Straightening Laws on Modules and Their Symmetric Algebras

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Several modules M over algebras with straightening law A have a structure which is similar to the structure of A itself: M has a system of generators endowed with a natural partial order, a standard basis over the ring B of coefficients, and the multiplication $A \times M \to A$ satisfies a "straightening law". We call them *modules with straightening law*, briefly MSLs.

In section 1 we recall the notion of an algebra with straightening law together with those examples which will be important in the sequel. Section 2 contains the basic results on MSLs, whereas section 3 is devoted to examples: (i) powers of certain ideals and residue class rings with respect to them, (ii) "generic" modules defined by generic, alternating or symmetric matrices of indeterminates, (iii) certain modules related to differentials and derivations of determinantal rings. The essential homological invariant of a module is its depth. We discuss how to compute the depth of an MSL in section 4. The main tool are filtrations related to the MSL structure.

The last section contains a natural strengthening of the MSL axioms which under certain circumstances leads to a straightening law on the symmetric algebra. The main examples of such modules are the "generic" modules defined by generic and alternating matrices.

The notion of an MSL was introduced by the author in [Br.3] and discussed extensively during the workshop. The main differences of this survey to [Br.3] are the more detailed study of examples and the treatment of the depth of MSLs which is almost entirely missing in [Br.3]

1. Algebras with Straightening Laws

An algebra with straightening law is defined over a ring B of coefficients. In order to avoid problems of secondary importance in the following sections we will assume throughout that B is a noetherian ring.

Definition. Let A be a B-algebra and $\Pi \subset A$ a finite subset with partial order $\leq$. A is an *algebra with straightening law on* Π (*over* B) if the following conditions are satisfied:
(ASL-0) $A = \bigoplus_{i \geq 0} A_i$ is a graded B-algebra such that $A_0 = B$, Π consists of homogeneous elements of positive degree and generates A as a B-algebra.
(ASL-1) The products $\xi_1 \cdots \xi_m$, $m \geq 0$, $\xi_1 \leq \cdots \leq \xi_m$ are a free basis of A as a B-module. They are called *standard monomials*.
(ASL-2) (*Straightening law*) For all incomparable $\xi, \upsilon \in \Pi$ the product $\xi\upsilon$ has a representation

$$\xi\upsilon = \sum a_\mu \mu, \qquad a_\mu \in B, a_\mu \neq 0, \quad \mu \quad \text{standard monomial},$$

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satisfying the following condition: every μ contains a factor $\zeta \in \Pi$ such that $\zeta \le \xi$, $\zeta \le \upsilon$. (It is of course allowed that $\xi\upsilon = 0$, the sum $\sum a_\mu \mu$ being empty.)

The theory of ASLs has been developed in [Ei] and [DEP.2]; the treatment in [BV.1] also satisfies our needs. In [Ei] and [BV.1] B-algebras satisfying the axioms above are called graded ASLs, whereas in [DEP.2] they figure as graded ordinal Hodge algebras.

In terms of generators and relations an ASL is defined by its poset and the straightening law:

(1.1) Proposition. *Let A be an ASL on Π. Then the kernel of the natural epimorphism*

$$B[T_\pi \colon \pi \in \Pi] \longrightarrow A, \qquad T_\pi \longrightarrow \pi,$$

is generated by the relations required in (ASL-2), *i.e. the elements*

$$T_\xi T_\upsilon - \sum a_\mu T_\mu, \qquad T_\mu = T_{\xi_1} \cdots T_{\xi_m} \quad \textit{if} \quad \mu = \xi_1 \cdots \xi_m.$$

See [DEP.2, 1.1] or [BV.1, (4.2)].

(1.2) Proposition. *Let A be an ASL on Π, and $\Psi \subset \Pi$ an ideal, i.e. $\psi \in \Psi$, $\phi \le \psi$ implies $\phi \in \Psi$. Then the ideal $A\Psi$ is generated as a B-module by all the standard monomials containing a factor $\psi \in \Psi$, and $A/A\Psi$ is an ASL on $\Pi \setminus \Psi$ ($\Pi \setminus \Psi$ being embedded into $A/A\Psi$ in a natural way.)*

This is obvious, but nevertheless extremely important. First several proofs by induction on $|\Pi|$, say, can be based on (1.2), secondly the ASL structure of many important examples is established this way.

(1.3) Examples. (a) Let X be an $m \times n$ matrix of indeterminates over B, and $\mathrm{I}_{r+1}(X)$ denote the ideal generated by the $r+1$-minors (i.e. the determinants of the $r+1 \times r+1$ submatrices) of X. For the investigation of the ideals $\mathrm{I}_{r+1}(X)$ and the residue class rings $A = B[X]/\mathrm{I}_{r+1}(X)$ one makes $B[X]$ an ASL on the set $\Delta(X)$ of all minors of X. Denote by $[a_1, \dots, a_t | b_1, \dots, b_t]$ the minor with row indices $a_1, \dots, a_t$ and column indices $b_1, \dots, b_t$. The partial order on $\Delta(X)$ is given by

$$[a_1, \dots, a_u | b_1, \dots, b_u] \le [c_1, \dots, c_v | d_1, \dots, d_v] \qquad \Longleftrightarrow$$
$$u \ge v \quad \text{and} \quad a_i \le c_i,\ b_i \le d_i,\ i = 1, \dots, v.$$

Then $B[X]$ is an ASL on $\Delta(X)$; cf. [BV.1], Section 4 for a complete proof. Obviously $\mathrm{I}_{r+1}(X)$ is generated by an ideal in the poset $\Delta(X)$, so A is an ASL on the poset $\Delta_r(X)$ consisting of all the i-minors, $i \le r$.

(b) Another example needed below is given by "pfaffian" rings. Let X_{ij}, $1 \le i < j \le n$, be a family of indeterminates over B, $X_{ji} = -X_{ij}$, $X_{ii} = 0$. The pfaffian of the alternating matrix $(X_{i_u i_v} \colon 1 \le u, v \le t)$, t even, is denoted by $[i_1, \dots, i_t]$. The polynomial ring $B[X]$ is an ASL on the set $\Phi(X)$ of the pfaffians $[i_1, \dots, i_t]$, $i_1 < \dots < i_t$, $t \le n$. The pfaffians are partially ordered in the same way as the minors in (b). The residue class ring $A = B[X]/\mathrm{Pf}_{r+2}(X)$, $\mathrm{Pf}_{r+2}(X)$ being generated by the $(r+2)$-pfaffians, inherits its ASL structure from $B[X]$ according to (1.2). The poset underlying A is denoted $\Phi_r(X)$. Note that the rings A are Gorenstein rings over a Gorenstein B—in fact factorial over a factorial B, cf. [Av.1], [KL].

(c) A non-example: If X is a symmetric $n \times n$ matrix of indeterminates, then $B[X]$ can *not* be made an ASL on $\Delta(X)$ in a natural way. Nevertheless there is a standard monomial theory for this ring based on the concept of a *doset*, cf. [DEP.2]. Many results which can be derived from this theory were originally proved by Kutz [Ku] using the method of principal radical systems. —

For an element $\xi \in \Pi$ we define its rank by

$$\operatorname{rk}\xi = k \qquad \iff \qquad \begin{array}{l}\text{there is a chain } \xi = \xi_k > \xi_{k-1} > \cdots > \xi_1,\ \xi_i \in \Pi,\\ \text{and no such chain of greater length exists.}\end{array}$$

For a subset $\Omega \subset \Pi$ let

$$\operatorname{rk}\Omega = \max\{\operatorname{rk}\xi\colon \xi \in \Omega\}.$$

The preceding definition differs from the one in [Ei] and [DEP.2] which gives a result smaller by 1. In order to reconcile the two definitions the reader should imagine an element $-\infty$ added to Π, vaguely representing $0 \in A$.

(1.4) Proposition. *Let A be an ASL on Π. Then*

$$\dim A = \dim B + \operatorname{rk}\Pi \quad \textit{and} \quad \operatorname{ht} A\Pi = \operatorname{rk}\Pi.$$

Here of course $\dim A$ denotes the Krull dimension of A and $\operatorname{ht} A\Pi$ the height of the ideal $A\Pi$. A quick proof of (1.4) may be found in [BV.1, (5.10)].

2. Straightening Laws on Modules

It occurs frequently that a module M over an ASL A has a structure closely related to that of A: the generators of M are partially ordered, a distinguished set of "standard elements" forms a B-basis of M, and the multiplication $A \times M \to A$ satisfies a straightening law similar to the straightening law in A itself. In this section we introduce the notion of a module with straightening law whereas the next section contains a list of examples.

Definition. Let A be an ASL over B on Π. An A-module M is called a *module with straightening law* (MSL) on the finite poset $\mathcal{X} \subset M$ if the following conditions are satisfied:

(MSL-1) For every $x \in \mathcal{X}$ there exists an ideal $\mathcal{I}(x) \subset \Pi$ such that the elements

$$\xi_1 \cdots \xi_n x, \qquad x \in \mathcal{X}, \quad \xi_1 \notin \mathcal{I}(x), \quad \xi_1 \le \cdots \le \xi_n, \quad n \ge 0,$$

constitute a B-basis of M. These elements are called *standard elements.*

(MSL-2) For every $x \in \mathcal{X}$ and $\xi \in \mathcal{I}(x)$ one has

$$\xi x \in \sum_{y<x} Ay.$$

It follows immediately by induction on the rank of x that the element ξx as in (MSL-2) has a standard representation

$$\xi x = \sum_{y<x} \Bigl(\sum b_{\xi x \mu y}\mu\Bigr) y, \qquad b_{\xi x \mu y} \in B,\ b_{\xi x \mu y} \neq 0,$$

in which each μy is a standard element.

(2.1) Remarks. (a) Suppose M is an MSL, and $\mathcal{T} \subset \mathcal{X}$ an ideal. Then the submodule of M generated by $\mathcal{T}$ is an MSL, too. This fact allows one to prove theorems on MSLs by noetherian induction on the set of ideals of $\mathcal{X}$.

(b) It would have been enough to require that the standard elements are linearly independent. If just (MSL-2) is satisfied then the induction principle in (a) proves that M is generated as a B-module by the standard elements. —

The following proposition helps to detect MSLs:

(2.2) Proposition. *Let M, M_1, M_2 be modules over an ASL A, connected by an exact sequence*

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0.$$

Let M_1 and M_2 be MSLs on $\mathcal{X}_1$ and $\mathcal{X}_2$, and choose a splitting f of the epimorphism $M \to M_2$ over B. Then M is an MSL on $\mathcal{X} = \mathcal{X}_1 \cup f(\mathcal{X}_2)$ ordered by $x_1 < f(x_2)$ for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, and the given partial orders on $\mathcal{X}_1$ and the copy $f(\mathcal{X}_2)$ of $\mathcal{X}_2$. Moreover one chooses $\mathcal{I}(x)$, $x \in \mathcal{X}_1$, as in M_1 and $\mathcal{I}(f(x)) = \mathcal{I}(x)$ for all $x \in \mathcal{X}_2$.

The proof is straightforward and can be left to the reader.

In terms of generators and relations an ASL is defined by its generating poset and its straightening relations, cf. (1.1). This holds similarly for MSLs:

(2.3) Proposition. *Let A be an ASL on Π over B, and M an MSL on $\mathcal{X}$ over A. Let e_x, $x \in \mathcal{X}$, denote the elements of the canonical basis of the free module $A^{\mathcal{X}}$. Then the kernel $K_{\mathcal{X}}$ of the natural epimorphism*

$$A^{\mathcal{X}} \longrightarrow M, \qquad e_x \longrightarrow x,$$

is generated by the relations required for (MSL-2)*:*

$$\rho_{\xi x} = \xi e_x - \sum_{y<x} a_{\xi x y} e_y, \qquad x \in \mathcal{X},\ \xi \in \mathcal{I}(x).$$

Proof: We use the induction principle indicated in (2.1), (a). Let $\widetilde{x} \in \mathcal{X}$ be a maximal element. Then $\mathcal{T} = \mathcal{X} \setminus \{\widetilde{x}\}$ is an ideal. By induction $A\mathcal{T}$ is defined by the relations $\rho_{\xi x}$, $x \in \mathcal{T}$, $\xi \in \mathcal{I}(x)$. Furthermore (MSL-1) and (MSL-2) imply

$$M/A\mathcal{T} \cong A/A\mathcal{I}(\widetilde{x}) \tag{1}$$

If $a_{\widetilde{x}}\widetilde{x} - \sum_{y \in \mathcal{T}} a_y y = 0$, one has $a_{\widetilde{x}} \in A\mathcal{I}(\widetilde{x})$ and subtracting a linear combination of the elements $\rho_{\xi\widetilde{x}}$ from $a_{\widetilde{x}} e_{\widetilde{x}} - \sum_{y \in \mathcal{T}} a_y e_y$ one obtains a relation of the elements $y \in \mathcal{T}$ as desired. —

The kernel of the epimorphism $A^{\mathcal{X}} \to M$ is again an MSL:

(2.4) Proposition. *With the notations and hypotheses of* (2.3) *the kernel $K_{\mathcal{X}}$ of the epimorphism $A^{\mathcal{X}} \to M$ is an MSL if we let*

$$\mathcal{I}(\rho_{\xi x}) = \{\pi \in \Pi \colon \pi \not\geq \xi\}$$

and

$$\rho_{\xi x} \le \rho_{\upsilon y} \qquad \Longleftrightarrow \qquad x < y \quad \text{or} \quad x = y,\ \xi \le \upsilon.$$

PROOF: Choose $\widetilde{x}$ and $\mathcal{T}$ as in the proof of (2.3). By virtue of (2.3) the projection $A^{\mathcal{X}} \to Ae_{\widetilde{x}}$ with kernel $A^{\mathcal{T}}$ induces an exact sequence

$$0 \longrightarrow K_{\mathcal{T}} \longrightarrow K_{\mathcal{X}} \longrightarrow A\mathcal{I}(\widetilde{x}) \longrightarrow 0.$$

Now (2.2) and induction finish the argument. —

If a module M is given in terms of generators and relations, it is in general more difficult to establish (MSL-1) than (MSL-2). For (MSL-2) one "only" has to show that elements $\rho_{\xi x}$ as in the proof of (2.3) can be obtained as linear combinations of the given relations. In this connection the following proposition may be useful: it is enough that the module generated by the $\rho_{\xi x}$ satisfies (MSL-2) again.

(2.5) Proposition. *Let the data $M, \mathcal{X}, \mathcal{I}(x), x \in \mathcal{X}$, be given as in the definition, and suppose that* (MSL-2) *is satisfied. Suppose that the kernel $K_{\mathcal{X}}$ of the natural epimorphism $A^{\mathcal{X}} \to M$ is generated by the elements $\rho_{\xi x} \in A^{\mathcal{X}}$ representing the relations in* (MSL-2)*. Order the $\rho_{\xi x}$ and choose $\mathcal{I}(\rho_{\xi x})$ as in* (2.4)*. If $K_{\mathcal{X}}$ satisfies* (MSL-2) *again, M is an MSL.*

PROOF: Let $\widetilde{x} \in \mathcal{X}$ be a maximal element, $\mathcal{T} = \mathcal{X} \setminus \{\widetilde{x}\}$. We consider the induced epimorphism

$$A^{\mathcal{T}} \longrightarrow A\mathcal{T}$$

with kernel $K_{\mathcal{T}}$. One has $K_{\mathcal{T}} = K_{\mathcal{X}} \cap A^{\mathcal{T}}$. Since the $\rho_{\xi x}$ satisfy (MSL-2), every element in $K_{\mathcal{X}}$ can be written as a B-linear combination of standard elements, and only the $\rho_{\xi \widetilde{x}}$ have a nonzero coefficient with respect to $e_{\widetilde{x}}$. The projection onto the component $Ae_{\widetilde{x}}$ with kernel $A^{\mathcal{T}}$ shows that $K_{\mathcal{T}}$ is generated by the $\rho_{\xi x}$, $x \in \mathcal{T}$. Now one can argue inductively, and the split-exact sequence

$$0 \longrightarrow A\mathcal{T} \longrightarrow M \longrightarrow M/A\mathcal{T} \cong A/A\mathcal{I}(\widetilde{x}) \longrightarrow 0$$

of B-modules finishes the proof. —

Modules with a straightening law have a distinguished filtration with cyclic quotients; by the usual induction this follows immediately from the isomorphism (1) above:

(2.6) Proposition. *Let M be an MSL on $\mathcal{X}$ over A. Then M has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that each quotient M_{i+1}/M_i is isomorphic with one of the residue class rings $A/A\mathcal{I}(x)$, $x \in \mathcal{X}$, and conversely each such residue class ring appears as a quotient in the filtration.*

It is obvious that an A-module with a filtration as in (2.6) is an MSL. It would however not be adequate to replace (MSL-1) and (MSL-2) by the condition that M has such a filtration since (MSL-1) and (MSL-2) carry more information and lend themselves to natural strengthenings, see section 5.

In section 4 we will base a depth bound for MSLs on (2.6). Further consequences concern the annihilator, the localizations with respect to prime ideals $P \in \operatorname{Ass} A$, and the rank of an MSL.

(2.7) Proposition. *Let M be an MSL on $\mathcal{X}$ over A, and*

$$J = A\Big(\bigcap_{x \in \mathcal{X}} \mathcal{I}(x)\Big).$$

Then

$$J \supset \operatorname{Ann} M \supset J^n, \qquad n = \operatorname{rk} \mathcal{X}.$$

PROOF: Note that $A(\bigcap \mathcal{I}(x)) = \bigcap A\mathcal{I}(x)$ (as a consequence of (1.2)). Since $\operatorname{Ann} M$ annihilates every subquotient of M, the inclusion $\operatorname{Ann} M \subset J$ follows from (2.6). Furthermore (MSL-2) implies inductively that

$$J^i M \subset \sum_{\operatorname{rk} x \le \operatorname{rk} \Pi - i} Ax$$

for all i, in particular $J^n M = 0$. —

(2.8) Proposition. *Let M be an MSL on $\mathcal{X}$ over A, and $P \in \operatorname{Ass} A$.*
(a) *Then $\{\pi \in \Pi \colon \pi \notin P\}$ has a single minimal element σ, and σ is also a minimal element of* Π.
(b) *Let $\mathcal{Y} = \{x \in \mathcal{X} \colon \sigma \notin \mathcal{I}(x)\}$. Then $\mathcal{Y}$ is a basis of the free A_P-module M_P. Furthermore $(K_{\mathcal{X}})_P$ is generated by the elements $\varrho_{\sigma x}$, $x \notin \mathcal{Y}$.*

PROOF: (a) If π_1, π_2, $\pi_1 \neq \pi_2$, are minimal elements of $\{\pi \in \Pi : \pi \notin P\}$, then, by (ASL-2), $\pi_1\pi_2 \in P$. So there is a single minimal element σ. It has to be a single minimal element of Π, too, since otherwise P would contain all the minimal elements of Π whose sum, however, is not a zero-divisor in A ([BV.1, (5.11)]).

(b) Consider the exact sequence

$$0 \longrightarrow A\mathcal{T} \longrightarrow M \longrightarrow A/A\mathcal{I}(\widetilde{x}) \longrightarrow 0$$

introduced in the proof of (2.3). If $\widetilde{x} \notin \mathcal{Y}$, then $\widetilde{x} \in A_P\mathcal{T}$ by the relation $\varrho_{\sigma\widetilde{x}}$, and we are through by induction. If $\widetilde{x} \in \mathcal{Y}$, then σ and all the elements of $\mathcal{I}(\widetilde{x})$ are incomparable, so they are annihilated by σ (because of (ASL-2)). Consequently $(A/A\mathcal{I}(\widetilde{x}))_P \cong A_P$, $\widetilde{x}$ generates a free summand of M_P, and induction finishes the argument again. —

We say that a module M over A has rank r if $M \otimes L$ is free of rank r as an L-module, L denoting the total ring of fractions of A. Cf. [BV.1, 16.A] for the properties of this notion.

(2.9) Corollary. *Let M be an MSL on $\mathcal{X}$ over the ASL A on* Π. *Suppose that* Π *has a single minimal element π, a condition satisfied if A is a domain. Then*

$$\operatorname{rank} M = |\{x \in \mathcal{X} \colon \mathcal{I}(x) = \emptyset\}|.$$

3. Examples

In this section we list some of the examples of MSLs. The common patterns in their treatment in [BV.1], [BV.2], and [BST] were the author's main motivation in the creation of the concept of an MSL. We start with a very simple example:

(3.1) Example. A itself is an MSL if one takes $\mathcal{X} = \{1\}$, $\mathcal{I}(1) = \emptyset$. Another choice is $\mathcal{X} = \Pi \cup \{1\}$, $\mathcal{I}(\xi) = \{\pi \in \Pi \colon \pi \not\geq \xi\}$, $\mathcal{I}(1) = \Pi$, $1 > \pi$ for each $\pi \in \Pi$. The relations necessary for (MSL-2) are then given by the identities $\pi 1 = \pi$, the straightening relations

$$\xi\upsilon = \sum b_\mu \mu, \qquad \xi, \upsilon \quad \text{incomparable},$$

and the Koszul relations

$$\xi\upsilon = \upsilon\xi, \qquad \xi < \upsilon.$$

By (2.1),(a) for every poset ideal $\Psi \subset \Pi$ the ideal $A\Psi$ is an MSL, too.

(3.2) MSLs derived from powers of ideals. (a) Suppose that Ψ as in (3.1) additionally satisfies the following condition: Whenever $\phi, \psi \in \Psi$ are incomparable, then every standard monomial μ in the standard representation $\phi\psi = \sum a_\mu \mu$, $a_\mu \neq 0$, contains at least two factors from Ψ. This condition appears in [Hu], [EH], and in [BV.1, Section 9] where the ideal $I = A\Psi$ is called *straightening-closed*. See [BST] for a detailed treatment of straightening-closed ideals. As a consequence of (b) below the powers I^n of $I = A\Psi$ are MSLs. Observe in particular that the condition above is satisfied if every μ a priori contains at most two factors and Ψ consists of the elements in Π of highest degree.

(b) In order to prove and to generalize the statements in (a) let us consider an MSL M on $\mathcal{X}$ and an ideal $\Psi \subset \Pi$ such that $I = A\Psi$ is straightening-closed and the following condition holds:

$(*)$ The standard monomials in the standard representation of a product ψx, $\psi \in \Psi$, $x \in \mathcal{X}$, all contain a factor from Ψ.

Then it is easy to see that IM is again an MSL on the set $\{\psi x \colon x \in \mathcal{X},\ \psi \in \Psi \setminus \mathcal{I}(x)\}$ partially ordered by

$$\psi x \leq \phi y \qquad \Longleftrightarrow \qquad x < y \quad \text{or} \quad x = y,\ \psi \leq \phi,$$

if one takes

$$\mathcal{I}(\psi x) = \{\pi \in \Pi \colon \pi \ngeq \psi\}.$$

Furthermore $(*)$ holds again. Thus $I^n M$ is an MSL for all $n \geq 1$, and in particular one obtains (b) from the special case $M = A$.

The residue class module M/IM also carries the structure of an MSL on the set $\overline{\mathcal{X}}$ of residues of $\mathcal{X}$ if we let

$$\mathcal{I}(\overline{x}) = \mathcal{I}(x) \cup \Psi.$$

Combining the previous arguments we get that $I^n M/I^{n+1}M$ is an MSL for all $n \geq 0$. Arguing by (2.2) one sees that all the quotients $I^n M/I^{n+k}M$ are MSLs.

In the situation just considered the associated graded ring $\operatorname{Gr}_I A$ is an ASL on the set Π^* of leading forms (ordered in the same way as Π), cf. [BST] or [BV.1,(9.8)], and obviously $\operatorname{Gr}_I M$ is an MSL on $\mathcal{X}^*$.

(c) If an ideal $I = A\Psi$ is not straightening-closed, one cannot make the associated graded ring an ASL in a natural way. Under certain circumstances there is however a "canonical" substitute, the *symbolic associated graded ring*

$$\operatorname{Gr}_I^{()}(A) = \bigoplus_{i=0}^{\infty} I^{(i)}/I^{(i+1)}.$$

Suppose that every standard monomial in a straightening relation of A contains at most two factors and that Ψ consists of all the elements of Π whose degree is at least d, d fixed. Furthermore put

$$\gamma(\pi) = \begin{cases} 0 & \text{if } \deg \pi < d, \\ \deg \pi - d + 1 & \text{else,} \end{cases} \qquad \text{and} \qquad \gamma(\pi_1 \ldots \pi_m) = \sum \gamma(\pi_i)$$

for an element $\pi \in \Pi$ and a standard monomial $\pi_1 \ldots \pi_m$ (deg denotes the degree in the graded ring A). Then it is not difficult to show that the B-submodule J_i generated by the standard monomials μ such that $\gamma(\mu) \geq i$ is an ideal of A and that $\bigoplus J_i/J_{i+1}$ is (a well-defined B-algebra and) an ASL over B on the poset given by the leading forms of the elements of Π cf. [DEP.2, Section 10]. Therefore J_i and J_i/J_{i+1} have standard B-bases and one easily establishes that they are MSLs.

For $B[X]$, B a domain, X a generic matrix of indeterminates or an alternating matrix of indeterminates, J_i indeed is the i-th symbolic power of the ideal I generated by all minors or pfaffians resp. of size d, [BV.1, 10.A] or [AD]. Consequently $\operatorname{Gr}_I^{()}(A)$ is an ASL, and $I^{(i)}$, $I^{(i)}/I^{(i+1)}$ are MSLs for all i.

(3.3) MSLs derived from generic maps. (a) Let $A = B[X]/\operatorname{I}_{r+1}(X)$ as in (1.3), (a), $0 \leq r \leq \min(m,n)$ (so $A = B[X]$ is included). The matrix x over A whose entries are the residue classes of the indeterminates defines a map $A^m \to A^n$, also denoted by x. The modules $\operatorname{Im} x$ and $\operatorname{Coker} x$ have been investigated in [Br.1]. A simplified treatment has been given in [BV.1, Section 13], from where we draw some of the arguments below. Let $d_1, \dots, d_m$ and $e_1, \dots, e_n$ denote the canonical bases of A^m and A^n. Then we order the system $\overline{e}_1, \dots, \overline{e}_n$ of generators of $M = \operatorname{Coker} x$ linearly by

$$\overline{e}_1 > \cdots > \overline{e}_n.$$

Furthermore we put

$$\mathcal{I}(\overline{e}_i) = \begin{cases} \{\, \delta \in \Delta_r(X) : \delta \not\geq [1, \dots, r | 1, \dots, \widehat{i}, \dots, r+1] \,\} & \text{for } i \leq r, \\ \emptyset & \text{else,} \end{cases}$$

if $r < n$, and in the case in which $r = n$

$$\mathcal{I}(\overline{e}_i) = \{\, \delta \in \Delta_r(X) : \delta \not\geq [1, \dots, r-1 | 1, \dots, \widehat{i}, \dots, r] \,\}.$$

(where $\widehat{i}$ denotes that i is to be omitted). We claim: M is an MSL with respect to these data.

Suppose that $\delta \in \mathcal{I}(\overline{e}_i)$. Then

$$\delta = [a_1, \dots, a_s | 1, \dots, i, b_{i+1}, \dots, b_s], \qquad s \leq r.$$

The element

$$\sum_{j=1}^{s} (-1)^{j+i} [a_1, \dots, \widehat{a_j}, \dots, a_s | 1, \dots, i-1, b_{i+1}, \dots, b_s] x(d_{a_j})$$

of $\operatorname{Im} x$ is a suitable relation for (MSL-2):

$$\text{(1)} \qquad \delta \overline{e}_i = \sum_{k=i+1}^{n} \pm [a_1, \dots, a_s | 1, \dots, i-1, k, b_{i+1}, \dots, b_s] \overline{e}_k.$$

Rearranging the column indices $1, \dots, i-1, k, b_{i+1}, \dots, b_s$ in ascending order one makes (1) the standard representation of $\delta\overline{e}_i$, and observes the following fact recorded for later purpose:

(2) $\quad \delta \notin \mathcal{I}(\overline{e}_k)$ for all $k \geq i+1$ such that $[a_1, \dots, a_s | 1, \dots, i-1, k, b_{i+1}, \dots, b_s] \neq 0$.

In order to prove the linear independence of the standard elements one may assume that $r < n$ since $\mathrm{I}_n(X)$ annihilates M. Let

$$\widetilde{M} = \sum_{i=r+1}^{n} A\overline{e}_i, \quad \Psi = \{\, \delta \in \Delta_r(X)\colon \delta \not\geq [1,\dots,r|1,\dots,r-1,r+1] \,\} \quad \text{and} \quad I = A\Psi.$$

We claim:
(i) $\widetilde{M}$ is a free A-module.
(ii) $M/\widetilde{M}$ is (over A/I) isomorphic to the ideal generated by the minors $[1,\dots,r|1,\dots,\widehat{i},\dots,r+1]$, $1 \leq i \leq r$, in A/I.
In fact, the minors just specified form a linearly ordered ideal in the poset $\Delta_r(X) \setminus \Psi$ underlying the ASL A/I, and the linear independence of the standard elements follows immediately from (i) and (ii).

Statement (i) simply holds since $\operatorname{rank} x = r$, and the r-minor in the left upper corner of x, being the minimal element of $\Delta_r(X)$, is not a zero-divisor in A. For (ii) one applies (2.3) to show that $M/\widetilde{M}$ and the ideal in (ii) have the same representation given by the matrix

$$\begin{pmatrix} x_{11} & \dots & x_{1r} \\ \vdots & & \vdots \\ x_{m1} & \dots & x_{mr} \end{pmatrix},$$

the entries taken in A/I: The assignment $\overline{e}_i \to (-1)^{i+1}[1,\dots,r|1,\dots,\widehat{i},\dots,r+1]$ induces the isomorphism. The computations needed for the application of (2.5) are covered by (1).

By similar arguments one can show that $\operatorname{Im} x$ is also an MSL, see [BV.1, proof of (13.6)] where a filtration argument is given which shows the linear independence of the standard elements. Such a filtration argument could also have been applied to prove (MSL-1) for M, cf. (c) below.

(b) Another example is furnished by the modules defined by generic alternating maps. Recalling the notations of (1.3), (b) we let $A = B[X]/\operatorname{Pf}_{r+2}(X)$ and M be the cokernel of the linear map

$$x\colon F \longrightarrow F^*, \qquad F = A^n.$$

In complete analogy with the preceding example M is an MSL on $\{\overline{e}_1,\dots,\overline{e}_n\}$, the canonical basis of F^*, $\overline{e}_1 > \dots > \overline{e}_n$, if one puts

$$\mathcal{I}(\overline{e}_i) = \begin{cases} \{\, \pi \in \Phi_r(X) : \pi \not\geq [1,\dots,\widehat{i},\dots,r+1] \,\} & \text{for} \quad i \leq r, \\ \emptyset & \text{else,} \end{cases}$$

if $r < n$, and in the case in which $r = n$

$$\mathcal{I}(\overline{e}_i) = \begin{cases} \{\, \pi \in \Phi(X) : \pi \not\geq [1,\dots,\widehat{i},\dots,r-1] \,\} & \text{for} \quad i \leq n-1, \\ \{\, [1,\dots,n] \,\} & \text{for} \quad i = n. \end{cases}$$

The straightening law (1) is replaced by the equation

$$(1') \qquad \pi\overline{e}_i = \sum_{k=i+1}^{n} \pm[1,\dots,i-1,k,b_{i+1},\dots,b_s]\,\overline{e}_k,$$

obtained from Laplace type expansion of pfaffians as (1) has been derived from Laplace expansion of minors. Observe that the analogue (2′) of (2) is satisfied. The linear independence of the standard elements is proved in entire analogy with (d). With $\widetilde{M} = \sum_{i=r+1}^{n} A\overline{e}_i$ and $I = A[1, \ldots, r]$ one has in the essential case $r < n$:
(i′) $\widetilde{M}$ is a free A-module.
(ii′) $M/\widetilde{M}$ is (overA/I) isomorphic to the ideal generated by the pfaffians $[1, \ldots, \widehat{i}, \ldots, r+1]$, $1 \le i \le r$, in A/I.

A notable special case is n odd, $r = n-1$. In this case $\operatorname{Coker} x \cong \operatorname{Pf}_r(X)$ is an ideal of grade 2 and projective dimension 2 [BE] and generated by a linearly ordered poset ideal in $\Phi(X)$.

(c) The two previous examples suggest to discuss the case of a symmetric matrix of indeterminates as in (1.3),(c), too. As mentioned there, the ring $A = B[X]/\mathrm{I}_{r+1}(X)$ is not an ASL. Nevertheless the cokernel M of the map $x\colon F \to F^*$, $F = A^n$, has the same structure relative to A as the modules in the two previous examples. With respect to what is known about the rings A, it is easier to work with slightly different arguments which could have been applied in (a) and (b), too, and were in fact applied in [BV.1] to the modules of (a).

Taking analogous notations as in (b), we put $M_i = \sum_{j=i+1}^{n} A\overline{e}_j$, $\overline{e}_j$ denoting the residue class in M of the j-th canonical basis element of F^*. One has a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_r.$$

We claim:
(i) M_r is a free A-module.
(ii) The annihilator J_i of M/M_i is the ideal generated by the i-minors of the first i columns of x.
(iii) The generator $\overline{e}_i$ of M_{i-1}/M_i is linearly independent over A/J_i.

Claim (i) is clear: $\operatorname{rank} x = r$, and the first r columns are linearly independent, hence $\operatorname{rank} M/M_r = 0 = \operatorname{rank} M - (n-r)$—none of the $r-$minors of x is a zero-divisor of A by the results of Kutz [Ku]. (This may not be found explicitely in [Ku] for arbitrary B, it is however enough to have it over a field B, cf. [BV.1, (3.15)]). Since M/M_i is represented by the matrix $(x \mid i)$ consisting of the first i columns of x, $\operatorname{Ann} M/M_i \supset J_i$. On the other hand the first $i-1$ columns of $(x \mid i)$ are linearly independent over A/J_i (again by [Ku]), and by the same argument as used for (i) one concludes (iii) and (ii).

Altogether M has a filtration by cyclic modules whose structure can be considered well-understood because of the results of [Ku] or the standard basis arguments based on the notion of a doset [DEP.2]. In particular M is a free B-module. Taking into account the remark below (2.6) one sees that one could call M an MSL relative to A. Of course the modules in (a) and (b) have an analogous filtration as follows from (2.6). —

(3.4) MSLs related to differentials and derivations. Let $A = B[X]/\mathrm{I}_{r+1}(X)$. The module $\Omega = \Omega_{A/B}$ of Kähler differentials of A and its dual Ω^*, the module of derivations, have been investigated in [Ve.1], [Ve.2], and [BV.1]. A crucial point in the investigation of Ω is a filtration which stems from an MSL structure on the first syzygy of Ω. In fact, with $I = \mathrm{I}_{r+1}(X)$, one has an exact sequence

$$0 \longrightarrow I/I^{(2)} \longrightarrow \Omega_{B[X]/B} \otimes A \longrightarrow \Omega \longrightarrow 0,$$

and it has been observed in (3.2),(c) that $I/I^{(2)}$ is an MSL.

The surjection $\Omega_{B[X]/B}\otimes A \longrightarrow \Omega$ induces an embedding $\Omega^* \longrightarrow (\Omega_{B[X]/B}\otimes A)^*$ whose cokernel is denoted N in [BV.1, Section 15]. It follows immediately from the filtration described in [BV.1, (15.3)] that N is an MSL. (It would take too much space to describe this filtration in such a detail that would save the reader to look up [BV.1].)

4. The depth of an MSL

As usual let A be an ASL over B on Π. For any A-module M we denote the length of a maximal M-sequence in $A\Pi$ by $\operatorname{depth} M$. An MSL M over A is free as a B-module, in particular flat. Let P be a prime ideal of A, $P \supset A\Pi$, and put $Q = P \cap B$, $\kappa(Q) = B_Q/QB_Q$. By [Ma, (21.B)] one has

$$\operatorname{depth} M_P = \operatorname{depth} B_Q + \operatorname{depth}(M \otimes \kappa(Q))_P.$$

Since all the prime ideals Q of B appear in the form $P \cap B$, it turns out that

$$\operatorname{depth} M = \min_P \operatorname{depth}(M \otimes \kappa(Q))_P, \qquad Q = P \cap B.$$

One sees easily that $M \otimes \kappa(Q)$ is an MSL over $A \otimes \kappa(Q)$, an ASL over $\kappa(Q)$. Therefore eventually

$$\operatorname{depth} M = \min_Q \operatorname{depth} M \otimes \kappa(Q).$$

This means: In computing $\operatorname{depth} M$ only the case in which B is a field is essential, and if the result does not depend on the particular field (as will be the case below) it holds automatically for arbitrary B. (Another possibility very often is the reduction to the case $B = \mathbf{Z}$ in order to apply results on generic perfection, cf. [BV.1], [BV.2].)

Every MSL has a natural filtration by (2.6). Applying the standard result on the behaviour of depth along short exact sequences one therefore obtains:

(4.1) Proposition. *Let M be an MSL on $\mathcal{X}$ over A. Then*

$$\operatorname{depth} M \geq \min\{\operatorname{depth} A/A\mathcal{I}(x) \colon x \in \mathcal{X}\}.$$

We specialize to ASLs over wonderful posets (cf. [Ei], [DEP.2], or [BV.1] for this notion and the properties of ASLs over wonderful posets).

(4.2) Corollary. *Let A be an ASL on the wonderful poset Π. If M is an MSL on $\mathcal{X}$ over A, then*

$$\operatorname{depth} M \geq \min\{\operatorname{rk}\Pi - \operatorname{rk}\mathcal{I}(x) \colon x \in \mathcal{X}\}.$$

Since M may be the direct sum of the quotients in its natural filtration there is no way to give a better bound for $\operatorname{depth} M$ in general. Even when (4.2) does not give the best possible result it may be useful as a "bootstrap". While it is sometimes possible to find a coarser filtration which preserves more of the structure of M, there are also examples for which the exact computation of $\operatorname{depth} M$ requires completely different, additional arguments. We now discuss the examples in the same order as in the preceding section.

(4.3) MSLs derived from powers of ideals. As in (3.2) let $I = A\Psi$ be straightening-closed. Applying (4.2) to I^n and changing to A/I^n then, one obtains:

(a) *Suppose that* Π *is wonderful. Then* $\min_i \operatorname{depth} A/I^i \geq \operatorname{rk}\Pi - \operatorname{rk}\Psi$.

Elementary examples show that (a) is by no means sharp in general: Take $A = B[X]$, X a 2×2 matrix, I the ideal generated by the elements in its first column. Then obviously $\operatorname{depth} A/I^i = 2$ for all i, and (a) gives the lower bound 2 if one takes $\Pi = \{X_{11}, X_{21}, X_{12}, X_{22}\}$, its elements ordered in the sequence given. On the other hand, the choice $\Pi = \Delta(X)$ gives the lower bound 1 only since Ψ then consists of X_{11}, X_{21}, and $[1\ 2|1\ 2]$, hence $\operatorname{rk}\Psi = 3$. Under special hypotheses the bound given by (a) is sharp however:

(b) *Suppose, in addition, that* Ψ *consists of elements of highest degree within* Π *and that the standard monomials in the straightening relations of* A *have at most two factors. Then* $\min_i \operatorname{depth} A/I^i = \operatorname{rk}\Pi - \operatorname{rk}\Psi$.

This is [BST, (3.3.3)]. We sketch its proof: First one reduces the problem to the case of a field B as above. Then one shows that $\operatorname{Gr}_I A/\Pi \operatorname{Gr}_I A$ is isomorphic to the sub-ASL of A generated by the elements of Ψ. The latter obviously has dimension $\operatorname{rk}\Psi$. Thus one knows the analytic spread $\ell(I)$ and obtains $\min_i \operatorname{depth} A/I^i = \dim A - \ell(I) = \operatorname{rk}\Pi - \operatorname{rk}\Psi$ since $\operatorname{Gr}_I A$ is a Cohen-Macaulay ring.

Completely analogous arguments can be applied to derive the same result for the ideals discussed in (3.2),(c):

(c) *Suppose that the monomials in the straightening relations of* A *have at most two factors, and let* Ψ *be the ideal of* Π *generated by the elements of degree at least* d*,* d *fixed. Then, with the notations of* (3.3),(c) *one has:* $\min_i \operatorname{depth} A/J_i = \operatorname{rk}\Pi - \operatorname{rk}\Psi$.

See [BV.1, 10.B] for the case $A = B[X]$, $\Pi = \Delta(X)$, $J = \mathrm{I}_d(X)$ in which, as mentioned in (3.3),(c) already, $J_i = \mathrm{I}_d(X)^{(i)}$.

It would be interesting to find natural filtrations on the modules I^i (or A/I^i or I^i/I^{i+1}) and J_i in order to obtain a good lower bound for the depth of each individual power. This may be possible in special cases only. The instances for which we know $\operatorname{depth} R/I^i$ precisely for all n have been discussed in [BV.1, (9.27)]. Note that these results are based on free resolutions rather than filtrations.

(4.4) MSLs derived from generic maps. (a) Let first X be an $m \times n$ matrix of indeterminates, and $A = \mathrm{R}_{r+1}(X)$. We consider the map $x\colon A^m \to A^n$ as in (3.3),(a) and its cokernel M. In determining $\operatorname{depth} M$ we assume rightaway that B is a field. Since $\mathrm{I}_n(X)$ annihilates M the case $r = n$ is covered by the case $r = n-1$; therefore one can restrict oneself to the case $r < n$. As shown in (3.3),(a) M fits into an exact sequence

$$0 \longrightarrow \widetilde{M} \longrightarrow M \longrightarrow J \longrightarrow 0$$

in which $\widetilde{M}$ is free over A and J is an ideal in A/I, I generated by the r-minors of the first r columns of x. It is not difficult to show via (1.4) that $\operatorname{depth} J = \operatorname{depth} A - 1$: A/I and $(A/I)/J$ are Cohen-Macaulay again, and the dimensions of A, A/I, and $(A/I)/J$ differ successively by 1, cf. [BV.1, proof of (13.4)]. This implies

$$\operatorname{depth} M \geq \operatorname{depth} A - 1.$$

It turns out that this inequality is an equation exactly when $m \ge n$, equivalently: $\operatorname{Ext}^1_R(M,\omega_A) = 0$ if and only if $m \ge n$. Fortunately the computations needed to prove this are not difficult—see [BV.1, 13.B] for the details.

(b) The case in which X is an alternating matrix and $A = B[X]/\operatorname{Pf}_{r+2}(X)$ is simpler such that we can give complete arguments relative to standard results on the rings A and ASLs in general.

There is one exceptional case: $n = r+1$. As stated in (3.3),(b) already, one has $M \cong \operatorname{Pf}_r(X)$, whence M is an ideal of grade 3 and projective dimension 2 in this case. In particular $\operatorname{depth} M = \operatorname{depth} A - 2$.

Similarly to (a) one can now restrict oneself to the case $r+1 < n$. Using the exact sequence analogous to the one in (a) we get $\operatorname{depth} M \ge \operatorname{depth} A - 1$: The defining ideal of $(A/I)/J$ as a residue class ring of A is generated by the pfaffians $\{\pi \in \Phi_r(X)\colon \pi \not\geq [1,\dots,r-1,r+2]\}$. Therefore $(A/I)/J$ is an ASL over a wonderful poset, cf. [BV.1, (5.10)]. Furthermore, computing the ranks of the underlying posets, one sees that the dimensions of A, A/I, and $(A/I)/J$ behave as in (a). (Note that in the exceptional case dealt with above $\dim A/I = \dim(A/I)/J + 2$.)

Since the matrix x is skew-symmetric, $M^* = \operatorname{Hom}_A(M,A) \cong \operatorname{Ker} x$, hence $\operatorname{depth} M^* \ge \min(\operatorname{depth} M + 2, \operatorname{depth} A) = \operatorname{depth} A$. Furthermore A is a Gorenstein ring (over any Gorenstein B), cf. [KL]. Since M^* is a maximal Cohen-Macaulay module, its dual M^{**} is also a maximal Cohen-Macaulay module. Now it follows that M itself is a maximal Cohen-Macaulay module over A, since M is reflexive: The inequality $\operatorname{depth} M \ge \operatorname{depth} A - 1$ carries over to all localizations of M and A. A well-known criterion for reflexivity (see [BV.1, 16.E] for example) therefore implies that it is enough to have M_P free over A_P for all prime ideals P of A such that $\operatorname{depth} A_P \le 2$. M_P is free if and only if P does not contain one of the r-pfaffians; the ideal generated by them in A has height $2(n-r)+1 \ge 5$.

(c) The main arguments in (a) and (b) are first the isomorphism $M/\widetilde{M} \cong J$ together with precise information on $\operatorname{depth} J$ and secondly a duality argument. While the isomorphism could be established in the case of a symmetric matrix X as well and the duality argument will be used below, one lacks information on $\operatorname{depth} J$. This forces us into a trickier line of proof which demonstrates the "bootstrap" function of a preliminary depth bound based on the filtration by cyclic modules as established in (3.3),(c). Again we assume that B is a field and that $r < n$.

(i) *If* $n \equiv r+1\ (2)$, *then* $\operatorname{depth} M = \operatorname{depth} A$.
(ii) *If* $n \not\equiv r+1\ (2)$, *then* $\operatorname{depth} M = \operatorname{depth} A - 1$.

Part (i) is almost as easy to prove as the same equation in (b). First we establish the depth bound based on the filtration by cyclic modules:

(iii) *For all* n *and all* r *one has*

$$\operatorname{depth} M \ge \operatorname{depth} A - r \ge \frac{1}{2}\operatorname{depth} A.$$

In fact, by [Ku]

$$\operatorname{depth} A \ge nr - r(r-1)/2,$$

implying the second inequality. In (3.3),(c) we established that M has a filtration with quotients A and A/J_i, $i = 1,\dots,r$. By [Ku] all these rings are Cohen-Macaulay, and $\dim A/J_i = \dim A + i - r - 1$. This proves the first inequality.

We now introduce a standard induction argument (which exists similarly under the conditions of (a) or (b) but was not necessary there). Take any prime ideal $Q \neq \mathrm{I}_1(x)$ in A. Then there is (1) an element $x_{ii} \notin Q$ or (2) a 2-minor $x_{ii}x_{jj} - (x_{ij})^2 \notin Q$, by symmetry $x_{11} \notin Q$ or $x_{11}x_{22} - (x_{12})^2 \notin Q$. Over $B[X][X_{11}^{-1}]$ one performs elementary row and column transformations to obtain

$$\begin{pmatrix} X_{11} & 0 & \dots & 0 \\ 0 & Y_{11} & \dots & Y_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Y_{1,n-1} & \dots & Y_{n-1,n-1} \end{pmatrix},$$

$Y_{ij} = Y_{ji} = X_{i+1,j+1}X_{11} - X_{1,i+1}X_{1,j+1}$. It is easy to see that the elements Y_{ij}, $1 \le i \le j \le n$, are algebraically independent over B and that $A[x_{11}^{-1}]$ is a Laurent polynomial extension of $B[Y]/\mathrm{I}_r(Y)$. A similar argument works in case (2), now reducing both n and r by 2.

(iv) *There are families Y_{ij}, $1 \le i \le j \le n-1$, and Z_{ij}, $1 \le i \le j \le n-2$, of algebraically independent elements over B such that $A[x_{11}^{-1}]$ is a Laurent polynomial extension of $S_{r-1} = B[Y]/\mathrm{I}_r(Y)$, and $A[(x_{11}x_{22} - x_{12}^2)^{-1}]$ is a Laurent polynomial extension of $T_{r-2} = B[Z]/\mathrm{I}_{r-1}(Z)$. In both cases M is the extension of the modules defined by Y and Z resp.*

Now we can already prove (i) under whose hypotheses A is a Gorenstein ring. Let $P \subset A$ be the irrelevant maximal ideal. Arguing inductively via (iv) one may suppose that M_Q is a maximal Cohen-Macaulay module for all primes Q different from P. Let $D = \operatorname{Coker} x^*$ be the Auslander-Bridger dual of M. Because x is symmetric, $D \cong M$. The assumptions so far imply that M_P is a d-th syzygy module, $d = \operatorname{depth} M_P$, hence

$$\operatorname{Ext}^i_{A_P}(M_P, A_P) = \operatorname{Ext}^i_{A_P}(D_P, A_P) = 0 \qquad \text{for} \quad i = 1, \dots, d,$$

(cf. [BV.1, 16.E] for example). On the other hand $\operatorname{depth} M_P \ge d$ is equivalent to

$$\operatorname{Ext}^i_{A_P}(M_P, A_P) = 0 \qquad \text{for} \quad i = \operatorname{depth} A_P - d + 1, \dots, \operatorname{depth} A_P$$

by local duality. Hence $\operatorname{Ext}^i_{A_P}(M_P, A_P) = 0$ for all $i > 0$, and M_P is a maximal Cohen-Macaulay module. This establishes (i).

Next we show that $\operatorname{depth} M < \operatorname{depth} A$ under the hypotheses of (ii). Again induction via (iv) can be applied to reduce to the case $r = 1$ first. Then $\operatorname{Ext}^1_A(M, \omega_A) \neq 0$ is obvious since ω_A is generated by the entries of the first row (or column) of x, cf. [Go].

It remains to verify that $\operatorname{depth} M \ge \operatorname{depth} A - 1$ in (ii). Since $\operatorname{depth} A/J_r = \operatorname{depth} A - 1$, it is enough to show the following statements which hold for all n and all r:

(v) *As an (A/J_r)-module M/M_r is reflexive.*
(vi) *Its dual over A/J_r is isomorphic to J_{r-1}/J_r.*
(vii) *M/M_r is a maximal Cohen-Macaulay module over A/J_r.*
(In order to include the case $r = 1$: A 0-minor has the value 1.)

To simplify the notation write $\overline{A}$ for A/J_r and $\overline{M}$ for M/M_r. Let us first observe that (vii) holds in case $n \equiv r + 1$ (2) since, as has just been proved, M is a maximal Cohen-Macaulay module over A.

Next one notices that the case $r = 1$ is indeed trivial, M/M_1 being free of rank 1 over A/J_1. Suppose that $r > 1$ and proceed by induction. Then, via M and (iv), it follows that $\overline{M}_P$ is a maximal Cohen-Macaulay module over $\overline{A}_P$ for all $P \in \operatorname{Spec} \overline{A}$, $P \not\supset \mathrm{I}_1(x)/J_r$.

For (v) it is enough to show that (1) $\overline{M}_P$ is free for all primes P such that depth $A_P \le 1$, and (2) depth $\overline{M}_P \ge 2$ for the remaining ones. (1) is clear: grade $\mathrm{I}_{r-1}(x\,|\,r)/J_r \ge 2$, and $\overline{M}_P$ is free if $P \not\supset \mathrm{I}_{r-1}(x\,|\,r)/J_r$. In order to verify (2) one may now assume that $n \ge r+2$, $r > 1$, and $P \supset \mathrm{I}_1(x)/J_r$. Then (iii) implies (2).

The dual of $\overline{M}$ is isomorphic to the kernel of the map $\overline{A}^r \to \overline{A}^n$ defined by the transpose y of $(x\,|\,r)$. Taking the determinantal relations of the rows of y, one sees that J_{r-1}/J_r is embedded in $\operatorname{Ker} y$ such that this embedding splits at all prime ideals not containing $\mathrm{I}_{r-1}(x\,|\,r)/J_r$, in particular at all primes P such that depth $A_P \le 1$. Since J_{r-1}/J_r is a maximal Cohen-Macaulay module over $\overline{A}$, (vi) follows easily.

It remains to prove (vii) for $n \not\equiv r+1$ (2). In this case J_r is the canonical module of A, so $\overline{A} = A/J_r$ is a Gorenstein ring, cf. [HK, 6.13]. By (vi) the dual of $\overline{M}$ is Cohen-Macaulay, so is $\overline{M}$ by (v).

The results of (b) and (c) are also contained in [BV.2].

(4.5) MSLs related to differentials and derivations. We resume the hypotheses and notations of (3.4). One obtains a first depth bound for $I/I^{(2)}$ from (4.3),(c) above which is already quite good; it suffices to prove that Ω is reflexive. In order to get a precise result one has however to work with a coarser filtration, cf. [BV.1, Section 14].

A similar filtration yields that depth $N \ge$ depth $A - 2$, so depth $\Omega^* \ge$ depth $A - 1$ for all values of m, n, and r. While Ω^* cannot be a maximal Cohen-Macaulay module for a determinantal ring A if A is a non-regular Gorenstein ring, i.e. when $m = n$, $1 \le r < \min(m,n)$, it has this property in all the other cases. Similar to (4.4),(a) this is shown by verifying that $\operatorname{Ext}^1_A(\Omega^*, \omega_A) = 0$. Unfortunately the details of this computation, for which we refer the reader to [BV.1, Section 15], are rather complicated.

5. Modules with a Strict Straightening Law

Some MSLs satisfy further natural axioms which strengthen (MSL-1) and (MSL-2). Let M be an MSL on $\mathcal{X}$ over A. The first additional axiom:

(MSL-3) For all $x, y \in \mathcal{X}$: $\quad x < y \;\Rightarrow\; \mathcal{I}(x) \subset \mathcal{I}(y)$.

The property (MSL-3) implies that $\Pi \cup \mathcal{X}$ is a partially ordered set if we order its subsets Π and $\mathcal{X}$ as given and all other relations are given by

$$x < \xi \qquad \Longleftrightarrow \qquad \xi \notin \mathcal{I}(x).$$

(MSL-3) simply guarantees transitivity. If it is satisfied, one can consider the following strengthening of (MSL-2):

(MSL-4) $\xi x = \sum_{y<x,\xi} a_{\xi x y} y \quad$ for all $x \in \mathcal{X}$, $\xi \in \mathcal{I}(x)$.

Definition. We say that M has a *strict straightening law* if it is an MSL satisfying (MSL-3) and (MSL-4).

An ideal $I \subset A$ generated by an ideal $\Psi \subset \Pi$ is a trivial example of a module with a strict straightening law, and the generic modules (3.3),(a) and (b) may be considered significant examples. On the other hand not every MSL has a strict straightening law. The following proposition which strengthens (2.7) excludes all the modules M/I^nM, $n \geq 2$, as in (3.2), in particular the residue class rings A/I^nA, $n \geq 2$, $I = A\Psi$ straightening-closed.

(5.1) Proposition. *Let M be a module with a strict straightening law on $\mathcal{X}$ over A. Then*

$$\operatorname{Ann} M = A(\bigcap_{x\in\mathcal{X}} \mathcal{I}(x)).$$

PROOF: In fact, if $\xi \in \bigcap \mathcal{I}(x)$, then $\xi x = 0$ for all $x \in \mathcal{X}$, since there is no element $y \in \mathcal{X}$, $y < \xi$. —

Suppose that $\mathcal{X}$ is linearly ordered. Then the straightening laws (MSL-4) and (ASL-2) constitute a set of straightening relations on $\Pi \cup \mathcal{X}$, and the following question suggests itself: Is the symmetric algebra $\mathrm{S}(M)$ an ASL over B? In general the answer is “no”, as the following example demonstrates: $A = B[X_1, X_2, X_3]$, $X_1 < X_2 < X_3$,

$$M = A^3/(A(X_1, 0, 0) + A(X_2, 0, 0) + A(0, X_1, X_3)),$$

the residue classes of the canonical basis ordered by $\overline{e}_1 > \overline{e}_2 > \overline{e}_3$. On the other hand $\mathrm{S}(I)$ is an ASL if I is generated by a linearly ordered poset ideal, cf. [BV.1, (9.13)] or [BST]; one uses that the Rees algebra $\mathcal{R}(I)$ of A with respect to I is an ASL, and concludes easily that the natural epimorphism $\mathrm{S}(I) \to \mathcal{R}(I)$ is an isomorphism. We will give a new proof of this fact below.

The following proposition may not be considered ultima ratio, but it covers the case just discussed and also the generic modules.

(5.2) Proposition. *Let M be a graded module with strict straightening law on the linearly ordered set $\mathcal{X} = \{x_1, \ldots, x_n\}$, $x_1 < \cdots < x_n$. Put $\mathcal{X}_i = \{x_1, \ldots, x_i\}$, $M_i = A\mathcal{X}_i$, $\overline{M}_{i+1} = M/M_i$, $i = 0, \ldots, n$. Suppose that for all $j > i$ and all prime ideals $P \in \operatorname{Ass}(A/A\mathcal{I}(x_j))$ the localization $(\overline{M}_i)_P$ is a free $(A/A\mathcal{I}(x_i))_P$-module, $i = 1, \ldots, n$.*
(a) *Then $\mathrm{S}(M)$ is an ASL on $\Pi \cup \mathcal{X}$.*
(b) *If $\mathcal{I}(x_1) = \emptyset$, then $\mathrm{S}(M)$ is a torsionfree A-module.*

PROOF: Since $\Pi \cup \mathcal{X}$ generates $\mathrm{S}(M)$ as a B-algebra (and $\mathrm{S}(M)$ is a graded B-algebra in a natural way) and (ASL-2) is obviously satisfied, it remains to show that the standard monomials containing k factors from $\mathcal{X}$ are linearly independent for all $k \geq 0$. Since $\mathrm{S}^0(M) = A$ this is obviously true for $k = 0$, and it remains true if $\operatorname{Ann} M = A\mathcal{I}(x_1)$ is factored out; since this does not affect the symmetric powers $\mathrm{S}^k(M)$, $k > 0$, we may assume that $\operatorname{Ann} M = 0$. If $n = 1$, then M is now a free A-module and the contention holds for trivial reasons.

The hypotheses indicate that an inductive argument is in order. Independent of the special assumptions on M_i and $\mathcal{I}(x_i)$ there is an exact sequence

$$(*) \qquad \mathrm{S}^k(M) \xrightarrow{g} \mathrm{S}^{k+1}(M) \xrightarrow{f} \mathrm{S}^{k+1}(M/Ax_1) \longrightarrow 0$$

in which f is the natural epimorphism and g is the multiplication by x_1. Let $P \in \operatorname{Ass} A$. By (2.8) x_1 generates a free direct summand of M_P. Therefore (5) splits over A_P, and $g \otimes A_P$ is injective. It is now enough to show that $\mathrm{S}^k(M)$ is torsionfree; then g is injective itself and $(*)$ splits as a sequence of B-modules as desired: By induction the standard elements in $\mathrm{S}^k(M)$ as well as in $\mathrm{S}^{k+1}(M/Ax_1)$ are linearly independent.

The linear independence of the standard elements in $\mathrm{S}^k(M)$ implies that $\mathrm{S}^k(M)$ is an MSL over A on the set of monomials of length k in $\mathcal{X}$ with respect to a suitable partial order and the choice

$$\mathcal{I}(x_{i_1}\cdots x_{i_k}) = \mathcal{I}(x_{i_k}), \qquad i_1 \le \cdots \le i_k.$$

Let $P \in \operatorname{Spec} A$, $P \notin \operatorname{Ass} A$. Then $P \notin \operatorname{Ass}(A/A\mathcal{I}(x_1))$, since $\mathcal{I}(x_1) = \emptyset$ by assumption. If $P \notin \operatorname{Ass}(A/A\mathcal{I}(x_j))$ for all $j = 2,\dots,n$, then $P \notin \operatorname{Ass}\mathrm{S}^k(M)$ by virtue of (2.6); otherwise $\mathrm{S}^k(M)_P$ is a free A_P-module by hypothesis. Altogether: $\operatorname{Ass}\mathrm{S}^k(M) = \operatorname{Ass} A$, and $\mathrm{S}^k(M)$ is torsionfree. —

(5.3) Corollary. *With the notations and hypotheses of* (5.2)*, the symmetric algebra* $\mathrm{S}(M_i)$ *is an ASL on* $\Pi \cup \mathcal{X}_i$ *for all* $i = 1,\dots,n$. $\mathrm{S}(M_i)$ *is a sub-ASL of* $\mathrm{S}(M)$ *in a natural way.*

Proof: There is a natural homomorphism $\mathrm{S}(M_i) \to \mathrm{S}(M)$ induced by the inclusion $M_i \to M$. Since $\mathrm{S}(M_i)$ satisfies (ASL-2), it is generated as a B-module by the standard monomials in $\Pi \cup \mathcal{X}_i$. Since these standard monomials are linearly independent in $\mathrm{S}(M)$, they are linearly independent in $\mathrm{S}(M_i)$, too, and $\mathrm{S}(M_i)$ is a subalgebra of $\mathrm{S}(M)$. —

The following corollary has already been mentioned:

(5.4) Corollary. *Let* A *be an ASL on* Π*, and* $\Psi \subset \Pi$ *a linearly ordered ideal. Then* $\mathrm{S}(A\Psi)$ *is an ASL on the disjoint union of* Π *and* Ψ*.*

Proof: For each $\psi \in \Psi$ the poset $\Pi \setminus \mathcal{I}(\psi)$ has ψ as its single minimal element. Let $\Psi = \{\psi_1,\dots,\psi_n\}$, $\psi_1 < \cdots < \psi_n$. If $P \in \operatorname{Ass}(A/A\mathcal{I}(\psi_j))$, then $\psi_j \notin P$ since ψ_j is not a zero-divisor of the ASL $A/A\mathcal{I}(\psi_j)$. Consequently $(A\Psi/(\sum_{k=1}^{i} A\psi_k))_P$ is isomorphic to $(A/\mathcal{I}(\psi_i))_P$ for all $i < j$. —

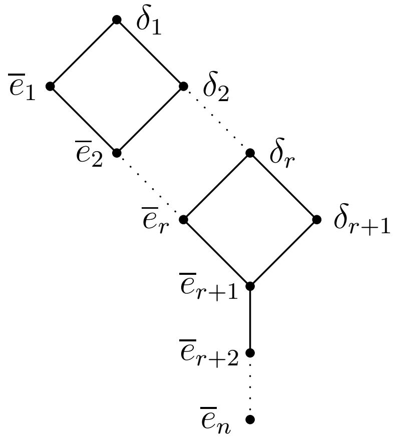
We want to apply (5.2) to the generic modules discussed in (3.3), (a), and recall the notations introduced there: $A = B[X]/\mathrm{I}_{r+1}(X)$ is an ASL on $\Delta_r(X)$, the set of all i-minors, $i \le r$, of X. M is the cokernel of the map $A^m \to A^n$ defined by the matrix x, $\overline{e}_1,\dots,\overline{e}_n$ are the residue classes of the canonical basis $e_1,\dots,e_n$ of A^n. (Thus M_k is the submodule of M generated by $\overline{e}_{n-k+1},\dots,\overline{e}_n$.)

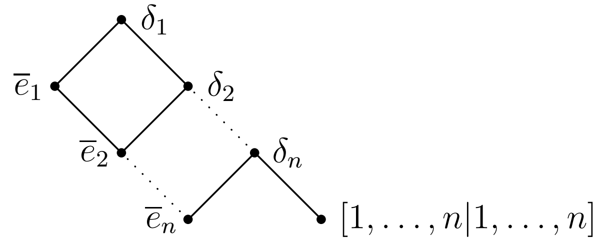
(5.5) Corollary. (a) *With the notations just recalled, the symmetric algebra of a generic module* M *is an ASL. If* $r+1 \le n$*,* $\mathrm{S}(M)$ *is torsionfree over* A*.*
(b) *Let* B *be a Cohen-Macaulay ring.* $\mathrm{S}(M)$ *is Cohen-Macaulay if and only if* $r+1 \le n$ *or* $r = m = n$*.*

Proof: (a) Factoring out the ideal generated by $\mathcal{I}(\overline{e}_n)$ we may suppose that $r < n$. Note that with the notations introduced in (3.3),(a) one has $\overline{e}_n < \cdots < \overline{e}_1$. Because of statement (ii) in (3.3),(a) the validity of the hypothesis of (5.2) for $i \ge n-r+1$ follows from the proof of (5.4).

Let $i \le n-r$, $j > i$, $k = n-j+1$, $\delta = [1,\dots,r|1,\dots,r]$ for $k \ge r+1$ and $\delta = [1,\dots,r|1,\dots,\widehat{k},\dots,r+1]$ for $k \le r$. Then δ is the minimal element of the poset underlying $A/\mathcal{I}(x_j) = A/\mathcal{I}(\overline{e}_k)$, thus not contained in an associated prime ideal of the latter. On the other hand $(\overline{M}_i)_P$ is free for every prime P not containing δ.

(b) in order to form the poset $\Pi \cup \{\overline{e}_1, \dots, \overline{e}_n\}$ one attaches $\{\overline{e}_1, \dots, \overline{e}_n\}$ to Π as indicated by the following diagrams for the cases $r+1 \le n$ and $r = m = n$ resp. In the first case we let $\delta_i = [1, \dots, r|1, \dots, \widehat{i}, \dots, r+1]$, in the second $\delta_i = [1, \dots, r-1|1, \dots, \widehat{i}, \dots, r]$.





It is an easy exercise to show that $\Pi \cup \{\overline{e}_1, \dots, \overline{e}_n\}$ and $\Pi \cup \{\overline{e}_{n-k+1}, \dots, \overline{e}_n\}$ are wonderful, implying the Cohen-Macaulay property for ASL's defined on the poset ([BV.1, Section 5] or [DEP.2]).

In the case in which $m > n = r$, the ideal $\mathrm{I}_n(X)\,\mathrm{S}(M)$ annihilates $\bigoplus_{i>0} \mathrm{S}^i(M)$, and $\dim \mathrm{S}(M)/\,\mathrm{I}_n(X) < \dim \mathrm{S}(M)$ by (1.3), excluding the Cohen-Macaulay property. —

Admittedly the preceding corollary is not a new result. In fact, let Y be an $n \times 1$ matrix of new indeterminates. Then

$$\mathrm{S}(M) \cong B[X, Y]/(\mathrm{I}_{r+1}(X) + \mathrm{I}_1(XY))$$

can be regarded as the coordinate ring of a variety of complexes, which has been shown to be a Hodge algebra in [DS]. The results of [DS] include part (b) of (5.5) as well as the fact that $\mathrm{S}(M)$ is a (normal) domain if $r + 1 \le n$ and B is a (normal) domain. The divisor class group of $\mathrm{S}(M)$ in case $r + 1 \le b$, B normal, has been computed in [Br.2]: $\mathrm{Cl}(\mathrm{S}(M)) = \mathrm{Cl}(B)$ if $m = r < n - 1$, $\mathrm{Cl}(\mathrm{S}(M)) = \mathrm{Cl}(B) \oplus \mathbf{Z}$ else. The algebras $\mathrm{S}(M)$, in particular for the cases $r + 1 > \min(m, n)$, i.e. $A = B[X]$, and $r + 1 = \min(m, n)$, have received much attention in the literature, cf. [Av.2], [BE], [BKM], and the references given there. Note that (5.5) also applies to the subalgebras $\mathrm{S}(M_k)$. In the case $A = B[X]$, $m \le n$, these rings have been analyzed in [BS].

The analogue (5.6) of (5.5) seems to be new however. We recall the notations of (3.3),(b): X is an alternating $n \times n$-matrix of indeterminates, $A = B[X]/\,\mathrm{Pf}_{r+2}(X)$, $F = A^n$, $x\colon F \to F^*$ given by the residue class of X, and $M = \operatorname{Coker} x$.

(5.6) Corollary. (a) *With the notations just recalled, the symmetric algebra of an "alternating" generic module M is an ASL. If $r < n$, $\mathrm{S}(M)$ is a torsionfree A-module.*
(b) *Let B be a Cohen-Macaulay ring. Then $\mathrm{S}(M)$ is Cohen-Macaulay if and only if $r < n$.*
(c) *Let B be a (normal) domain. Then $\mathrm{S}(M)$ is a (normal) domain if and only if $r < n$.*
(d) *Let B be normal and $r < n$. Then $\mathrm{Cl}(\mathrm{S}(M)) \cong \mathrm{Cl}(B) \oplus \mathbf{Z}$ if $r = n - 1$, and $\mathrm{Cl}(\mathrm{S}(M)) \cong \mathrm{Cl}(B)$ if $r < n - 1$. In particular $\mathrm{S}(M)$ is factorial if $r < n - 1$ and B is factorial.*

PROOF: (a) and (b) are proved in the same way as (5.5).

Standard arguments involving flatness reduce (c) to the case in which B is a field (cf. [BV.1, Section 3] for example). Thus we may certainly suppose that B is a normal domain.

In the case in which $r = n-1$ the module M is just $I = \mathrm{Pf}_{n-1}(X)$ as remarked above, an ideal generated by a linearly ordered poset ideal. Then (i) $\mathrm{Gr}_I A$ is an ASL, in particular reduced, and (ii) $\mathrm{S}(M)$ is the Rees algebra of A with respect to I (cf. [BST] for example). Thus we can apply the main result of [HV] to conclude (c) and (d).

Let $r \le n-2$ now. In the spirit of this paper a "linear" argument seems to be most appropriate: By [Fo, Theorem 10.11] and [Av.1] it is sufficient that all the symmetric powers of M are reflexive. Since M_P, hence $\mathrm{S}^k(M_P)$ is free for prime ideals $P \not\supset \mathrm{Pf}_r(x)$ it is enough to show that $\mathrm{Pf}_r(x)$ contains an $\mathrm{S}^k(M)$-sequence of length 2 for every k. Each $\mathrm{S}^k(M)$ is an MSL whose data $\mathcal{I}(\dots)$ coincide with those of M itself. Therefore (2.6) can be applied and we can replace the $\mathrm{S}^k(M)$ by the residue class rings A/I_i, $I_i = A\{\pi \in \Phi_r(x)\colon \pi \not\ge [1,\dots,\widehat{i},\dots,r+1]\}$, $i = 1,\dots,r$. One has $\mathrm{Pf}_r(X) \supset I_i$.

The poset Π underlying A/I_i is wonderful (cf. [DEP.2, Lemma 8.2] or [BV.1, (5.13)]). Therefore the elements

$$[1,\dots,\widehat{i},\dots,r+1] = \sum_{\substack{\pi\in\Pi \\ rk\pi=1}} \pi \qquad \text{and} \qquad \sum_{\substack{\pi\in\Pi \\ rk\pi=2}} \pi$$

form an A/I_i-sequence by [DEP.2, Theorem 8.1]. Both these elements are contained in $\mathrm{Pf}_r(x)$.

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