the analogous statement holds for the normalization $\overline{S}'$ of $S' = S_{\Delta(\ ')}$ with respect to φ'. (Note that z_{n+1} is the degree of z with respect to the natural grading of $\operatorname{gp}(S) = \operatorname{gp}(S')$.)

It is crucial in the following that we use the following observations:

$$\varphi(z) = \varphi'(\alpha(z)) \tag{i}$$

where

$$\alpha(z) = \Bigl(z_1, \dots, z_{n-1}, z_n + z_{n+1}\ell - (l_1 z_1 + \dots + l_{n-1} z_{n-1}), z_{n+1}\Bigr), \text{ with } l_i = \ell/\lambda_i$$

(note that α is an automorphism of $\mathbb{Z}^{n+1}$).

(ii) Suppose that $z \in \overline{S}$ with $\varphi(z) < \ell$; then $z' = \alpha(z) \in \overline{S}'$. Conversely if $z' \in \overline{S}'$ with $\varphi'(z') < \ell$, then $z = \alpha^{-1}(z') \in \overline{S}$.

The identity (i) is easily verified. In proving (ii) we have to pay attention only to the n-th entries of z' and z respectively, because all the other entries of z and z' coincide and, furthermore $\varphi(z) = \varphi'(z')$ by (i). Starting from z, let us assume $z'_n < 0$. Set $d = \varphi(z)$. Then

$$\ell(\lambda_n + \ell) z_{n+1} - (L'_1 z_1 + \dots + L'_{n-1} z_{n-1}) = d + z'_n \ell < 0$$

Multiplication by $\lambda_n/(\lambda_n + \ell)$ yields

$$\ell\lambda_n z_{n+1} - (L_1 z_1 + \dots + L_{n-1} z_{n-1}) < 0,$$

and, *a fortiori*, $\varphi(z) < 0$, which is a contradiction. Now we start from $z' \in \overline{S}'$. Assume that $z_n < 0$. Then

$$\ell\lambda_n z_{n+1} - (L_1 z_1 + \dots + L_{n-1} z_{n-1}) = d + z_n \ell < 0.$$

and we use the same argument as before, but in the other direction.

After these preparations we turn to the proof of the theorem itself. Assume first that $\boldsymbol{\lambda}$ is not normal, and choose $z \in \overline{S} \setminus S$ such that its degree z_{n+1} is minimal. Next we minimize $\varphi(z)$; then $d = \varphi(z) < \ell$, because otherwise we could replace z_n by $z_n + 1$. We claim that $z' = \alpha(z) \in \overline{S}' \setminus S'$.

Observation (ii) above implies $z' \in \overline{S}'$. Next we show $z' \notin S'$. Assume that $z' \in S'$. Then there exists $w' \in S'$ of degree $w'_{n+1} = 1$ for which $z' - w' \in S'$. We have $0 \leq e = \varphi'(z' - w') = d - \varphi'(w') \leq d < \ell$. Now set $w = \alpha^{-1}(w')$. By the choice of z_{n+1} we obtain a contradiction if we can verify that $w \in S$ and $z - w \in \overline{S}$. That $z - w \in \overline{S}$ follows immediately from (ii). Furthermore we have $\varphi'(w') = d - e < \ell$ so that also $w \in \overline{S}$ (again by (ii)). However, S and $\overline{S}$ coincide in degree 1.

It remains to show that the non-normality of $\boldsymbol{\lambda}'$ implies that of $\boldsymbol{\lambda}$. Since the observations (i) and (ii) are symmetric with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$, the same arguments can be used. $\square$

2. Some more rules

In the following proposition we have listed some more results about the normality of rectangular simplices. In conjunction with 1.6 (and 2.4 below) it yields plenty of examples of normal and non-normal rectangular simplices. If $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ is normal, then every subsequence of $\boldsymbol{\lambda}$ is also normal. Therefore, and since $\boldsymbol{\lambda}$ is always normal for $n = 2$, negative statements about normality are most interesting for $n = 3$, and these imply negative statements for larger n. This is the reason why we restrict ourselves to the case $n = 3$ in several parts of the following proposition.

Proposition 2.1. *Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$.*

(a) *If $n \leq 2$, then $\boldsymbol{\lambda}$ is normal.*
(b) *If $(\lambda_1, \dots, \lambda_{n-1})$ is normal, then $(\lambda_1, \dots, \lambda_{n-1}, 1+m\ell)$, $\ell = \operatorname{lcm}(\lambda_1, \dots, \lambda_{n-1})$, is normal for all $m \geq 0$.*
(c) *If $\lambda_1 \mid \lambda_2$, then $(\lambda_1, \lambda_2, \lambda_3)$ is normal.*
(d) *Suppose that $\lambda_1, \lambda_2, \lambda_3$ satify the following conditions:* (i) $\lambda_1, \lambda_2 \geq 2$, (ii) $\lambda_1, \lambda_2, \lambda_3$ *are pairwise prime, and* (iii) $0 < \lambda_3 < \lambda_1\lambda_2$. *Then exactly one of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and $\boldsymbol{\lambda}' = (\lambda_1, \lambda_2, \lambda_1\lambda_2 - \lambda_3)$ is almost 1-normal. In particular, at most one of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ is (1-)normal.*
(e) *Suppose that $\operatorname{lcm}(\lambda_1, \lambda_2) > \lambda_1, \lambda_2$. Then $(\lambda_1, \lambda_2, \operatorname{lcm}(\lambda_1, \lambda_2) - 1)$ is not almost 1-normal.*
(f) *A triple $(2, \lambda_2, \lambda_3)$ is normal if and only if it is almost 1-normal.*

Proof. (a) This follows from the normality of all polytopes of dimension at most 2. (See [3], 1.2.4.)

(b) In view of 1.6 it is enough to show that $(\lambda_1, \dots, \lambda_{n-1}, 1)$ is normal. Let $P = \Delta(\lambda_1, \dots, \lambda_{n-1})$ and $P' = \Delta(\lambda_1, \dots, \lambda_{n-1}, 1)$. Then $S_{P'} = S_P \oplus \mathbb{Z}_+$, whence $S_{P'}$ is normal if S_P is.

(c) Suppose that $1 < b_1/\lambda + b_2/(\kappa\lambda) + b_3/\mu \leq 2$. We may assume $b_1 < \lambda$, $b_2 < \lambda\kappa$, $b_3 < \mu$. If $b_1/\lambda + b_2/(\kappa\lambda) < 1$, we write $(b_1, b_2, b_3) = (b_1, b_2, 0) + (0, 0, b_3)$, and both summands are in $(S_\Delta)_1$. Otherwise $b_1/\lambda + b_2'/(\lambda\kappa) = 1$ for some $b_2' \leq b_2$. Then $(b_1, b_2, b_3) = (b_1, b_2', 0) + (0, b_2 - b_2', b_3)$.

(d) Set $L_1 = \lambda_2\lambda_3$, $L_2 = \lambda_1\lambda_3$, and $L_3 = \lambda_1\lambda_2$, and define the numbers L_i' similarly for the entries of $\boldsymbol{\lambda}'$. Then $\gcd(L_1, L_2, L_3) = 1$ and $\gcd(L_1', L_2', L_3') = 1$ so that the the equations

$$\lambda_1\lambda_2\lambda_3 - 1 = a_3\lambda_1\lambda_2 + a_2\lambda_1\lambda_3 + a_1\lambda_2\lambda_3, \tag{$*$}$$
$$\lambda_1\lambda_2(\lambda_1\lambda_2 - \lambda_3) - 1 = b_3\lambda_1\lambda_2 + b_2\lambda_1(\lambda_1\lambda_2 - \lambda_3) + b_1\lambda_2(\lambda_1\lambda_2 - \lambda_3). \tag{$\dagger$}$$

have solutions $a_i, b_i \in \mathbb{Z}$. Observe that a_1, a_2, b_1 and b_2 must be non-zero (because of condition (i)). Since we have the relations $\lambda_1 L_1 = \lambda_2 L_2 = \lambda_3 L_3$ and $\lambda_1 L_1' = \lambda_2 L_2' = \lambda_3' L_3'$ (with $\lambda_3' = \lambda_1\lambda_2 - \lambda_3$)), we can assume that $0 < a_1, b_1 < \lambda_1$, $0 < a_2, b_2 < \lambda_2$.

The claim in (d) amounts to the fact that exactly one of the numbers a_3 and b_3 is non-negative.

The equations above imply $-1 \equiv a_1\lambda_2\lambda_3 \bmod \lambda_1$ and $-1 \equiv -b_1\lambda_2\lambda_3 \bmod \lambda_1$. Together with the inequalities $0 < a_1, b_1 < \lambda_1$ this yields $b_1 = \lambda_1 - a_1$, and analogously one sees that $b_2 = \lambda_2 - a_2$. It follows that

$$a_1\lambda_2 + a_2\lambda_1 + (a_3 - b_3) = \lambda_1\lambda_2.$$

Multiplication by λ_3 and comparison with $(*)$ gives us

$$a_3(\lambda_1\lambda_2 - \lambda_3) + b_3\lambda_3 = -1.$$

Both $\lambda_1\lambda_2 - \lambda_3$ and λ_3 are positive. It follows that exactly one of a_3 and b_3 is non-negative.

(e) We write $\lambda_1 = \gamma\mu_1$, $\lambda_2 = \gamma\mu_2$ with coprime μ_1, μ_2. Assume that $(\lambda_1, \lambda_2, \operatorname{lcm}(\lambda_1, \lambda_2) - 1)$ is almost 1-normal. With the notation of 1.2, one has

$$L = \gamma\mu_1\mu_2(\gamma\mu_1\mu_2 - 1), \quad L_1 = \mu_2(\gamma\mu_1\mu_2 - 1), \quad L_2 = \mu_1(\gamma\mu_1\mu_2 - 1),$$

$L_3 = \gamma\mu_1\mu_2$, and, finally, $d = 1$. The equation

$$\gamma\mu_1\mu_2(\gamma\mu_1\mu_2 - 1) - 1 = a_1\mu_2(\gamma\mu_1\mu_2 - 1) + a_2\mu_1(\gamma\mu_1\mu_2 - 1) + a_3\gamma\mu_1\mu_2$$

implies $a_1\mu_2 + a_2\mu_1 \equiv 1 \bmod \gamma\mu_1\mu_2$. Note that $1 \le a_1 < \gamma\mu_1$ and $1 \le a_2 < \gamma\mu_2$ where, for the lower inequality, we use the hypothesis that $\mu_1, \mu_2 > 1$. Thus $a_1\mu_2 + a_2\mu_1 = \gamma\mu_1\mu_2 + 1$, and substitution into the equation above leads us to the contradiction $a_3 = -1$.

(f) If one of λ_2 or λ_3 is even, then $(2, \lambda_2, \lambda_3)$ is normal by virtue of (c). Thus assume that both λ_2 and λ_3 are odd, and that $1 < b_1/2 + b_2/\lambda_2 + b_3/\lambda_3 \le 2$ with $b_1 < 2$, $b_2 < \lambda_2$, $b_3 < \lambda_3$. Note that if $b_1 = 0$, $b_2/\lambda_2 \le 1/2$, or $b_3\lambda_3 \le 1/2$, then $\mathbf{b}$ decomposes into a sum of 2 elements of Δ. Next, since $b_2\lambda_2 + b_3\lambda_3 = 3/2$ is impossible, we have $b_1/2 + b_2/\lambda_2 + b_3/\lambda_3 < 2$. If $(2, \lambda_2, \lambda_3)$ is almost 1-normal, then $1 - 1/L = a_1/2 + a_2/\lambda_2 + a_3/\lambda_3$ $(L = \operatorname{lcm}(2, \lambda_2, \lambda_3))$. Since $a_1 \ne 0$, we have $a_2/\lambda_2 < 1/2$ and $b_3/\lambda_3 < 1/2$. It follows that $a_1 \le 1 = eb_1$, $a_2 \le b_2$, and $a_3 \le b_3$. Hence $\mathbf{b} - \mathbf{a} \in \Delta$ since $1 - dL$ is the biggest linear combination of $2, \lambda_2, \lambda_3$ that is < 1. Thus we obtain the desired decomposition $\mathbf{b} = (\mathbf{b} - \mathbf{a}) + \mathbf{a}$. $\square$

Examples 2.2. (a) As mentioned already, $\boldsymbol{\lambda} = (2, 3, 5)$ is not normal. (Without any calculation this follows from 2.1(e).) However, the semigroup ring $R = K[S_{\Delta(\ )}]$ is Cohen–Macaulay, and even Gorenstein, as can be easily checked using MACAULAY [1]. In fact, note that $\Delta(\boldsymbol{\lambda})$ is a simplicial semigroup for all $\boldsymbol{\lambda}$, and that the monomials corresponding to the vertices of $\Delta(\boldsymbol{\lambda})$ form a (multi-) homogeneous system of parameters $\mathbf{v}$. Thus R is Cohen–Macaulay if and only if $\mathbf{v}$ is an R-sequence, and it is Gorenstein if and only if $(\mathfrak{m} : (\mathbf{v}))/(\mathbf{v})$ is a cyclic R-module, where $\mathfrak{m}$ denotes the irrelevant maximal ideal of R. (See Goto, Suzuki, and Watanabe [5] for a discussion of simplicial semigroups.)

This example also shows that in general a ring $R = K[S_\Delta]$ is not normal on $\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Indeed, the previous example has dimension 4 and is Cohen–Macaulay, but not normal; therefore it must violate Serre's condition (R_1).

(b) It follows from (a) and 1.6 that $(3,11,2)$ is not normal. One can easily check that $\boldsymbol{\lambda} = (3,11,31)$ is also not normal, and therefore not 1-normal by 1.3 ($x = (2,10,13,2) \notin S = S_{\Delta(\ )}$, but x is an element of the normalization $\overline{S}$). Furthermore $(3,11,2)$ is not even almost 1-normal. In conjunction with 2.1(d) this implies that $(3,11,31)$ is almost 1-normal. This example shows that almost 1-normality is not sufficient for (1-)normality.

(c) A 1-normal, but not normal example is given by $\boldsymbol{\lambda} = (3,10,28)$. (The generator of the normalization is $(2,9,12,2)$.) This example shows that 1.3 can not be extended to arbitrary $\boldsymbol{\lambda}$.

Examples such as in (b) and (c) yield non-Cohen–Macaulay polytopal semigroup rings.

Proposition 2.3. *Suppose that $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3)$ is almost 1-normal, but not normal. Then $K[S_{\Delta(\ )}]$ is not Cohen–Macaulay.*

Proof. Set $\Delta = \Delta(\boldsymbol{\lambda})$. Let the linear form $\varphi\colon \mathbb{R}^4 \to \mathbb{R}$ be given by

$$\varphi(x_1,x_2,x_3,x_4) = Lx_4 - (L_1x_1 + L_2x_2 + L_3x_3)$$

where $L = \operatorname{lcm}(\lambda_1,\lambda_2,\lambda_3)$, and $L_i = L/\lambda_i$. Then $x \in \mathbb{Z}^4$ belongs to the normalization $\overline{S}_\Delta$ of S_Δ if and only if $x_i \geq 0$ for all i and $\varphi(x) \geq 0$.

Let $H = \{x\colon \varphi(x) = 0\}$, and suppose that $x \in \overline{S}_\Delta \cap H$. The vertices $v^{(1)} = (\lambda_1,0,0,1),\dots,v^{(3)} = (0,0,\lambda_3,1)$ span a triangle δ in the 2-dimensional hyperplane $H \cap \mathbb{R}^3$ (where, as usually, $\mathbb{R}^3$ is considered as a hyperplane in $\mathbb{R}^4$). Simce δ is a lattice polygon with respect to the 2-dimensional lattice $\mathbb{Z}^3 \cap H$, we have $\operatorname{gp}(S_\delta) = \mathbb{Z}^4 \cap H$ and that δ is normal. This implies $x \in S_\delta \subset S_\Delta$ for all $x \in \overline{S}_\Delta$ with $\varphi(x) = 0$.

We choose an element $x \in \overline{S}_\Delta \setminus S_\Delta$ such that first its degree x_4 is minimal and second $\varphi(x)$ is minimal. As just seen, $\varphi(x) > 0$. Furthermore $x_i < \lambda_i$ for $i = 1,2,3$, and $x_4 = 2$. Next it follows that $\varphi(x) < L_i$ for $i = 1,2,3$, because otherwise we could increase x_i by 1.

Set $d = \gcd(L_1,L_2,L_3)$. Then $\varphi(x) \geq d$, since d is the smallest positive element in the group generated by L_1,L_2,L_3 (and L). By hypothesis there exists $y \in S_\Delta$ with $y_4 = 1$ and $\varphi(y) = d$; in fact, this is the definition of almost 1-normality. If $x - y \in \mathbb{Z}_+^4$, then $x - y \in \overline{S}_\Delta$, and, by assumption on x it would follow that $x \in S_\Delta$. Therefore at least one of the differences $x_i - y_i$, $i = 1,2,3$, must be negative.

We claim that this holds for exactly one i. Suppose $x_i - y_i < 0$ for at least two indices i, say $i = 1,2$. Then

$$\begin{aligned}
\varphi(x) - \varphi(y) &= L - (x_1 - y_1)L_1 - (x_2 - y_2)L_2 - (x_3 - y_3)L_3 \\
&\geq L + L_1 + L_2 - (x_3 - y_3)L_3 \\
&> L_1 + L_2
\end{aligned}$$

and this is a contradiction. So we may assume that $x_1 - y_1 < 0$ and $x_i - y_i \geq 0$ for $i = 2,3$.

Set $z^{(0)} = x + v^{(0)}$ where $v^{(0)} = (0,0,0,1)$. Then obviously $z^{(0)} \in S_\Delta$. Next set $z^{(1)} = x + v^{(1)}$. Then $w = z^{(1)} - y \in \mathbb{Z}_+^4$, w has the same degree as x, namely 2, and

$\varphi(w) = \varphi(x) - \varphi(y) < \varphi(x)$. By assumption on x we have $w \in S_\Delta$, and therefore $z^{(1)} \in S_\Delta$.

The equation $z^{(0)}+v^{(1)} = z^{(1)}+v^{(0)}$ in conjunction with the fact that $z^{(0)} \notin v^{(0)}+S_\Delta$ implies that the monomials corresponding to $v^{(0)}$ and $v^{(1)}$ do not form a regular sequence in $K[S_\Delta]$. This excludes the Cohen–Macaulay property for $K[S_\Delta]$; cf. the discussion in 2.2(a). $\square$

One should note that every non-Cohen–Macaulay example R among our rings also yields an example R' such that R' is not even Cohen–Macaulay on $\operatorname{Spec}(R') \setminus \{\mathfrak{m}\}$ where $\mathfrak{m}$ is the irrelevant maximal ideal of R'; one simply passes from $\boldsymbol{\lambda}$ to $(\boldsymbol{\lambda}, 1)$, or, in other words, from R to the polynomial ring $R' = R[T]$.

We have singled out one more rule concerning normality since it uses a different argument and since the polytopes of 2.4(b) are Koszul. A graded K-algebra is called *Koszul* if K has a linear free resolution as an R-module. (The resolution is linear if all the entries of its matrices are forms of degree 1; see Backelin and Fröberg [2] for a discussion of the basic properties of Koszul algebras.)

Proposition 2.4. (a) *If $\boldsymbol{\lambda}$ is normal, then every multiple $\mu\boldsymbol{\lambda}$, $\mu \in \mathbb{N}$ is normal.*

(b) *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be natural numbers such that λ_i divides λ_{i+1} for $i = 1, \dots, n-2$ and λ_{n-2} divides λ_n. Then $\Delta(\lambda_1, \dots, \lambda_n)$ is normal and Koszul.*

(c) *Let $g = \gcd(\lambda_1, \dots, \lambda_n)$. If $g \geq n-1$, then $\Delta(\boldsymbol{\lambda})$ is normal, and if $g \geq n$, then it is also Koszul.*

Proof. By definition the semigroup $S_{\Delta(\mu\ )} \subset \mathbb{Z}^{n+1}$ is generated by the elements

$$\{(x, 1)\colon x \in \Delta(\mu\boldsymbol{\lambda}) \cap \mathbb{Z}^n\}.$$

It is clear that $S_{\Delta(\mu\ )}$ is isomorphic to the subsemigroup $H \subset \mathbb{Z}^{n+1}$ generated by

$$\{(y, \mu)\colon y \in \mu\Delta(\boldsymbol{\lambda}) \cap \mathbb{Z}^n\}.$$

Now, using the normality of $\boldsymbol{\lambda}$ and Proposition 1.1 one easily observes that H is the subsemigroup of $S_{\Delta(\ )}$ consisting of those elements whose last (i. e. $(n+1)$-th) coordinate is a multiple of μ. In other words, for any field K the subalgebra $K[H] \subset K[S_{\Delta(\ )}]$ is the μ-th Veronese subalgebra (with respect to the natural grading of $K[S_{\Delta(\ )}]$. Now the desired normality follows immediately. (One can also argue directly in terms of semigroups.)

The assertion about normality in (b) follows from (a) by induction on n if one observes that the case $n = 2$ is guaranteed by 2.1(a) and that the normality of $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ implies that of $(1, \lambda_1, \dots, \lambda_n)$; see 2.1(b).

The Koszul property follows by the same induction: a polynomial ring over a Koszul algebra R is obviously Koszul, and the Veronese subalgebras of a Koszul algebra are also Koszul [2]; for $n = 2$ we use that all rectangular triangles yield Koszul polygons, as follows from [3], 3.2.5.

Part (c) follows immediately from [3], Theorem 1.3.3 which says that $K[S_{cP}]$ is normal for $c \geq \dim P - 1$ and Koszul for $c \geq \dim P$ (P is an arbitrary lattice polytope with $\operatorname{gp}(S) = \mathbb{Z}^{n+1}$). Now observe that $\Delta(\boldsymbol{\lambda}) = g\Delta(\boldsymbol{\lambda}')$ where $\boldsymbol{\lambda}'$ is obtained from $\boldsymbol{\lambda}$ by dividing each of the entries by g. $\square$

The rectangular simplices Δ are in general non-Koszul; for example, it is not hard to see that $\boldsymbol{\lambda} = (3, 4, 5)$ does not yield a Koszul algebra: the defining ideal of $K[S_\Delta]$ needs a generator of degree 3. In fact, the points $x_1 = (3, 0, 0)$, $x_2 = (0, 0, 5)$, $x_3 = (0, 3, 1)$, and $y = (1, 1, 2)$ belong to Δ, and $x_1 + x_2 + x_3 = 3y$. However, an equation $v + w = 2y$ with lattice points $v, w \in \Delta$ is impossible.

3. Class groups

We conclude this section with the calculation of the divisor class groups for normal rectangular simplices.

Theorem 3.1. *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ and set $L_0 = \prod_{i=1}^{n} \lambda_i$ and $L_i = \frac{\lambda_i - 1}{\lambda_i} L_0$ for $i = 1, \ldots, n$. Denote by $\overline{S}_\Delta$ the normalization of S_Δ. Then for any field K the group $\operatorname{Cl}(K[\overline{S}_\Delta])$ is cyclic of the finite order*

$$L_0 / \gcd(L_0, L_1, \ldots, L_n).$$

That the divisor class group of the normalization of a simplicial affine semigroup ring is cyclic of finite order, is well known and not hard to prove. So the main point is finding the order, for which we shall use the following special case of Chouinard's general calculation of divisor class groups of Krull semigroup rings. (In the following an extension $S \subset S'$ of semigroups is called integral if for each $s' \in S'$ there exists $c \in \mathbb{N}$ with $cs' \in S$; this is the case if and only if the extension $K[S] \subset K[S']$ is integral.)

Proposition 3.2. *Let S be a finitely generated normal subsemigroup of $\mathbb{Z}_+^n$ for which the extension $S \subset \mathbb{Z}_+^n$ is integral. Assume that $S = \operatorname{gp}(S) \cap \mathbb{Z}_+^n$ (the intersection being considered in $\mathbb{Z}^n$) and that for each $i = 1, \ldots, n$ the set*

$$T_i = \{e_j + \operatorname{gp}(S) \colon j = 1, \ldots, n, j \neq i\}$$

generates $\mathbb{Z}^n / \operatorname{gp}(S)$ as a semigroup. Then one has $\operatorname{Cl}(K[S]) \cong \mathbb{Z}^n / \operatorname{gp}(S)$ for any field K.

This proposition follows from Gilmer [4], 16.9, 16.10.

Proof of Theorem 3.1. Consider the elements

$$\vartheta_0 = (0, 0, \ldots, 0, 0, 1/L_0),\ \vartheta_1 = (1, 0, \ldots, 0, 0, 1/\lambda_1), \ldots, \vartheta_n = (0, \ldots, 0, 1, 1/\lambda_n)$$

of $\mathbb{Q}_+^{n+1}$. Let F_ϑ denote the free subsemigroup of $\mathbb{Q}_+^{n+1}$ generated by $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$. Observe that $S_\Delta \subset F_\vartheta$. Indeed, S_Δ is generated by

$$\{(a_1, \ldots, a_n, 1) \colon a_1, \ldots, a_n \in \mathbb{Z}_+ \quad \text{and} \quad a_1/\lambda_1 + \cdots + a_n/\lambda_n \leq 1\}.$$

It suffices to show $(a_1, \ldots, a_n, 1) \in F_\vartheta$ whenever $a_1, \ldots, a_n \in \mathbb{Z}_+$ and $a_1/\lambda_1 + \cdots + a_n/\lambda_n \leq 1$. But the latter is evidently equivalent to the existence of $a_0 \in \mathbb{Z}_+$ for which

$$a_0/L_0 + a_1/\lambda_1 + \cdots + a_n/\lambda_n = 1;$$

then $(a_1, \ldots, a_n, 1) = a_0\vartheta_0 + a_1\vartheta_1 + \cdots + a_n\vartheta_n \in F_\vartheta$.

Since $L_0\vartheta_0, \lambda_1\vartheta_1, \ldots, \lambda_n\vartheta_n$ are the vertices of Δ we also have that the semigroup extension $S_\Delta \subset F_\vartheta$ is integral. Since F_ϑ is a free semigroup, it is normal. Therefore

$\mathrm{gp}(S_\Delta) \cap F_\vartheta$ (the intersection being formed in $\mathrm{gp}(F_\vartheta)$) is also normal. Since F_ϑ is furthermore integral over S_Δ, we have $\overline{S}_\Delta = \mathrm{gp}(S_\Delta) \cap F_\vartheta$, and especially $\overline{S}_\Delta \subset F_\vartheta$.

The group of differences $\mathrm{gp}(\overline{S}_\Delta) = \mathrm{gp}(S_\Delta)$ is obviously generated by

$$L_0\vartheta_0 = (0,0,\dots,0,1),\ \vartheta_1 + L_1\vartheta_0 = (1,0,\dots,0,1),\ \dots\ ,\vartheta_n + L_n\vartheta_0 = (0,0,\dots,1,1).$$

Now, set $d = \gcd(L_0, L_1, \dots, L_n)$ and consider the system of elements of $\mathbb{Q}_+^{n+1}$ given by

$$\varepsilon_0 = d\vartheta_0, \quad \varepsilon_i = \vartheta_i \quad \text{for } i = 1,\dots,n,$$

Then $S_\Delta \subset F_\varepsilon$, where F_ε denotes the free subsemigroup of $\mathbb{Q}_+^{n+1}$ generated by $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$. Indeed, we have

$$\mathrm{gp}(\overline{S}_\Delta) \subset \mathrm{gp}(F_\varepsilon)$$

and thus, using the inclusion $F_\varepsilon \subset F_\vartheta$,

$$\overline{S}_\Delta = \mathrm{gp}(S_\Delta) \cap F_\vartheta \subset \mathrm{gp}(F_\varepsilon) \cap F_\vartheta = F_\varepsilon.$$

In particular, the integrality of the extension $\overline{S}_\Delta \subset F_\vartheta$ implies that of the extension $S_\Delta \subset F_\varepsilon$. Hence we arrive at the equality

$$\overline{S}_\Delta = \mathrm{gp}(S_\Delta) \cap F_\varepsilon.$$

We also have that $\mathrm{gp}(\overline{S}_\Delta)$ is generated by the elements

$$L_0'\varepsilon_0,\ \varepsilon_1 + L_1'\varepsilon_0, \dots, \varepsilon_n' + L_n'\varepsilon_0,$$

where $L_i' = L_i/d$, $i = 1,\dots,n$. We will identify the quotient group $\mathrm{gp}(F_\varepsilon)/\mathrm{gp}(S_\Delta)$ with $\mathbb{Z}/L_0'\mathbb{Z}$ via the isomorphism ψ determined by the assignment

$$\psi\colon [\varepsilon_0] \mapsto [1], \quad \psi\colon [\varepsilon_i] \mapsto [-L_i'],\ i = 1,\dots,n.$$

We claim that any choice of n representatives from $\{[1], [-L_1'], \dots, [-L_n']\}$ generates $\mathbb{Z}/L_0'\mathbb{Z}$ as a semigroup. If the choice contains $[1]$ then this is obvious. Otherwise we have to show that $\{[-L_1'], \dots, [-L_n']\}$ generates $\mathbb{Z}/L_0'\mathbb{Z}$ as a semigroup. But this is an easy consequence of the condition $\gcd(L_0', L_1', \dots, L_n') = 1$. Therefore, by Proposition 3.2, we have that $\mathrm{Cl}(K[\overline{S}_\Delta]) = \mathbb{Z}/L_0'\mathbb{Z}$ for every field K. □

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